J. Hartwig

The Heisenberg VOA

Algebra/Combinatorics Seminar 4/17/2017

1 Formal Calculus

V v. sp. $/\mathbb{C}$

Associated to V are three vector spaces of infinite series:

$$V[[x]] = \left\{ \sum_{n=0}^{\infty} v_n x^n \,\middle|\, v_n \in V \right\} \quad \text{Power series}$$

$$\cap$$

$$V((x)) = \left\{ \sum_{n=-N}^{\infty} v_n x^n \,\middle|\, \begin{matrix} N \in \mathbb{Z} \\ v_n \in V \end{matrix} \right\} \quad \text{Laurent series}$$

$$\cap$$

$$V[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} v_n x^n \,\middle|\, v_n \in V \right\} \quad \text{doubly infinite Laurent series.}$$

A key role is played by the formal delta distribution/function

$$\delta(x) \in \mathbb{C}[[x, x^{-1}]], \quad \delta(x) = \sum_{n \in \mathbb{Z}} x^n$$

Properties

(1) $x^n \delta(x) = \delta(x)$

(2) $f(x)\delta(x) = f(1)\delta(x) \quad \forall f(x) \in V[x, x^{-1}]$

To state the next property we need the

Binomial Expansion Convention:

$$(x+y)^n = x^n(1+y/x)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k} = \begin{cases} 0 & k<0 \\ \dfrac{n(n-1)\cdots(n-k+1)}{k!}, & k \geq 0 \end{cases}$

That is, we always expand in non-negative integral powers of the second variable.

$\Rightarrow (x+y)^n \neq (y+x)^n$ if $n<0$.

(3) $$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)$$

2 Vertex (operator) algebras

Def A vertex algebra is a triple $(V, Y, \mathbb{1})$ where

- V is a vector space over $\mathbb{C}$ (state space)
- Y is a linear map (state-field correspondence)

$$V \longrightarrow (\mathrm{End}\, V)[[x, x^{-1}]]$$
$$v \longmapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

- $\mathbb{1} \in V$ (vacuum vector)

such that the following four axioms are satisfied:

(i) Truncation condition: For all $u, v \in V$ there exists $N \in \mathbb{Z}$ such that $u_n v = 0$ for all $n > N$. (Equivalently, for all $u, v \in V$: $Y(u,x)v \in V((x))$.)

(ii) Vacuum property: $Y(\mathbb{1}, x) = 1 = \mathrm{Id}_V$

(iii) Creation property: For any $v \in V$:

$$Y(v, x)\mathbb{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbb{1} = v$$

(iv) <u>Jacobi Identity</u>: For all $u, v \in V$:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{x_0}\right)Y(v,x_2)Y(u,x_1) =$$

$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v, x_2)$$

or, which can be shown to be equivalent,

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}u_{m+l-i}v_{n+i} - (-1)^l\sum_{i\geq 0}(-1)^i\binom{l}{i}v_{n+l-i}u_{m+i} =$$

$$= \sum_{i\geq 0}\binom{m}{i}(u_{l+i}v)_{m+n-i}$$

<u>Remark</u>. The Jacobi Identity for vertex algebras is analogous to the Jacobi Identity for Lie algebras L which can be written

$$(\mathrm{ad}\, u)\circ(\mathrm{ad}\, v) - (\mathrm{ad}\, v)\circ(\mathrm{ad}\, u) = \mathrm{ad}\left((\mathrm{ad}\, u)(v)\right)$$

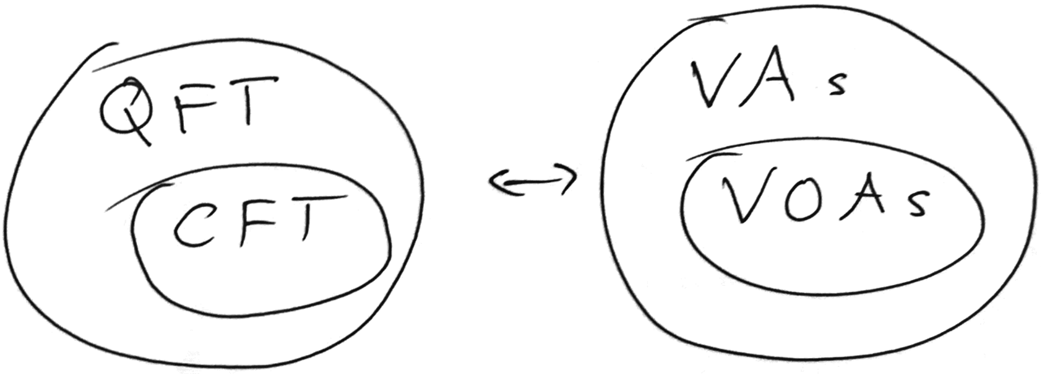
for $u, v \in L$, with $\mathrm{ad}\, u = [u, \cdot]$.

<u>Remark</u> By the creation property, the map Y is necessarily injective.

<u>Example</u> Let A be any commutative associative algebra, with identity 1.
Define for $a, b \in A$

$$Y(a, x)b = ab, \qquad \mathbb{1} = 1.$$

Then $(A, Y, \mathbb{1})$ is a vertex algebra.
Jacobi Identity follows from the delta function identity



<u>Def</u> A <u>vertex operator algebra</u> $(V, Y, \mathbb{1}, \omega)$ is a vertex algebra $(V, Y, \mathbb{1})$ together with a vector $\omega \in V$ such that

- $V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$, $\dim V_{(n)} < \infty \ \forall n$, $V_{(n)} = 0 \quad n << 0$

  and $\omega \in V_{(2)}$

- writing $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2} = \sum_{n \in \mathbb{Z}} \omega_n x^{-n-1}$

the <u>Virasoro relations</u> hold:

$$[L(m), L(n)] = (m-n) L(m+n) + \frac{1}{12}(m^3 - m)\,\delta_{m+n,0}\, c_V$$

for some $c_V \in \mathbb{C}$ (<u>central charge</u>)

- $L(0)v = nv$ for $v \in V_{(n)}$

- $Y(L(-1)v, x) = \frac{d}{dx} Y(v, x)$

(the $L(-1)$-<u>derivative property</u>.)

3 Heisenberg VOA

Let $\mathfrak{h}$ be an abelian Lie alg/$\mathbb{C}$ with a nondeg symm bil form $(\cdot|\cdot)$

Let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c$ be the corr. affinization

$$[x \otimes t^m, y \otimes t^n] = (x|y) m \delta_{m+n} c \quad , \quad c \text{ central}$$

Let $\lambda \in \mathfrak{h}^*$ and consider the Verma module

$$M(\lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c)} \mathbb{C}e^{\lambda}$$

where
$$\begin{cases} a \otimes t^n e^{\lambda} = 0 & n > 0 \\ c e^{\lambda} = 1 e^{\lambda} & \\ a \otimes 1 \, e^{\lambda} = \lambda(a) e^{\lambda} & \end{cases}$$

Thm

There exists a unique vertex a'g str on $M(1,0)$ s.t. $\mathbb{1} = e^{\lambda}$

and $Y(a,x) = a(x) \in (\mathrm{End}\, M(1,0))[[x,x^{-1}]]$

for $a \in \mathfrak{h}$. $a(x) = \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}$, $a(n) = a\otimes t^n$

Explicity

$$Y(a^{(1)}(n_1)\cdots a^{(r)}(n_r)\mathbb{1}, x) =$$

$$= a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} e^{\lambda}$$

Moreover this is a VOA with

$$\omega = \frac{1}{2}\sum_{i=1}^{d} u^{(i)}(-1)\, u^{(i)}(-1)\, \mathbb{1}$$

$d = \dim \mathfrak{h}$, $\{u^{(i)}\}$ ON basis for $\mathfrak{h}$.