Algebra univers. **55** (2006) 387–406
0002-5240/06/020387 – 20
DOI 10.1007/s00012-006-2005-x

Algebra Universalis

Permutation representations of left quasigroups

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This paper is dedicated to Walter Taylor.

Abstract. The concept of a permutation representation has recently been extended from groups to quasigroups. Following a suggestion of Walter Taylor, the concept is now further extended to left quasigroups. The paper surveys the current state of the theory, giving new proofs where necessary to cover the general case of left quasigroups. Both the Burnside Lemma and the Burnside algebra appear in this new context.

1. Introduction

One of the major programs in the study of quasigroups has been the extension to them of the permutation representation theory of groups. After hearing the author talk about some aspects of quasigroup permutation representations during an AMS meeting at the University of Colorado in October 2003, Walter Taylor asked how much of the theory would extend to left quasigroups. This paper is intended to answer that question. It provides a record of the many results that carry over to left quasigroups, including new proofs where a reformulation of the published proofs for quasigroups is required. Section 2 recalls the combinatorial and equational definitions of quasigroups and left quasigroups. Section 3 introduces the permutation groups on the underlying set of a left quasigroup that result from the left quasigroup structure. Section 4 presents the basic concept of a left quasigroup homogeneous space, whose underlying set $P\backslash Q$ is the set of orbits of the relative left multiplication group of a subquasigroup P of a left quasigroup Q. In the group case, this is just the set of cosets of P in Q, and the group Q acts on $P\backslash Q$ by permutations specified by permutation matrices. In the left quasigroup case, the action matrices of Q on $P\backslash Q$ are stochastic or Markov matrices, the action being probabilistic rather than combinatorial. (Recall that a matrix is *stochastic* if its entries are non-negative real numbers, and each row sum is 1.) Section 5 describes general finite

Presented by J. Hyndman.
Received August 9, 2005; accepted in final form March 7, 2006.
2000 *Mathematics Subject Classification*: 20N05.
Key words and phrases: quasigroup, left quasigroup, right quasigroup, multiplication group, permutation representation, iterated function system, IFS, coalgebra, covariety, Burnside lemma, Burnside algebra.

sets acted upon by a Q-indexed set of Markov matrices as Q-IFS or iterated function systems in the sense of fractal geometry. However, the most satisfactory general description is in terms of coalgebras, which are summarized briefly in Section 6. The Q-IFS are then interpreted as certain coalgebras in Section 7. Following the technical Section 8 describing irreducible coalgebras, the Q-sets or permutation representations of a finite left quasigroup Q are defined in Section 9 as the members of the covariety of coalgebras generated by the homogeneous spaces of Q.

Structurally, Q-sets decompose as sums of disjoint orbits, homomorphic images of homogeneous spaces. For groups, the classical Burnside Lemma gives the number of orbits in a finite group permutation representation as the average number of points fixed by each group element. Section 10 presents Burnside's Lemma for left quasigroup permutation actions, with a proof specializing to a new proof of Burnside's Lemma for group permutation representations. The general formulation gives the number of orbits as the average trace of the action matrices. (Recall that the number of fixed points of a permutation is the trace of the corresponding permutation matrix.) Section 11 extends the concept of a Burnside algebra from groups to left quasigroups. The final Section 12 illustrates the Burnside algebra of the 3-element left quasigroup with right projection as the multiplication. This left quasigroup Q exhibits two characteristic features: a homomorphic image of a homogeneous space need not be isomorphic to a homogeneous space, and the underlying set of a product of Q-sets need not be the product of the underlying sets of the factors.

2. Quasigroups and left quasigroups

Quasigroups and left quasigroups may be defined combinatorially or equationally. Combinatorially, a *quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. A *left quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary multiplication such that for all x and z, there is a unique element y such that

$$x \cdot y = z \,. \tag{2.1}$$

Equationally, a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{llll} \text{(SL)} & x \cdot (x\backslash z) = z\,; & \text{(SR)} & z = (z/x) \cdot x\,; \\ \text{(IL)} & x\backslash(x \cdot z) = z\,; & \text{(IR)} & z = (z \cdot x)/x\,. \end{array}$$

A left quasigroup $(Q, \cdot, \backslash)$ is a set Q equipped with binary operations of multiplication and left division $\backslash$, satisfying the identities (SL) and (IL). These identities correspond respectively to the existence and uniqueness of the solution y to (2.1). When considering subsets of a left quasigroup, the term *subquasigroup* will be used in place of the cumbersome "sub-left-quasigroup."

3. Left multiplication groups

For each element x of a left quasigroup Q, consider the *right multiplication*

$$R(x)\colon Q \to Q;\ y \mapsto y \cdot x$$

and *left multiplication*

$$L(x)\colon Q \to Q;\ y \mapsto x \cdot y\,.$$

The left multiplications are elements of the group $Q!$ of bijections from the set Q to itself. The identity (SL) says that each $L(x)$ surjects, while (IL) gives the injectivity of $L(x)$. The *left multiplication group* of Q is the subgroup $\mathrm{LMlt}\,Q$ of $Q!$ generated by

$$\{L(q) \mid q \in Q\}. \tag{3.1}$$

For a subquasigroup P of a left quasigroup Q, the *relative left multiplication group* of P in Q is the subgroup $\mathrm{LMlt}_Q(P)$ of $\mathrm{LMlt}\,Q$ generated by

$$L_Q(P) = \{L(p)\colon Q \to Q \mid p \in P\}. \tag{3.2}$$

If Q is a group and P is nonempty, then the set of orbits of $\mathrm{LMlt}_Q P$ on Q is the set

$$P\backslash Q = \{Px \mid x \in Q\} \tag{3.3}$$

of cosets of P.

4. Homogeneous spaces

The construction of a homogeneous space for a finite left quasigroup is analogous to the permutation representation of a group Q (with subgroup P) on the homogeneous space $P\backslash Q$ by the actions

$$R_{P\backslash Q}(q)\colon P\backslash Q \to P\backslash Q\,;\ Px \mapsto Pxq \tag{4.1}$$

for elements q of Q. (See [17] and [18] for the quasigroup case.) Let P be a subquasigroup of a finite left quasigroup Q. Let $P\backslash Q$ denote the set of orbits of the permutation group $\mathrm{LMlt}_Q P$ on the set Q. If Q is a group, this notation is consistent with (3.3). Let A be the incidence matrix of the membership relation between the set Q and the set $P\backslash Q$ of subsets of Q. The pseudoinverse A^+ [13] of the incidence matrix A_P or A is the $|P\backslash Q| \times |Q|$ matrix whose entry in the row indexed by $\mathrm{LMlt}_Q P$-orbit X and column indexed by Q-element x is given by

$$\textbf{if } x \in X \textbf{ then } |X|^{-1} \textbf{ else } 0\,. \tag{4.2}$$

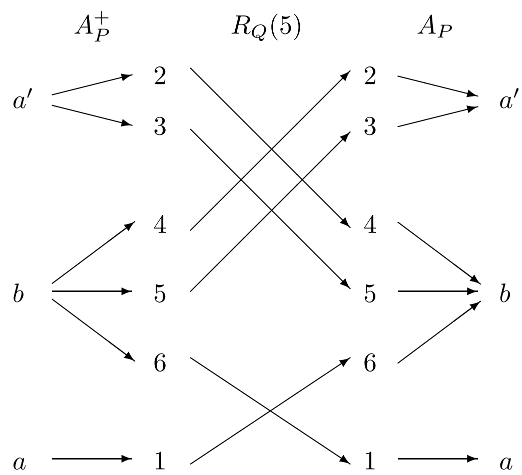

FIGURE 1. The Markov chain $R_{P\backslash Q}(5)$.

For each element q of Q, right multiplication in Q by q yields a map from the set Q to itself. Let $R_Q(q)$ be the corresponding matrix. Define the *action matrix*

$$R_{P\backslash Q}(q) = A^+ R_Q(q) A\,. \tag{4.3}$$

For a graph-theoretical interpretation of this product, see Figure 1.

Definition 4.1. Let P be a subquasigroup of a finite left quasigroup Q. Then the *homogeneous space* $(P\backslash Q, Q)$ is defined to be the set $P\backslash Q$ equipped with the set

$$\{R_{P\backslash Q}(q) \mid q \in Q\}$$

of action matrices.

Theorem 4.2. *For each element q of Q,* (4.3) *yields a Markov chain with transition matrix $R_{P\backslash Q}(q)$ on the state space $P\backslash Q$ of orbits of the permutation group* $\mathrm{LMlt}_Q P$ *on the set Q. The probability of transition from an orbit X to an orbit Y is given as*

$$|X \cap R(q)^{-1}(Y)| \,\big/\, |X|\ . \tag{4.4}$$

Proof. By (4.3), one has

$$\begin{aligned}
[R_{P\backslash Q}(q)]_{XY} &= \sum_{x\in Q}\sum_{y\in Q} A^{+}_{Xx}R(q)_{xy}A_{yY} \\
&= \sum_{x\in Q} A^{+}_{Xx}A_{(xq)Y} = \sum_{x\in X} A^{+}_{Xx}A_{(xq)Y} \\
&= |X|^{-1}|\{x \mid x\in X,\ xq\in Y\}| \\
&= |X|^{-1}|X\cap R(q)^{-1}(Y)|\ ,
\end{aligned}$$

giving (4.4). Summing (4.4) over all elements Y of $P\backslash Q$ yields the value 1. □

Corollary 4.3. *In the group case, the matrix* (4.3) *is just the permutation matrix given by the permutation* (4.1)*.*

Proof. Here, the numerator of (4.4) is $|X|$ if $XR(q) = Y$, and zero otherwise. □

Remark 4.4. With the uniform distribution on the left quasigroup Q, the quotient (4.4) becomes the conditional probability of the event $xq \in Y$ given $x \in X$. One might use this to define the action matrices directly, but (4.3) is more convenient for explicit computations.

The set of convex combinations of the states from $P\backslash Q$ forms a complete metric space, and the actions (4.3) of the left quasigroup elements form an iterated function system (IFS) in the sense of fractal geometry [1].

Example 4.5. Consider the quasigroup Q whose multiplication table is displayed in Table 1.

Q	1	2	3	4	5	6
1	1	3	2	5	6	4
2	3	2	1	6	4	5
3	2	1	3	4	5	6
4	4	5	6	1	2	3
5	5	6	4	2	3	1
6	6	4	5	3	1	2

Table 1. The quasigroup Q.

Let P be the singleton subquasigroup $\{1\}$. Note that $\mathrm{LMlt}_Q P$ is the cyclic subgroup of $Q!$ generated by $(23)(456)$. Thus

$$P\backslash Q = \{\{1\},\{2,3\},\{4,5,6\}\},$$

yielding

$$A_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_P^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

The basic definition (4.3) of the action matrix gives

$$R_{P\backslash Q}(5) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}, \tag{4.5}$$

as illustrated in Figure 1.

5. The IFS category

Let Q be a finite set. Define a Q-IFS (X,Q) as a finite set X together with an *action map*

$$R\colon Q \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}X);\ q \mapsto R_X(q) \tag{5.1}$$

from Q to the set of endomorphisms of the complex vector space with basis X (identified with their matrices with respect to the basis X), such that each *action matrix* $R_X(q)$ is stochastic. (It is convenient to work within the context of complex matrices, in order to facilitate connections with character theory such as [11].) For finite Q, the *total matrix* of (X,Q) is the sum

$$S_{(X,Q)} = \sum_{q\in Q} R_X(q) \tag{5.2}$$

of the action matrices of the elements of Q. For non-empty Q, the *Markov matrix* of (X,Q) is the mean

$$M_{(X,Q)} = \frac{1}{|Q|}\sum_{q\in Q} R_X(q) \tag{5.3}$$

of the action matrices of the elements of Q. Note that the Markov matrix of a Q-IFS is stochastic. If P is a subquasigroup of a finite non-empty left quasigroup Q, then the homogeneous space $P\backslash Q$ is a Q-IFS with the action map specified by (4.3).

A *morphism*

$$\phi\colon (X,Q) \to (Y,Q) \tag{5.4}$$

from a Q-IFS (X,Q) to a Q-IFS (Y,Q) is a function $\phi\colon X \to Y$, whose graph has incidence matrix F, such that the intertwining relation

$$R_X(q)F = FR_Y(q) \tag{5.5}$$

is satisfied for each element q of Q. It is readily checked that the class of morphisms (5.4), for a fixed finite set Q, forms a concrete category $\mathbf{IFS}_Q$.

Proposition 5.1. *Let Q be a finite group.*

(a) *The category of finite Q-sets forms the full subcategory of* $\mathbf{IFS}_Q$ *consisting of those objects for which the action map* (5.1) *is a monoid homomorphism.*
(b) *A Q-IFS* (X,Q) *is a Q-set if and only if it is isomorphic to a Q-set* (Y,Q) *in* $\mathbf{IFS}_Q$.

Proof. For (a), suppose that the action map (5.1) of a Q-IFS (X,Q) is a monoid homomorphism. Let A be in the image of (5.1). Then A is a stochastic matrix with $A^r = I$ for some positive integer r. It follows that A is a permutation matrix (cf. §XV.7 of [4]). Part (b) follows from part (a): if the morphism $\phi\colon (X,Q) \to (Y,Q)$ is an isomorphism whose graph has incidence matrix F, then the action map of (X,Q) is the composite of the action map of (Y,Q) with the monoid isomorphism $R_Y(q) \mapsto FR_Y(q)F^{-1}$ given by (5.5). □

For a fixed finite set Q, the category $\mathbf{IFS}_Q$ has finite sums or coproducts. Consider objects (X,Q) and (Y,Q) of $\mathbf{IFS}_Q$. Their *sum* or *disjoint union* $(X+Y,Q)$ consists of the disjoint union $X+Y$ of the sets X and Y together with the action map

$$q \mapsto R_X(q) \oplus R_Y(q) \tag{5.6}$$

sending each element q of Q to the direct sum of the matrices $R_X(q)$ and $R_Y(q)$. One obtains an object of $\mathbf{IFS}_Q$, since the direct sum of stochastic matrices is stochastic. The disjoint union, equipped with the appropriate insertions, yields a sum or coproduct in $\mathbf{IFS}_Q$. The *tensor product* $(X \otimes Y, Q)$ of (X,Q) and (Y,Q) is the direct product $X \times Y$ of the sets X and Y together with the action map

$$q \mapsto R_X(q) \otimes R_Y(q)$$

sending each element q of Q to the tensor (or Kronecker) product of the matrices $R_X(q)$ and $R_Y(q)$. Again, one obtains an object of $\mathbf{IFS}_Q$, since the tensor product of stochastic matrices is stochastic. The abstract significance of the tensor product is given by Corollary 7.9 below. (Contrary to an erroneous claim in [19], it does not give a product in $\mathbf{IFS}_Q$.)

6. Coalgebras and covarieties

This section summarises the basic coalgebraic concepts required. For more details, readers may consult [5], [6] or [16]. Crudely speaking, coalgebras are just the duals of algebras: coalgebras in a category $\mathcal{C}$ are algebras in the dual category $\mathcal{C}^{\mathrm{op}}$.

Let $F\colon \mathbf{Set} \to \mathbf{Set}$ be an endofunctor on the category of sets and functions. Then an F-*coalgebra*, or simply a *coalgebra* if the endofunctor is implicit in the context, is a set X equipped with a function α_X or $\alpha\colon X \to XF$. This function is known as the *structure map* of the coalgebra X. (Of course, for complete precision, one may always denote a coalgebra by its structure map.) A function $f\colon X \to Y$ between coalgebras is a *homomorphism* if $f\alpha_Y = \alpha_X f^F$. A subset S of a coalgebra X is a *subcoalgebra* if it is itself a coalgebra such that the embedding of S in X is a homomorphism. A coalgebra Y is a *homomorphic image* of a coalgebra X if there is a surjective homomorphism $f\colon X \to Y$. A *bisimulation* between coalgebras X and Y is a binary relation $R \subseteq X \times Y$ affording a coalgebra structure such that the two set product projections $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ restrict to respective coalgebra homomorphisms $R \to X$ and $R \to Y$.

Let $(X_i \mid i \in I)$ be a family of coalgebras. Then the *sum* of this family is the disjoint union of the sets of the family, equipped with a coalgebra structure map α given as follows. Let $\iota_i\colon X_i \to X$ insert X_i as a summand in the disjoint union X of the family. For each i in I, let α_i be the structure map of X_i. Then the restriction of α to the subset X_i of X is given by $\alpha_i \iota_i^F$. (More generally, the forgetful functor from coalgebras to sets creates colimits — cf. Proposition 1.1 of [2].)

A *covariety* of coalgebras is a class of coalgebras closed under the operations H of taking homomorphic images, S of taking substructures, and Σ of taking sums. If $\mathcal{K}$ is a class of F-coalgebras, then the smallest covariety containing $\mathcal{K}$ is given by $\mathsf{SH\Sigma}(\mathcal{K})$ (cf. [5, Th. 7.5] or [6, Th. 3.3]). Since homomorphic images are dual to substructures, while sums are dual to products, this result is dual to the usual characterisation of the variety generated by a class $\mathcal{K}$ of algebras as $\mathsf{HSP}(\mathcal{K})$, where P denotes the operation of taking products.

7. Actions as coalgebras

For a finite set Q, the Q-IFS are realised as coalgebras for the Q-th power of the endofunctor B sending a set to (the underlying set of) the free barycentric algebra that it generates. It is helpful to recall some basic facts about barycentric algebras. For more details, readers may consult [14] or [15]. Let I° denote the open unit interval $]0, 1[$. For p, q in I°, define $p' = 1 - p$ and $p \circ q = (p'q')'$.

Definition 7.1. A *barycentric algebra* A or (A, I°) is an algebra of type $I^\circ \times \{2\}$, equipped with a binary operation

$$\underline{p}\colon A \times A \to A;\ (x, y) \mapsto xy\,\underline{p}$$

for each p in I°, satisfying the identities

$$xx\,\underline{p} = x \tag{7.1}$$

of *idempotence* for each p in I°, the identities

$$xy\,\underline{p} = yx\,\underline{p'} \tag{7.2}$$

of *skew-commutativity* for each p in I°, and the identities

$$\big(xy\,\underline{p}\big)\ z\,\underline{q} = x\ \big(yz\,\underline{q/p \circ q}\big)\ \underline{p \circ q} \tag{7.3}$$

of *skew-associativity* for each p, q in I°. The variety of all barycentric algebras, construed as a category with the homomorphisms as morphisms, is denoted by $\mathbf{B}$. The corresponding free algebra functor is $B\colon \mathbf{Set} \to \mathbf{B}$.

A convex set C forms a barycentric algebra (C, I°), with $xy\,\underline{p} = (1-p)x + py$ for x, y in C and p in I°. A semilattice $(S, \cdot)$ becomes a barycentric algebra on setting $xy\,\underline{p} = x \cdot y$ for x, y in S and p in I°, so that semilattices form the variety of barycentric algebras satisfying the identities $xy\,\underline{p} = xy\,\underline{q}$ for p, q in I°.

For the following result, see [12], [14, §2.1], [15, §5.8].

Theorem 7.2. *Let X be a set. Then the free barycentric algebra XB on X is realized by the set of all finitely-supported probability distributions on X.*

Let P denote the covariant power set functor, and let $P_{<\omega}$ denote the free semilattice functor (realizing the free semilattice $XP_{<\omega}$ on a set X as the set of finite subsets of X). For each set X, let $\eta_X\colon XB \to XP_{<\omega}$ denote the replication, and let $\sigma_X\colon XB \to XP$ denote the composition of η_X with the embedding of $XP_{<\omega}$ in XP. In terms of Theorem 7.2, σ_X sends a finite probability distribution to its support. The σ_X form the components of a natural transformation $\sigma\colon B \to P$.

Standard "coalgebraic" properties of the functor B^Q are listed for reference in the following proposition.

Proposition 7.3. *Let Q be a finite set.*

(a) *The functor B^Q preserves weak pullbacks.*
(b) *The functor B^Q is bounded.*
(c) *Each covariety of B^Q-coalgebras is bicomplete.*
(d) *The functor B^Q preserves infinite intersections and preimages (pullbacks of monomorphisms).*

Proof. (a) By Appendix A of [23], the functor B preserves weak pullbacks. Thus the finite power B^Q of B also preserves weak pullbacks (compare [5, Lemma 8.11]).
(b) See the proof of [23, Th. 4.6].
(c) Since B^Q is bounded, the result follows according to [5, §7.4].
(d) The natural transformation $\sigma\colon B \to P$ is subcartesian (sends monomorphisms to weak pullbacks of monomorphisms), so B preserves infinite intersections and preimages by [7, §8]. The corresponding property for B^Q follows. □

Definition 7.4. Let Q be a finite set. The functor $B^Q\colon \mathbf{Set} \to \mathbf{Set}$ sends a set X to the set XB^Q of functions from Q to the free barycentric algebra XB over X. For a function $f\colon X \to Y$, its image under the functor B^Q is the function $fB^Q\colon XB^Q \to YB^Q$ defined by

$$fB^Q\colon (Q \to XB; q \mapsto w) \mapsto (Q \to YB; q \mapsto wf^B).$$

Theorem 7.5. *Let Q be a finite set. Then the category $\mathbf{IFS}_Q$ is isomorphic with the category of finite B^Q-coalgebras.*

Proof. Given a Q-IFS (X, Q) with action map R as in (5.1), define a B^Q-coalgebra $L_X\colon X \to XB^Q$ with structure map

$$L_X\colon X \to XB^Q;\ x \mapsto (Q \to XB; q \mapsto xR_X(q)). \tag{7.4}$$

(Note the use of Theorem 7.2 interpreting the vector $xR_X(q)$, lying in the simplex spanned by X, as an element of XB.) Given a Q-IFS morphism $\phi\colon (X, Q) \to (Y, Q)$ as in (5.4), with incidence matrix F, one has

$$xL_X.\phi B^Q\colon Q \to YB;\ q \mapsto xR_X(q)F \tag{7.5}$$

for each x in X, by Definition 7.4. On the other hand, one also has

$$x\phi L_Y\colon Q \to YB;\ q \mapsto xFR_Y(q). \tag{7.6}$$

By (5.5), it follows that the maps (7.5) and (7.6) agree. Thus $\phi\colon X \to Y$ is a coalgebra homomorphism. These constructions yield a functor from $\mathbf{IFS}_Q$ to the category of finite B^Q-coalgebras.

Conversely, given a finite B^Q-coalgebra with structure map $L_X\colon X \to XB^Q$, define a Q-IFS (X, Q) with action map

$$R_X\colon Q \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}X);\ q \mapsto (x \mapsto qL_X(x)), \tag{7.7}$$

well-defined by Theorem 7.2. Let $\phi\colon X \to Y$ be a coalgebra homomorphism with incidence matrix F. Then the maps (7.5) and (7.6) agree for all x in the basis X of $\mathbb{C}X$, whence (5.5) holds and $\phi\colon (X, Q) \to (Y, Q)$ becomes a Q-IFS morphism. In this way one obtains mutually inverse functors between the two categories. □

Corollary 7.6. *Each homogeneous space over a finite left quasigroup Q yields a B^Q-coalgebra.*

Example 7.7. Consider the structure map of the coalgebra corresponding to the homogeneous space presented in Example 4.5. In accordance with (4.5), the image of the state $\{4, 5, 6\}$ sends the element 5 of Q to the convex combination weighting the state $\{1\}$ with 1/3 and the state $\{2, 3\}$ with 2/3.

Corollary 7.8. *Let Q be a finite group. Then the category of finite Q-sets embeds faithfully as a full subcategory of the category of all B^Q-coalgebras.*

Proof. Apply Theorem 7.5 and Proposition 5.1. □

Corollary 7.9. [21, Cor. 5.5] *Let Q be a finite set. Let (X, Q) and (Y, Q) be objects of $\mathbf{IFS}_Q$, with corresponding B^Q-coalgebras $X \to XB^Q$ and $Y \to YB^Q$ under the isomorphism of Theorem* 7.5*. Then the tensor product $(X \otimes Y, Q)$ corresponds to a bisimulation between $X \to XB^Q$ and $Y \to YB^Q$.*

Corollary 7.10. [21, Cor. 5.6] *Let Q be a finite set. Let (X, Q) and (Y, Q) be objects of $\mathbf{IFS}_Q$, with corresponding B^Q-coalgebras $X \to XB^Q$ and $Y \to YB^Q$ under the isomorphism of Theorem* 7.5*. Then the tensor product $X \otimes Y$ forms a subcoalgebra of the product $X \times Y$ of X and Y in the category of all B^Q-coalgebras.*

8. Irreducibility and the regular representation

Let Q be a finite set. Let Y be a B^Q-coalgebra with structure map $L\colon Y \to YB^Q$. For elements y, y' of Y, the element y' is said to be *reachable* from y in Y if there is an element q of Q such that y' appears in the support of the distribution $qL(y)$ on Y. The *reachability graph* of Y is the directed graph of the reachability relation on Y. The coalgebra Y is said to be *irreducible* if its reachability graph is strongly connected.

Proposition 8.1. *If $P\backslash Q$ is a homogeneous space over a finite left quasigroup Q, realised as a B^Q-coalgebra according to Corollary* 7.6*, then $P\backslash Q$ is irreducible.*

Proof. Let H be the relative left multiplication group of P in Q. For an arbitrary pair x, x' of elements of Q, consider the corresponding elements xH and $x'H$ of $P\backslash Q$. For $q = x\backslash x'$ in Q, the element $x'H$ then appears in the support of $qL(xH)$. □

Corollary 8.2. *Let Q be a finite left quasigroup. Suppose that Y is a B^Q-coalgebra that is a homomorphic image of a homogeneous space S over Q. Then Y is irreducible.*

Proof. Since S and Y are finite, one may use the correspondence of Theorem 7.5. Let $\phi\colon S \to Y$ be the homomorphism, with incidence matrix F. Consider elements y and y' of Y. Suppose x and x' are elements of S with $x\phi = y$ and $x'\phi = y'$. By

Proposition 8.1, there is an element q of Q with x' in the support of the distribution $xR_S(q)$. Then $yR_Y(q) = xFR_Y(q) = xR_S(q)F$, so the support of $yR_Y(q)$, as the image of the support of $xR_S(q)$ under ϕ, contains $x'\phi = y'$. □

For a left quasigroup Q, the *regular* homogeneous space or permutation representation is the homogeneous space (Q, Q) or $(\varnothing\backslash Q, Q)$. Recall that the relative left multiplication group of the empty subquasigroup is trivial. A finite, non-empty left quasigroup Q may be recovered from its regular representation. For example, the multiplication table of $(Q, \backslash)$ may be realised as the formal sum $\Sigma_{q\in Q} qR_{\varnothing\backslash Q}(q)$ of multiples of the action matrices of $\varnothing\backslash Q$.

For a group Q, each homogeneous space $(P\backslash Q, Q)$ is obtained as a homomorphic image of the regular permutation representation. The following considerations show that the corresponding property does not hold for quasigroups, let alone left quasigroups.

Definition 8.3. Let Q be a finite set. A Q-IFS (X, Q) is said to be *crisp* if, for each q in Q, the action matrix $R_X(q)$ is a 0-1-matrix. A B^Q-coalgebra $L\colon X \to XB^Q$ is said to be *crisp* if its structure map corestricts to $L\colon X \to X^Q$.

Note that crisp Q-IFS and finite crisp B^Q-coalgebras correspond under the isomorphism of Theorem 7.5.

Proposition 8.4. *Homomorphic images of finite crisp B^Q-coalgebras are crisp.*

Proof. Using Theorem 7.5, it is simpler to work in the category $\mathbf{IFS}_Q$. Consider a surjective $\mathbf{IFS}_Q$-morphism $\phi\colon X \to Y$ with incidence matrix F and crisp domain. For an element y of Y, suppose that x is an element of X with $x\phi = y$. Then for each element q of Q, one has $yR_Y(q) = x\phi R_Y(q) = xFR_Y(q) = xR_X(q)F$, using (5.5) for the last step. Since X is crisp, there is an element x' of X with $xR_X(q) = x'$. Then $yR_Y(q) = x'F = y'$ for the element $y' = x'\phi$ of Y. Thus Y is also crisp. □

For each finite left quasigroup Q, the regular permutation representation is crisp. On the other hand, the quasigroup homogeneous space exhibited in Example 4.5 is not crisp. Proposition 8.4 shows that such spaces are not homomorphic images of the regular representation.

9. The covariety of Q-sets

Definition 9.1. Let Q be a finite left quasigroup. Then the *category $\underline{\underline{Q}}$ of Q-sets* or of *permutation representations* of Q is defined to be the covariety of B^Q-coalgebras generated by the (finite) set of homogeneous spaces over Q.

For a finite left quasigroup Q, the terms "Q-set" or "permutation representation of Q" are used for objects of the category of Q-sets, and also for those Q-IFS which correspond to finite Q-sets via Theorem 7.5. The term "Q-homomorphism" is used for morphisms of the category of Q-sets, and for morphisms between corresponding Q-IFS. The structure of Q-sets is described in a more general coalgebraic context.

Theorem 9.2. *Let $F\colon \mathbf{Set} \to \mathbf{Set}$ be an endofunctor on the category of sets and functions, preserving infinite intersections and preimages. Suppose that for each member K of a class $\mathcal{K}$ of F-coalgebras, K has no proper, non-empty subcoalgebras. Then the variety of F-coalgebras generated by $\mathcal{K}$ is $\Sigma\mathsf{H}(\mathcal{K})$.*

Proof. Intersections of F-subcoalgebras are subcoalgebras. Thus by [8, Prop. 2.4], the covariety generated by $\mathcal{K}$ is $\mathsf{HS}\Sigma(\mathcal{K})$. By [8, Prop. 2.5], the operators S and Σ commute. Thus the covariety generated by $\mathcal{K}$ becomes $\mathsf{H}\Sigma(\mathcal{K})$. By [8, Prop. 2.4(iii)], one has $\Sigma\mathsf{H}(\mathcal{K}) \subseteq \mathsf{H}\Sigma(\mathcal{K})$. It thus remains to be shown that each homomorphic image Y of a sum X of $\mathcal{K}$-coalgebras is a sum of homomorphic images of $\mathcal{K}$-coalgebras.

Suppose that X is the sum $\sum_{i\in I} X_i$ of coalgebras X_i from $\mathcal{K}$, with insertions $\iota\colon X_i \to X$ for each i in I. Then suppose that $\varphi\colon X \to Y$ is a surjective homomorphism. For each i in I, the composite $\iota_i\varphi\colon X_i \to Y$ corestricts to a surjective homomorphism $\varphi_i\colon X_i \to Y_i$ onto a subcoalgebra Y_i of Y. For (distinct) i, j in I, suppose that $Y_i \neq Y_j$. Let S be the proper subcoalgebra $Y_i \cap Y_j$ of Y_i. Then $\varphi^{-1}(S)$ is a proper subcoalgebra of the $\mathcal{K}$-coalgebra X_i [10, Th. 3.2]. It follows that S is empty, so that Y is the sum of homomorphic images Y_i of $\mathcal{K}$-coalgebras X_i. □

Corollary 9.3. *For a finite left quasigroup Q, the Q-sets are precisely the sums of homomorphic images of homogeneous spaces.*

Proof. By Proposition 7.3(d), the functor B^Q satisfies the hypothesis on the endofunctor F in Theorem 9.2. By Proposition 8.1, the class $\mathcal{H}$ of homogeneous spaces for Q satisfies the hypothesis on the class $\mathcal{K}$ in Theorem 9.2. □

Corollary 9.4. *A finite left quasigroup Q has only finitely many isomorphism classes of irreducible Q-sets.*

Proof. By Corollary 9.3, the irreducible Q-sets are precisely the homomorphic images of homogeneous spaces. Since Q is finite, it has only finitely many homogeneous spaces. The (First) Isomorphism Theorem for coalgebras (cf. [5, Th. 4.15]) then shows that each of these homogeneous spaces has only finitely many isomorphism classes of homomorphic images. □

Corollary 9.5. *For a finite group Q, the left quasigroup Q-sets coincide with the group Q-sets.*

Proof. For a group Q, each homomorphic image of a homogeneous space is isomorphic to a homogeneous space, and each group Q-set is isomorphic to a sum of homogeneous spaces. □

Section 12 exhibits a left quasigroup homogeneous space having a proper, nontrivial homomorphic image that is not a homogeneous space.

10. Burnside's Lemma

Definition 10.1. For a Q-set Y over a finite left quasigroup Q, the irreducible summands of Y given by Corollary 9.3 are called the *orbits* of Y. For an element y of Y, the smallest subcoalgebra of Y containing y is called the *orbit* of y.

Burnside's Lemma concerns itself with finite permutation representations. In the left quasigroup case, its formulation (and proof) rely on the identification given by Theorem 7.5. Recall that the classical Burnside Lemma for a finite group Q (compare say Theorem 3.1.2 in [22, Ch. I]) states that the number of orbits in a finite Q-set X is equal to the average number of points of X fixed by elements q of Q. The number of points fixed by such an element q is equal to the trace of the permutation matrix of q on X. In the IFS terminology of §3, this permutation matrix is the action matrix $R_X(q)$ of q on the corresponding Q-IFS (X, Q). Thus Theorem 10.3 below, the left quasigroup Burnside Lemma, specializes to the classical Burnside Lemma in the group case.

Lemma 10.2. *Let P be a subquasigroup of a left quasigroup Q. Then each row of the Markov matrix of the Q-IFS $P\backslash Q$ takes the form*

$$(|P_1|/|Q|, \ldots, |P_r|/|Q|), \tag{10.1}$$

where $P_1, \ldots, P_r$ are the orbits of the relative left multiplication group of P in Q.

Proof. By the combinatorial definition (2.1) of left quasigroups, the total matrix of the regular space $\varnothing \setminus Q$ is the $|Q| \times |Q|$ all-ones matrix J. According to (4.3), the total matrix of $P \setminus Q$ is then given by

$$A^+JA. \tag{10.2}$$

Consider the row of (10.2) indexed by a state S of $P\backslash Q$ having size s, and consider a fixed entry in this row, corresponding to a state T of $P \setminus Q$ having size t. According to (10.2), this entry is

$$\frac{1}{s} \cdot s \cdot t = t,$$

independently of the choice of S. □

Theorem 10.3 (Burnside's Lemma for left quasigroups)**.** *Let X be a finite Q-set over a finite, non-empty left quasigroup Q. Then the trace of the Markov matrix of X is equal to the number of orbits of X.*

Proof. Consider the Q-IFS (X, Q). By Theorem 7.5, Corollary 9.3 and (5.6), its Markov matrix decomposes as a direct sum of the Markov matrices of its orbits. Thus it suffices to show that the trace of the Markov matrix of a homomorphic image of a homogeneous space is equal to 1.

Consider a Q-set $Y = \{y_1, \ldots, y_m\}$ which is the image of a homogeneous space $P\backslash Q$ under a surjective homomorphism $\phi\colon P\backslash Q \to Y$ with incidence matrix F. Let F^+ be the pseudoinverse of F. Note that each row sum of F^+ is 1. Suppose that the Markov matrix Π of $P\backslash Q$ is given by Lemma 10.2. By (5.5), one has

$$R_Y(q) = F^+ R_{P\backslash Q}(q) F$$

for each q in Q. Thus the trace of the Markov matrix of Y is given by

$$\begin{aligned} tr(F^+\Pi F) &= \sum_{i=1}^{m}\sum_{j=1}^{r}\sum_{k=1}^{r} F^+_{ij}\Pi_{jk}F_{ki} \\ &= |Q|^{-1}\sum_{i=1}^{m}(\sum_{j=1}^{r} F^+_{ij})(\sum_{k=1}^{r}|P_k|F_{ki}) \\ &= |Q|^{-1}\sum_{k=1}^{r}|P_k| = 1, \end{aligned}$$

the penultimate equality following since for each $1 \leq k \leq r$, there is exactly one index i (corresponding to $P_k\phi = y_i$) such that $F_{ki} = 1$, the other terms of this type vanishing. □

Remark 10.4. Burnside's Lemma may fail for a Q-IFS which does not correspond to a Q-set. For example, the tensor square $P\backslash Q \otimes P\backslash Q$ of the quasigroup homogeneous space $P\backslash Q$ of Example 4.5 (in the category of Q-IFS) has a 9×9 Markov matrix of trace 1.875, which is not even integral.

11. The Burnside algebra

For a finite left quasigroup Q, the category $\underline{\underline{Q}}$ of Q-sets is closed under coalgebra sums. Thus the sum of two sums of images of homogeneous spaces is immediately obtained as a new sum of images of homogeneous spaces. In particular, the underlying set of a sum of Q-sets is the disjoint union of their underlying sets. However, as shown by examples such as that of Remark 10.4, the tensor product of two homogeneous spaces over Q in $\mathbf{IFS}_Q$ need not decompose as a sum of images of homogeneous spaces. By Corollary 7.10, it also follows that the direct product will not decompose as such a sum either. Nevertheless, by Proposition 7.3(c) the category $\underline{\underline{Q}}$ is bicomplete. Limits in the covariety $\underline{\underline{Q}}$ are constructed by a procedure dual to that used for the construction of colimits in a (pre)variety of τ-algebras of a given type τ (compare [22, §IV.2.2]). That procedure first builds the corresponding colimit L in the category $\underline{\underline{\tau}}$ of all τ-algebras, and then takes the replica of L in the (pre)variety, its largest homomorphic image lying in the (pre)variety. Given a B^Q-coalgebra L, its replica in $\underline{\underline{Q}}$ is obtained dually as the largest subcoalgebra of L that lies in the covariety $\underline{\underline{Q}}$. In particular, given two finite Q-sets X and Y, their *product* $X \times Y$ in $\underline{\underline{Q}}$ is formed as the largest $\underline{\underline{Q}}$-subcoalgebra contained in the product of X and Y in the category of all B^Q-coalgebras. Note that the underlying set of a product of Q-sets is not necessarily the product of their underlying sets (compare Section 12).

For a finite Q-set X, let $[X]$ denote its isomorphism type in the category $\underline{\underline{Q}}$. Let $A^+(Q)$ denote the set of all such isomorphism types. Let B be the set of so-called *basic* types, the isomorphism types of homomorphic images of homogeneous spaces over Q. It is often convenient to consider each element b of B as represented by a specified Q-set H_b. Now

$$\forall\ [X] \in A^+(Q),\ \forall\ b \in B,\ \exists\ n_b \in \mathbb{N}, \quad [X] = \sum_{b \in B} n_b b. \tag{11.1}$$

An inner product is defined on $A^+(Q)$ by

$$\Big\langle \sum_{b \in B} m_b b, \sum_{b \in B} n_b b \Big\rangle = \sum_{b \in B} m_b n_b. \tag{11.2}$$

With respect to this inner product, the set of basic types is orthonormal. The equation of (11.1) may then be rewritten as

$$[X] = \sum_{b \in B} \langle b, [X] \rangle b. \tag{11.3}$$

Theorem 11.1. *Let Q be a finite left quasigroup.*

(1) *The set $A^+(Q)$ forms a commutative unital semiring, with zero $[\varnothing]$ and unit $[\{1\}]$, under the sum $[X] + [Y] = [X + Y]$ and the* product $[X] \cdot [Y] = [X \times Y]$.
(2) *The $\mathbb{N}$-semimodule $A^+(Q)$ is free over the basis B.*

The mark concept introduced for left quasigroups in the following definition is a natural extension of Burnside's original [3, §180].

Definition 11.2. Let Q be a finite left quasigroup, and let X be a Q-set. For each basic Q-set type $b = [H_b]$, the *mark* of b in X or $x = [X]$ is defined to be the cardinality

$$Z_{xb} = \big|\underline{\underline{Q}}(H_b, X)\big| \tag{11.4}$$

of the set of Q-homomorphisms from H_b to X. The *mark matrix* or *Z-matrix* Z or Z_Q of Q is the $|B| \times |B|$ matrix $[Z_{bc}]$ for b and c in B.

Proposition 11.3. *For x, y in $A^+(Q)$ and b in B:*

(1) $Z_{(x \cdot y)b} = Z_{xb} Z_{yb}$;
(2) $Z_{(x+y)b} = Z_{xb} + Z_{yb}$;
(3) $Z_{xb} = \sum_{a \in B} \langle a, x \rangle Z_{ab}$.

Proof. Suppose $x = [X]$ and $y = [Y]$.
(1) is an immediate consequence of the definition (11.4) and the universality property of products:

$$\left|\underline{\underline{Q}}(H_b, X \times Y)\right| = \left|\underline{\underline{Q}}(H_b, X)\right| \cdot \left|\underline{\underline{Q}}(H_b, Y)\right| .$$

(2): The image of a Q-homomorphism from H_b to $X + Y$ is either a summand of X, or else a summand of Y. Thus

$$\left|\underline{\underline{Q}}(H_b, X + Y)\right| = \left|\underline{\underline{Q}}(H_b, X)\right| + \left|\underline{\underline{Q}}(H_b, Y)\right|.$$

(3) follows directly from (2) and (11.3). □

Corollary 11.4. *The product of two finite Q-sets is finite.*

Proof. Using notation as in the proof of Proposition 11.3, suppose that X and Y are finite. Then for each basic type b, the marks Z_{xb} and Z_{yb} are finite. By (1) of Proposition 11.3, the mark $Z_{(x \cdot y)b}$ is finite, so that $X \times Y$ can only contain finitely many summands of type b. □

Remark 11.5. Corollary 11.4 contrasts with examples such as that of [9, §9], where for a bounded endofunctor F weakly preserving pullbacks, it may still happen that a product of finite F-coalgebras is infinite.

Proposition 11.6. *With notation as in Definition* 11.2*:*

(1) *The set B may be ordered so that Z is triangular.*
(2) *The Z-matrix is invertible over $\mathbb{Q}$.*

Proof. (1): Linearly order B by increasing order of the cardinality of the representing Q-set, so that $b = [H] \leq [K] = c$ in B iff $|H| \leq |K|$. Then for $b > c \in B$, one has $|K| \leq |H|$. Suppose that

$$0 < Z_{bc} = \left|\underline{\underline{Q}}(K, H)\right|. \tag{11.5}$$

Now H is irreducible, so there can be a Q-homomorphism $f \colon K \to H$ in (11.5) only if $|H| = |K|$ and f bijects. Let F be the invertible incidence matrix of f. Then

$$\forall\, q \in Q,\, FR_K(q) = R_H(q)F \quad \Rightarrow \quad \forall\, q \in Q,\ R_K(q)F^{-1} = F^{-1}R_H(q),$$

so that f is a Q-isomorphism. This yields the contradiction $b = [H] = [K] = c$ to the hypothesis $b > c$ of (11.5). Thus with the given ordering of B, the Z-matrix is upper triangular.
(2): For $b = [H] \in B$, the identity map 1_H lies in $\underline{\underline{Q}}(H, H)$, so the diagonal entries of the triangular matrix Z are all non-zero. □

Theorem 11.7. *Let Q be a finite left quasigroup, with set B of basic types of Q-set. Then the* mark map

$$(A^+(Q), +, \cdot) \to \mathbb{Q}^B;\ x \mapsto (b \mapsto Z_{xb}) \tag{11.6}$$

is an embedding of semirings.

Proof. By Proposition 11.3(1)(2), the mark map is a semiring homomorphism. To see that it injects, use Proposition 11.3(3) to consider it in the equivalent form

$$(A^+(Q), +, \cdot) \to \mathbb{Q}^B;\ x \mapsto \Big(b \mapsto \sum_{a \in B} \langle a, x\rangle Z_{ab}\Big). \tag{11.7}$$

Apply Proposition 11.6 and note that

$$\sum_{c \in B} \sum_{b \in B} \Big(\sum_{a \in B} \langle a, x\rangle Z_{ab}\Big) Z_{bc}^{-1} c = \sum_{c \in B} \langle c, x\rangle c = x$$

by (11.3). □

Corollary 11.8. *Define $A(Q)$ as the $\mathbb{Q}$-vector space with basis B. Note that $A(Q)$ contains the free $\mathbb{N}$-semimodule $A^+(Q)$ of Theorem* 11.1(2) *as a subreduct. Then $A(Q)$ carries a $\mathbb{Q}$-algebra structure $(A(Q), +, \cdot)$ such that:*

(1) *The semiring $(A^+(Q), +, \cdot)$ is identified as a subreduct of the $\mathbb{Q}$-algebra*

$$(A(Q), +, \cdot);$$

(2) *The mark map* (11.6) *extends to a $\mathbb{Q}$-algebra isomorphism*

$$(A(Q), +, \cdot) \to \mathbb{Q}^B;\ \sum_{a \in B} r_a a \mapsto \Big(b \mapsto \sum_{a \in B} r_a Z_{ab}\Big). \tag{11.8}$$

Definition 11.9. For a finite left quasigroup Q, the (*rational*) *Burnside algebra* is defined to be the $\mathbb{Q}$-algebra $(A(Q), +, \cdot)$ of Corollary 11.8.

Proposition 11.10. *Let Q be a finite group. Then the Burnside algebra of Q in the left quasigroup sense of Definition* 11.9 *coincides with the Burnside algebra of Q in the classical group sense.*

Proof. By Corollary 9.5, the left quasigroup actions of Q coincide with the group actions of Q. □

12. Projections

Consider the left quasigroup $Q = \{1, 2, 3\}$ with right projection

$$Q \times Q \to Q;\ (x, y) \mapsto y$$

as the multiplication. Let $X = \{\{1\}, \{2\}, \{3\}\}$ be the regular space, with

$$\{x\}R_X(q) = \{q\}$$

for $\{x\}$ in X and q in Q. The regular space is the only homogeneous space. If $\phi\colon X \to Y$ is a surjective Q-homomorphism, (5.5) implies that $yR_Y(q) = \{q\}\phi$ for y in Y and q in Q. Thus for each equivalence relation or partition ρ on X, there is a Q-set Y_ρ for which the natural projection $\phi_\rho\colon X \to Y_\rho$ is a surjective Q-homomorphism. As an example of Burnside's Lemma (Theorem 10.3), note that the Markov matrix of the space $Y_{12.3}$ with respect to the ordered basis $(\{1, 2\}, \{3\})$ is

$$\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix},$$

with trace 1. Except for identity mappings and projections to the trivial space, the only homomorphisms amongst the spaces are the ϕ_ρ. Let x_1, x_2, x_3, x_4, x_5 denote the respective isomorphism types of the trivial space, $Y_{12.3}$, $Y_{13.2}$, $Y_{23.1}$, and X. With the convention of Proposition 11.6 (1), the Z-matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By Corollary 11.8, the non-trivial relations in the Burnside algebra are $x_i^2 = x_i$ and $x_i x_j = x_5$ with $1 < i, j \leq 5$. For example, the respective projections from the product $(Y_{12.3} \times X) = X$ to its factors are $\phi_{12.3}\colon X \to Y_{12.3}$ and $1_X\colon X \to X$. If an irreducible space W is the domain of Q-homomorphisms $f\colon W \to Y_{12.3}$ and $g\colon W \to X$, then (up to isomorphism) $W = X$, $f = \phi_{12.3}$ and $g = 1_X$, so the product map $f \times g\colon W \to X$ is just 1_X.

Acknowledgement. The author is grateful to a referee for many helpful comments on an earlier version of this paper, and in particular for proposing the general coalgebraic formulation of Theorem 9.2.

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