Algebra Universalis, 34 (1995) 110–117 0002-5240/95/010110-08\$01.50 + 0.20/0

Vector lattices and rooted trees

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Dedicated to the memory of Alan Day

Abstract. There is a 1–1-correspondence between isomorphism classes of finite dimensional vector lattices and finite rooted unlabelled trees. Thus the problem of counting isomorphism classes of finite dimensional vector lattices reduces to the well-known combinatorial problem of counting these trees. The correspondence is used to identify the class of congruence lattices of finite-dimensional vector lattices as the class of finite dual relative Stone algebras, in partial answer to a question posed by Birkhoff.

1. Introduction

A *vector lattice* is an algebra $(V, \mathbf{R}, +, \wedge, \vee)$ that satisfies the following conditions:

(1) $(V, \mathbf{R}, +)$ is a real vector space;
(2) $(V, \wedge, \vee)$ is a lattice;
(3) $r(v \vee 0) \vee 0 = r(v \vee 0)$ for all $r \geq 0$ in $\mathbf{R}$ and all v in V;
(4) $a + (x \wedge y) = (a + x) \wedge (a + y)$ for all a, x, y in V.

It is clear from the definition that the class of all vector lattices forms an arithmetical variety. It will be shown in the next section that Birkhoff's definition of a vector lattice [1] is equivalent to the definition given above. The problem of counting the isomorphism classes of finite-dimensional vector lattices of a given dimension is reduced to the well known combinatorial problem of counting finite rooted unlabelled trees with a given number of nodes. The lattice of congruences of a finite-dimensional vector lattice is a distributive lattice, determined, using the duality theory of finite distributive lattices and finite posets, by the poset of meet

Presented by J. Sichler.
Received December 7, 1992; accepted in final form October 10, 1994.

irreducibles. The poset is shown to be a forest of rooted trees. Conversely, a forest of rooted trees may be considered as the poset of meet irreducibles of a distributive lattice which is the congruence lattice of a vector lattice determined uniquely up to isomorphism.

2. Preliminary results

Birkhoff [1, XV.1] defines a vector lattice as an algebra $(V, \mathbf{R}, +, \wedge, \vee)$ such that:

(1) $(V, \mathbf{R}, +)$ *is a real vector space*;
(2) $(V, \wedge, \vee)$ is a lattice;
(3) $x \geq 0$ in V and $r \geq 0$ in $\mathbf{R}$ implies $rx \geq 0$ in V;
(4) $x \leq y$ in V implies $(a + x) \leq (a + y)$ for all a in V.

The third condition can be replaced with an infinite set of equations, namely

$$r(x \vee 0) \vee 0 = r(x \vee 0)$$

for each $r \geq 0$ in $\mathbf{R}$. It is easy to see the equivalence of these two conditions. The fourth condition can be replaced by one of the distributive laws:

$$a + (x \wedge y) = (a + z) \wedge (a + y).$$

To see this, note that Birkhoff's fourth condition says that every group translation is an order isomorphism. Each translation has an inverse translation which is also an order isomorphism. Thus each group translation is a lattice isomorphism. This is precisely our fourth condition. The converse is obvious.

The ordinal product, $H \circ G$, of two vector lattices H and G is defined as the vector space $H \times G$ together with the order relation given as:

$$(h_1, g_1) \leq (h_2, g_2) \text{ in } H \circ G \quad \Leftrightarrow \quad (h_1 < h_2 \text{ in } H \text{ or } (h_1 = h_2 \text{ and } g_1 \leq g_2 \text{ in } G)).$$

The ordinal product $H_{\circ} G$ of two vector lattices will be a vector lattice if H is simply ordered or G is zero-dimensional.

The next theorem and its corollary give a decomposition property of finite-dimensional vector lattices. Recall that an *l-ideal* N in a vector lattice V is the kernel of a vector lattice homomorphism with domain V.

THEOREM 1 (1, XV.2. Thm. 4). *Each n-dimensional vector lattice V is either a direct product of vector lattices of lower dimension, or the ordinal product $V = \mathbf{R} \circ M$ of* **R** *with the unique maximal proper l-ideal M of V.* □

NOTE. In the directly irreducible vector lattices $V = \mathbf{R} \circ M$ of Theorem 1, the unique maximal proper *l-ideal* M consists of elements inseparable from 0 under homomorphisms to **R**. Such elements may be considered to be "infinitesimal".

COROLLARY 1 (1, XV.2. Cor. 1). *Each finite-dimensional vector lattice can be built up from copies of* **R** *by direct products and ordinal products.* □

Now recall that isomorphism classes of finite-dimensional vector lattices have the unique factorization property under direct products [1, VII.7. Cor. 1]. Thus a finite-dimensional vector lattice V can be represented uniquely as a product $\Pi_{i=1}^{m} W_i$ of directly indecomposable vector lattices W_i. Moreover, since the variety of vector lattices is arithmetical, (i.e congruence-permutable and congruence-distributive), $Con(H \times G) \cong Con(H) \times Con(G)$ for vector lattices H and G. Let $VL(n)$ be the set of isomorphism classes of n-dimensional vector lattices. The following notation will be used: if W and V are vector lattices then $[W]$ will be the class of vector lattices isomorphic to W. The class containing $\mathbf{R} \circ W$ will be the class $[\mathbf{R} \circ W]$ and the class $[V] \times [W]$ will be the class $[V \times W]$.

The symbol $\oplus$ will be used to denote the ordinal sum of two posets. That is, given two posets H and G, the poset $H \oplus G$ is the disjoint union $H \amalg G$ equipped with the order given as follows:

$$x \leq y \text{ in } H \oplus G \quad \Leftrightarrow \quad (x \leq y \text{ in } H, \text{ or } x \leq y \text{ in } G, \text{ or } x \in H \text{ and } y \in G).$$

Recall that an element x in a lattice L is *join-irreducible* if x is not the maximal element of L and for all a, b in L such that $a, b < x$, then $a \vee b < x$. Meet-irreducible elements are defined dually. Let $Ji(L)$ and $Mi(L)$ be the posets of join-irreducible and meet-irreducible elements of L. If L_1 and L_2 are distributive lattices then $Mi(L_1 \times L_2)$ is the disjoint union poset $Mi(L_1) \amalg Mi(L_2)$ [4, 8.9(3)]. In particular, since $Con(V)$ is a distributive lattice, if $V \cong H \times G$ then $MiCon(V) \cong MiCon(H) \amalg MiCon(G)$. This proves the following:

LEMMA 1. *If H and G are vector lattices then $MiCon(H \times G) \cong MiCon(H) \amalg MiCon(G)$.* □

Let L be an l-ideal in V with $L \neq V$. Then L is contained in the unique maximal ideal M of V. Thus $Con(V) \cong Con(M) \oplus \{1\}$. This proves the following:

LEMMA 2. *If V is a directly indecomposable n-dimensional vector lattice, with $n \geq 1$, then $V \cong \mathbf{R} \circ M$ for some $(n-1)$-dimensional vector lattice. Moreover $Con(V) \cong Con(M) \oplus \{1\}$.* □

The maximal element of $Con(M)$, which was not meet-irreducible in the lattice $Con(M)$, becomes meet-irreducible in $Con(V)$. This proves the following:

LEMMA 3. *If $V \cong \mathbf{R} \circ M$, then $MiCon(V) \cong MiCon(M) \oplus \{1\}$.* □

3. Rooted trees

A *rooted unlabelled* tree is an ordered set described as an ordered triple $T = (N(T), r(T), L(T))$, where $N(T)$ is the set of nodes of T, $r(T) \in N(T)$ is the root of T, and $L(T)$ is the set ("forest") of "immediate" rooted subtrees of T obtained by deleting the root and the edges incident to it. The order on $N(T)$ is defined (inductively!) as $[\coprod_{\tau \in L(T)} \tau] \oplus \{\Delta\}$. Let $RUT(n)$ be the set of isomorphism classes of rooted (unlabelled) trees with n nodes. For a tree τ, let $[\tau]$ denote the isomorphism class of τ. If F is a forest of rooted trees, let $T[F] = (\{\Delta\} \amalg \bigcup_{\tau \in F} \tau, \Delta, F)$ be the *tree generated by the forest F.* If τ_1 and τ_2 are trees, define $[T([\tau_1] \amalg [\tau_2])]$ to be the class $[T(\tau_1 \amalg \tau_2)]$.

THEOREM 2. *For each nonnegative integer n, the maps*

$$\phi_n : VL(n) \to RUT(n+1); [W] \mapsto [TMiCon(W)]$$

and

$$\psi_n : RUT(n+1) \to VL(n); [(N, r, F)] \mapsto \Pi_{\tau \in F} \mathbf{R} \circ \psi_{|N(\tau)|-1}([\tau])$$

give mutually inverse bijections between $VL(n)$ and $RUT(n+1)$.

Proof. By induction on n. For $n = 0$ there is only one isomorphism class of zero-dimensional vector lattices and there is only one isomorphism class of rooted trees with one node.

Now suppose that ϕ_k and ψ_k are mutually inverse bijections for each $k < n$.

The map ϕ_n is well-defined: Since $V \cong \Pi_{i=1}^m W_i$ with each W_i indecomposable we have

$$Con(V) \cong \Pi_{i=1}^m Con(W_i),$$

whence

$$MiCon(V) \cong \coprod_{i=1}^{m} MiCon(W_i).$$

We conclude that

$$TMiCon(V) \cong T(\coprod_{i=1}^{m} MiCon(W_i)). \tag{1}$$

If $m > 1$, by induction $TMiCon(W_i)$ has $dim(W_i) + 1$ nodes, so that $MiCon(W_i)$ has $dim(W_i)$ nodes. Then since

$$dim(V) = \sum_{i=1}^{m} dim(W_i),$$

$MiCon(V)$ has $dim(V)$ nodes and $TMiCon(V)$ has $n+1$ nodes. Since $MiCon(W_i)$ is a forest of trees by induction, $TMiCon(V)$ is a tree. If $m = 1$, $V \cong \mathbf{R} \circ U$ so that $TMiCon(V) \cong T(MiCon(U) \oplus \{1\}) \cong T(TMiCon(U))$. By induction $TMiCon(U)$ has n nodes and is a tree so $TMiCon(V)$ is a tree with $n+1$ nodes.

The map ψ_n is well-defined: Let $S = [(N, r, F)]$ be an isomorphism class of trees with $n+1$ nodes. Then

$$\psi_n(S) = \Pi_{\tau \in F} \mathbf{R} \circ \psi_{|N(\tau)|-1}([\tau])$$

is an isomorphism class of vector lattices of dimension $\sum_{\tau \in F} |N(\tau)| = n$.

To show that these maps are inverses, we have

$$\begin{aligned}
\phi_n\psi_n(S) &= \phi_n(\Pi_{\tau \in F} \mathbf{R} \circ \psi_{|N(\tau)|-1}([\tau])) \\
&= [TMiCon(\Pi_{\tau \in F} \mathbf{R} \circ \psi_{|N(\tau)|-1}([\tau]))] \\
&= [T(\coprod_{\tau \in F} MiCon(\mathbf{R} \circ \psi_{|N(\tau)|-1}([\tau])))] \text{ by (1)} \\
&= [T(\coprod_{\tau \in F} TMiCon(\psi_{|N(\tau)|-1}([\tau])))] \text{ by Lemma 2} \\
&= [T(\coprod_{\tau \in F} \phi_{|N(\tau)|-1}\psi_{|N(\tau)|-1}([\tau]))] \\
&= [T(\coprod_{\tau \in F} [\tau])] \\
&= [T(\coprod_{\tau \in F} \tau)] \\
&= S,
\end{aligned}$$

and (with notation as above)

$$\begin{aligned}
\psi_n\phi_n([V]) &\cong \psi_n\phi_n([\Pi_{i=1}^m W_i]) \\
&= \psi_n([TMiCon(\Pi_{i=1}^m W_i)]) \\
&\cong \phi_n([T(\coprod_{i=1}^m MiCon(W_i))]) \text{ by (1)} \\
&= \Pi_{i=1}^m \mathbf{R} \circ \psi_{dim(W_i)-1}([MiCon(W_i)]) \\
&\cong \Pi_{i=1}^m \mathbf{R} \circ \psi_{dim(U_i)}([TmiCon(U_i)]) \text{ by Lemma 2} \\
&= \Pi_{i=1}^m \mathbf{R} \circ \psi_{dim(U_i)}\phi_{dim(U_i)}([U_i]) \\
&= \Pi_{i=1}^m \mathbf{R} \circ [U_i] \\
&= \Pi_{i=1}^m [\mathbf{R} \circ U_i] \\
&= \Pi_{i=1}^m [W_i] \\
&= [\Pi_{i=1}^m W_i] \\
&= [V].
\end{aligned}$$
□

EXAMPLE. The tree $(\{0, 1, 2, 3\}, 0, \{(\{1\}, 1, \emptyset), (\{2, 3\}, 2, \{(\{3\}, 3, \emptyset)\})\})$ corresponds to the vector lattice $\mathbf{R} \times (\mathbf{R} \circ \mathbf{R})$.

COROLLARY 2. *The number* $|VL(n)|$ *of isomorphism classes of vector lattices of dimension* n *is determined by*

$$\sum_{n=1}^{\infty} |VL(n)|x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-|VL(n-1)|}. \tag{2}$$

Proof. By the theorem, $|VL(n)| = |RUT(n+1)|$. Formula (2) is then given by [6, (3.17.2)]. □

The first few terms of the sequence $|VL(n)|$ are

1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486, 32973, 87811, 235381, 634847, 1721159, 4688676, 12826228, 35221832, 97055181, ... [5, 454].

This corrects and extends [1, Ex. XV.3.5].

4. Relative Stone algebras

The correspondence of Theorem 2 associates a forest $MiCon(V)$ with each vector lattice V. By duality for finite distributive lattices and posets, we can consider the lattice $O(MiCon(V))$ of up-sets in the poset $MiCon(V)$, which is isomorphic to the lattice $Con(V)$. Conversely, each finite distributive lattice D with forest MiD of meet irreducibles is the congruence lattice of a vector lattice in $\psi_n(TMiD)$ for $n = |MiD|$. As a consequence, the theorem determines all congruence lattices of finite-dimensional vector lattices.

These lattices can best be described using work of Bordalo. Recall that a *Heyting algebra* is an algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ satisfying the following conditions:

(1) $(H, \wedge, \vee)$ is a lattice;
(2) $x \rightarrow y \geq z$ if and only if $x \wedge z \leq y$ for all $x, y, z \in H$.

A *Stone algebra* is a Heyting algebra that satisfies the additional condition:

$$(x \rightarrow 0) \vee ((x \rightarrow 0) \rightarrow 0) = 1.$$

A *relative Stone algebra* is a distributive lattice in which every interval is a Stone algebra, i.e.

$$(x \rightarrow a) \vee ((x \rightarrow a) \rightarrow a) = b \text{ for all } a \leq x \leq b.$$

A distributive lattice is a *dual relative Stone algebra* if its dual is a relative Stone algebra. Bordalo has characterized relative Stone algebra as follows.

THEOREM 3 (2, Thm. 13). *Let L be a finite distributive lattice with n atoms. The following are equivalent:*
(1) *L is a relative Stone algebra;*
(2) *The poset Ji(L) is the dual of a disjoint union of n trees, the maximal element of each tree being an atom of L;*
(3) *L is the direct product of n lattices, each of which is a relative Stone algebra with a unique atom.* □

Dualizing we obtain:

COROLLARY 3. *Let L be a finite distributive lattice with n coatoms. The following are equivalent:*
(1) *L is a dual relative Stone algebra;*
(2) *The poset Mi(L) is a disjoint union of n trees, the maximal element of each tree being a coatom of L;*
(3) *L is the direct product of n lattices, each of which is a dual relative Stone algebra with a unique coatom.* □

We thus obtain the following answer to Birkhoff's question [1, Problem 130] for the finite-dimensional case.

THEOREM 4. *A lattice is the congruence lattice of a finite-dimensional vector lattice if and only if it is a finite dual relative Stone algebra.* □

Conrad, Harvey, and Holland [3] have shown that a necessary condition for a lattice to be a lattice of congruences of a vector lattice is that its prime elements form a "root system," a forest in our language. Thus their condition also turns out to be sufficient in the finite-dimensional case.

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