TRIALITY AND THE HOMOGENEOUS LINEARIZATION OF QUASIGROUPS

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Abstract. In this paper, triality refers to the S_3-symmetry of the language of quasigroups, which is related to, but distinct from, the notion of triality as the S_3-symmetry of the Dynkin diagram D_4. The paper investigates a homogeneous method for rendering the linearization of quasigroups (over a commutative ring) naturally invariant under the action of the triality group, on the basis of an appropriate algebra generated by three invertible, non-commuting coefficient variables that is isomorphic to the group algebra of the free group on two generators. The algebra has a natural quotient given by setting the square of each generating variable to be -1. The quotient is an algebra of quaternions over the underlying ring, in a way reminiscent of how symmetric groups appear as quotients of braid groups on declaring the generators to be involutions. The corresponding quasigroups (which are described as quaternionic) are characterized by three equivalent pairs of quasigroup identities, permuted by the triality symmetry. The three pairs of identities are logically independent of each other. Totally symmetric quasigroups (such as Steiner triple systems) are quaternionic.

1. Introduction

1.1. **Background and motivation.** An *algebra* (A, F) in the sense of general or universal algebra is usually taken as a set A equipped with a set F of (*basic*) *operations*

$$f\colon A \times \ldots \times A \to A \tag{1.1}$$

mapping to A from a finite Cartesian power of A. The power taken is described as the *arity* of the operation. The algebra is said to be (S-)*linear* if A is a module over a commutative, unital ring S (and by default, an abelian group, so with $S = \mathbb{Z}$), while each basic operation (1.1) (of arity r) takes the *linearized form*

$$f\colon A \oplus \ldots \oplus A \to A; (x_1, \ldots, x_r) \mapsto x_1 X_1^f + \ldots + x_r X_r^f \tag{1.2}$$

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with endomorphisms $X_1^f, \dots, X_r^f$ of the S-module A that are described as *coefficients*. In slightly more sophisticated terms, a linear algebra in this sense is a general algebra in the Cartesian symmetric monoidal category $\big(\underline{\underline{S}}, \oplus, \{0\}\big)$ of S-modules, with the direct sum as monoidal product and the trivial module as the identity object of the monoidal structure (compare [7, §3.5]). It is important to bear in mind that linearity puts special emphasis on the zero element 0 of the module A, making $\{ 0 \}$ a subalgebra.

Identities that hold in general algebras imply relations among the coefficients of the linearized forms (1.2) of operations that appear in the identities. For example, an *entropic law* expressing the fact that basic binary operations f and g are mutually homomorphic reduces to the commutativity of the elements of $\{ X_1^f, X_2^f \}$ with the elements of $\{ X_1^g, X_2^g \}$.

A *quasigroup* is normally interpreted as a universal algebra $(Q, \cdot, /, \backslash)$ with three basic binary operations that satisfy the identities (2.1). Since there are no constants — no basic nullary operations — the empty set supports a quasigroup structure $(\varnothing, \cdot, /, \backslash)$. It transpires that the language of quasigroups is invariant under the natural *triality* action of the symmetric group S_3 permuting the 3-element set $\{ \cdot, /, \backslash \}$ of basic operations (§2.2). Along with these three basic operations, it is also useful to consider their respective *opposite* derived operations: $x \circ y = y \cdot x$, $x/\!/y = y/x$, and $x\backslash\!\backslash y = y\backslash x$. For the study of triality and the related question of semisymmetrization (compare [20, 23, 25]), the most natural choices of sets of basic quasigroup operations are actually the orbits $\{ \cdot, /\!/, \backslash\!\backslash \}$ or $\{ \circ, /, \backslash \}$ of the 3-element cyclic subgroup C_3 of the triality group S_3. Choice of the latter orbit as the basic set will predominate in the main part of this paper.

The paper is devoted to the study of linear quasigroups, and their triality. Since the zero element of a linear quasigroup constitutes a singleton subquasigroup, it will be an *idempotent* element, satisfying $e = e \cdot e = e/e = e\backslash e$. Murdoch's "Second Structure Theorem" [14, Th. 9] states that each entropic quasigroup containing an idempotent element is linear. (Murdoch used the term "abelian" for "entropic".) In this context, it should be emphasized that the idempotent element is not generally defined by the quasigroup structure. Indeed, Murdoch provides an example to exhibit nonempty subquasigroups devoid of idempotent elements within a linear quasigroup [14, pp. 403–4]. As a result, it seems inappropriate to describe quasigroup representations of this nature as "structure theorems". Rather, they should be considered as *coordinatizations* of the quasigroup about a chosen idempotent.

In order to gain structural access to an idempotent element, we use the concept of a *pique*,[1] a general algebra $(P, \cdot, /, \backslash, e)$ in which $(P, \cdot, /, \backslash)$ is a quasigroup and e is (an idempotent element selected by) a nullary operation $e \colon P^0 \to P$ [21, §1.5]. Thus, we construe a linear quasigroup $(A, \cdot, /, \backslash)$ as a linear pique $(A, \cdot, /, \backslash, 0)$. Linear piques are equivalent to modules for the integral group algebra $\mathbb{Z}\langle R, L\rangle$ of the free group $\langle R, L\rangle$ over a two-element set $\{ R, L \}$, with the quasigroup operations $x \cdot y = xR + yL$ and $x/y = xR^{-1} - yLR^{-1}$ (§3.1). In particular, linear representations of two-generated groups (including all the finite simple groups [1, Th. B]) are captured by linear pique structures.

The underlying undirected graph of the Cayley graph of the free group $\langle R, L\rangle$ with respect to its generators R and L (compare [24, Fig. I.1.1], for example) is a homogeneous tree of valency 4. As such, its vertex set, the free group, is a metric space under the usual graph distance. However, while the distance between the coefficients R and L of $x \cdot y$ is 2, the distance between the (unsigned) coefficients R^{-1} and LR^{-1} of x/y is 3 (3.5). The discrepancy means that the representation of linear piques furnished by the free group algebra $\mathbb{Z}\langle R, L\rangle$ does not behave well under the triality action. It is this issue which provides the primary motivation for the current paper:

> How can linear piques be represented geometrically and algebraically in a way which is fully compatible with the triality symmetry of the language of quasigroups?

Preliminary observations in this direction (summarized in §§3.2–3.3) appeared in [25], inspired by consideration of the reversible automata of Gvaramiya and Plotkin [4, 5].

1.2. **Plan of the paper.** Chapter 2 establishes the background for quasigroups and their triality. Chapter 3 introduces the *homogeneous algebra* $\mathfrak{H}_S$ over a commutative, unital ring S that presents the free group algebra $S\langle R, L\rangle$ in a form which is naturally invariant under quasigroup triality, thereby answering the motivating question raised above. Here, (3.6) expresses the basic quasigroup operations $\circ, /, \backslash$ in terms of the set $\{ T_1, T_2, T_3 \}$ of generating invertible indeterminates of $\mathfrak{H}_S$. As displayed in Figure 2, this expression is equivariant under the respective triality of the quasigroup operations and the permutation of the algebra generators and their inverses.

Given the failure of the homogeneous tree of valency four to reflect triality, it is natural to look for implementations of triality within the

[1] The name is taken from an acronym for a <u>P</u>ointed <u>I</u>dempotent <u>QU</u>asigroup<u>E</u>, using the French spelling of "quasigroup".

symmetries of other geometrical structures. As an example, Section 3.5 examines the Euclidean cube from this point of view. Thus, (3.19) exhibits the triality group S_3 as a subgroup of the group S_4 of rotational symmetries of the cube in three-dimensional Euclidean space.

Over a commutative ring S, the homogeneous algebra is presented in the compact form

$$\mathfrak{H}_S = S\left\langle T_i \mid 1 + T_i T_{i-1} T_{i-2} \right\rangle_{i \in \mathbb{Z}/3}$$

(Remark 3.9). Chapter 4 is devoted to the quotient

$$\mathbb{H}_S = S\left\langle T_i \mid 1 + T_i T_{i-1} T_{i-2}, 1 + T_i^2 \right\rangle_{i \in \mathbb{Z}/3}$$

of $\mathfrak{H}_S$ obtained by imposing the additional power relations $T_i^2 = -1$ for $1 \leq i \leq 3$. (The relationship between $\mathfrak{H}_S$ and $\mathbb{H}_S$ is somewhat analogous to the relationship between a braid group and a symmetric group, where the latter is obtained by declaring the usual braid group generators to be involutions.) In Definition 4.1, the algebra $\mathbb{H}_S$ is described as a *quaternion algebra* over the ring S, based on the nature of the generating matrices that appear. The connection between this matrix definition and the presentation is made in Theorem 4.7.

Definition 4.13 identifies a linear pique as being *quaternionic* if its $\mathfrak{H}_S$-module structure descends to a $\mathbb{H}_S$-module structure. Theorem 4.15 characterizes quaternionic piques by three identities that are written entirely in the language of quasigroups, without appeal to the pointed idempotent. These three quasigroup identities are paired with their equivalent opposite identities. The full set of six identities is permuted by triality, as displayed in Figure 3. The final result, Theorem 4.20, shows that the three equivalent pairs of identities are independent of each other, even within the class of linear piques.

1.3. **Notational conventions.** The notation and conventions of [24] are generally followed in the paper. In particular, diagrammatic or algebraic notation is used by default: functions follow their arguments, either on the line (as in $n!$ for the factorial function) or in a superfix (as in x^2 for the squaring function).

2. Quasigroups and triality

2.1. **Quasigroups.** A *combinatorial quasigroup* $(Q, \cdot)$ is a set equipped with a binary operation $\cdot$ for which knowledge of any two of x, y, z from Q in the equation $x \cdot y = z$ specifies the third uniquely. An *equational quasigroup* $(Q, \cdot, /, \backslash)$ is a set that is equipped with three *basic* binary operations of multiplication $\cdot$, right division $/$, and left division $\backslash$ such that for all $x, y \in Q$, the following identities are satisfied:

$$\begin{array}{llll} \text{(SL)} & x \cdot (x\backslash y) = y\,; & \text{(SR)} & y = (y/x) \cdot x\,; \\ \text{(IL)} & x\backslash(x \cdot y) = y\,; & \text{(IR)} & y = (y \cdot x)/x\,. \end{array} \tag{2.1}$$

The product $x \cdot y$ is often simply written as a juxtaposition xy. In a combinatorial quasigroup $(Q, *)$, consider the left multiplication

$$L_*(x) \colon Q \to Q; y \mapsto x * y \tag{2.2}$$

and right multiplication

$$R_*(x) \colon Q \to Q; y \mapsto y * x \tag{2.3}$$

for $x \in Q$. If $(Q, *) = (Q, \cdot)$, two special conventions apply: A default convention $L_*(x) = L(x), R_*(x) = R(x)$, and an enlargement of the dot to $L_*(x) = L_\bullet(x), R_*(x) = R_\bullet(x)$ for enhanced readability. These conventions are used in the following paragraph.

The identities (IL) and (IR) give the respective injectivity of $L_\bullet(x)$ and $R_\bullet(x)$, while the identities (SL), (SR) give their surjectivity. Thus an equational quasigroup $(Q, \cdot, /, \backslash)$ yields a combinatorial quasigroup $(Q, \cdot)$. Conversely, a combinatorial quasigroup $(Q, \cdot)$ will support an equational quasigroup $(Q, \cdot, /, \backslash)$ with right division $x/y = xR(y)^{-1}$ and left division $x\backslash y = yL(x)^{-1}$. Equational quasigroups may be viewed as *Skolemizations* of combinatorial quasigroups (compare [18]). For example, the existence of the unique solution z of the equation $z \cdot y = x$ in a combinatorial quasigroup is expressed functionally as $z = x/y$ in the corresponding equational quasigroup.

In an equational quasigroup $(Q, \cdot, /, \backslash)$, the three equations

$$x_1 \cdot x_2 = x_3\,, \qquad x_3/x_2 = x_1\,, \qquad x_1\backslash x_3 = x_2 \tag{2.4}$$

that involve the basic operations are equivalent. Now consider the *opposite* operations

$$x \circ y = y \cdot x\,, \qquad x/\!/y = y/x\,, \qquad x\backslash\!\backslash y = y\backslash x$$

on Q. Then the equations (2.4) are further equivalent to the equations

$$x_2 \circ x_1 = x_3\,, \qquad x_2/\!/x_3 = x_1\,, \qquad x_3\backslash\!\backslash x_1 = x_2\,.$$

Thus each of the basic and opposite operations

$$(Q, \cdot), \quad (Q, /), \quad (Q, \backslash), \quad (Q, \circ), \quad (Q, /\!/), \quad (Q, \backslash\!\backslash) \tag{2.5}$$

furnishes a (combinatorial) quasigroup. In particular, note that the identities (IR) in $(Q, \backslash)$ and (IL) in (Q,/) yield the respective identities

$$\begin{array}{ll} \text{(DL)} & x/(y\backslash x) = y\,, \\ \text{(DR)} & y = (x/y)\backslash x \end{array} \tag{2.6}$$

in the basic quasigroup divisions. The six quasigroups (2.5) are known as the *conjugates*, "parastrophes" [16] or "derived quasigroups" [9] of $(Q, \cdot)$.

2.2. **Triality.** The six binary operations are displayed in Figure 1. In each box, one of the six binary operations is used to express an equivalent version of the multiplication relation $x_1 \cdot x_2 = x_3$ for elements x_1, x_2, x_3 of a quasigroup Q.

$$
\begin{tikzcd}
\boxed{x_1 \cdot x_2 = x_3} \arrow[r, Leftrightarrow] \arrow[d, leftrightarrow] & \boxed{x_1 \backslash x_3 = x_2} \arrow[r, leftrightarrow] & \boxed{x_2 /\!/ x_3 = x_1} \arrow[d, Leftrightarrow] \\
\boxed{x_2 \circ x_1 = x_3} \arrow[r, Leftrightarrow] & \boxed{x_3 \backslash\!\backslash x_1 = x_2} \arrow[r, leftrightarrow] & \boxed{x_3 / x_2 = x_1}
\end{tikzcd}
$$

FIGURE 1. Symmetry of the quasigroup operations.

Disregarding the content of the boxes, Figure 1 may be interpreted as the *Cayley graph* of the symmetric group S_3 of permutations of the index set $\{1, 2, 3\}$ with respect to the transpositions $(1\ 2)$ and $(2\ 3)$. This is the construction $\Gamma\big(S_3, \{(1\ 2), (2\ 3)\}\big)$ of [17, pp. 16–17]. In Figure 1, the respective involutive actions of these transpositions appear as single-shafted double-headed arrows $\overset{(1\ 2)}{\longleftrightarrow}$ and double-shafted double-headed arrows $\overset{(2\ 3)}{\Longleftrightarrow}$.

The left-right duality that is represented by the left hand vertical in Figure 1 is the symmetry S_2 of the language of groups. The entire figure represents the *triality* or S_3-action which is the richer symmetry of the language of quasigroups. A deeper discussion of triality may be found in [21, §1.8], [22]. Note that this version of triality or S_3-action is related to, but distinct from, the triality or S_3-action that arises from the geometric symmetry of the Dynkin diagram D_4, as it appears in Moufang loops or algebras (compare [6, 10], say).

3. The homogeneous algebra

3.1. **Representing linear piques.** Recall that a *pique* is a quasigroup with a pointed idempotent element. An algebra Q is called *central* if the diagonal $\widehat{Q} = \{(x,x) \mid x \in Q\}$ is a normal subalgebra (and thus a congruence class) in the direct square $Q \times Q$. The following result shows that central piques have an equivalent linear structure, with 0 as the pointed idempotent. For this reason, central piques are often described synonymously as *linear piques*.

Proposition 3.1. [19, §3.2],[21, Prop. 11.1] *Let $\langle R, L\rangle$ be the free group on the two-element set $\{R, L\}$. Then central piques are equivalent to right modules over the group algebra $\mathbb{Z}\langle R, L\rangle$, with*

$$x \circ y = xL + yR\,, \tag{3.1}$$

$$x/y = xR^{-1} - yLR^{-1}\,, \tag{3.2}$$

$$x\backslash y = -xRL^{-1} + yL^{-1}\,, \tag{3.3}$$

$$e = 0 \tag{3.4}$$

as the respective opposite multiplication, right division, left division, and pointed idempotent.

Corollary 3.2. *In Proposition 3.1, $R = L_\circ(0)$ and $L = R_\circ(0)$.*

Corollary 3.3. *The free linear (or central) pique over a base set B has a faithful representation as the subpique of $\big(\mathbb{Z}\langle R, L\rangle B, \cdot, /, \backslash, 0\big)$ generated by the subset B of the free right $\mathbb{Z}\langle R, L\rangle$-module $\mathbb{Z}\langle R, L\rangle B$.*

Although the models of free central piques given by Corollary 3.3 are faithful, they do have a serious disadvantage when it comes to studying triality symmetry: the polynomials in $R^{\pm 1}, L^{\pm 1}$ giving the divisions are not homogeneous. For an alternative outlook on the situation, consider the fragment

$$\begin{tikzcd}
\boxed{LR^{-1}} \arrow[r, dash, "R"] & \boxed{L} \arrow[d, dash, "L"] & & {} & \\
\boxed{R^{-1}} \arrow[r, dash, "R"] & \boxed{1} \arrow[r, dash, "R"] \arrow[d, dash, "L"] & \boxed{R} \arrow[d, dash, "L"] & {} \arrow[u, "L"] \arrow[r, "R"] & {} \\
& \boxed{L^{-1}} & \boxed{RL^{-1}} & &
\end{tikzcd} \tag{3.5}$$

of the homogeneous tree of valency 4 (as in [24, §I.1.4], for example) recording the right regular permutation representation of the free group

$\langle R, L \rangle$. Here, the graph distance separating the unsigned coefficients L, R of the multiplication (3.1) is 2. On the other hand, the graph distance that separates the respective unsigned coefficients R^{-1}, LR^{-1} of the right division (3.2) or RL^{-1}, L^{-1} of the left division (3.3) are 3. Thus, in the representation of Proposition 3.1, the action of triality does not correspond to an isometry of the homogeneous tree whose fragment is displayed in (3.5).

3.2. **The homogeneous representation.** Consider a linear pique P. Taking the operations as

$$\begin{cases} x \circ y &= xT_1^{-1} + yT_2 \,, \\ x/y &= xT_2^{-1} + yT_3 \,, \\ x\backslash y &= xT_3^{-1} + yT_1 \end{cases} \tag{3.6}$$

for $x, y \in P$, with

$$T_3T_2T_1 = T_2T_1T_3 = T_1T_3T_2 = -1 \,, \tag{3.7}$$

gives a homogeneous representation of linear pique words. Here, the invertible elements T_i are described as the *coefficient* (*variables*). The equations (3.6) are taken from [25, (6.10)], where they arose naturally in the context of a semisymmetrization (compare [20, 23]) and the reversible automata of Gvaramiya and Plotkin [4, 5].

The corresponding quasigroup operations from Figure 1 are displayed in Figure 2.

$$\begin{array}{ccccc}
\boxed{\begin{array}{c} x \cdot y = \\ xT_2 + yT_1^{-1} \end{array}} & \Longleftrightarrow & \boxed{\begin{array}{c} x\backslash y = \\ xT_3^{-1} + yT_1 \end{array}} & \longleftrightarrow & \boxed{\begin{array}{c} x//y = \\ xT_3 + yT_2^{-1} \end{array}} \\
\updownarrow & & & & \Updownarrow \\
\boxed{\begin{array}{c} x \circ y = \\ xT_1^{-1} + yT_2 \end{array}} & \Longleftrightarrow & \boxed{\begin{array}{c} x\backslash\backslash y = \\ xT_1 + yT_3^{-1} \end{array}} & \longleftrightarrow & \boxed{\begin{array}{c} x/y = \\ xT_2^{-1} + yT_3 \end{array}}
\end{array}$$

FIGURE 2. Homogeneous representation of quasigroup operations in a central pique.

Example 3.4. Abelian groups are represented by $R = L = 1$ in terms of Proposition 3.1. Correspondingly, they are given by $T_1 = T_2 = 1$ and $T_3 = -1$ in terms of (3.6).

In Figure 1, $\leftrightarrow$ stood for right multiplication by (1 2), while $\Leftrightarrow$ stood for right multiplication by (2 3). Now, in Figure 2, $\leftrightarrow$ means inverting the coefficient variables and applying (1 2) to their suffices. Similarly, $\Leftrightarrow$ means inverting the coefficient variables and applying (2 3) to their suffices. These actions may be displayed explicitly as

$$(1\ 2)\colon T_1 \mapsto T_2^{-1},\ T_2 \mapsto T_1^{-1},\ T_3 \mapsto T_3^{-1}$$

and

$$(2\ 3)\colon T_1 \mapsto T_1^{-1},\ T_2 \mapsto T_3^{-1},\ T_3 \mapsto T_2^{-1}\,.$$

The relations (3.7) are invariant under these actions, demonstrating the homogeneity of the representation under the triality group S_3.

In connection with (3.6) and (3.7) respectively, it is worth recording the following observations. Within the context of the homogeneous representation, the first provides an analogue of Corollary 3.2.

Lemma 3.5. *In a linear pique* $(P, \cdot, /, \backslash, 0)$ *with quasigroup operations presented by* (3.6)*, the multiplications*

$$\begin{aligned} T_1^{-1} &= L_\bullet(0) = R_\circ(0)\,, & T_1 &= L_\backslash(0) = R_{\backslash\backslash}(0)\,, \\ T_2^{-1} &= L_{//}(0) = R_/(0)\,, & T_2 &= L_\circ(0) = R_\bullet(0)\,, \\ T_3^{-1} &= L_{\backslash\backslash}(0) = R_\backslash(0)\,, & T_3 &= L_/(0) = R_{//}(0) \end{aligned}$$

by the pointed idempotent element give interpretations of the coefficient variables and their inverses.

Lemma 3.6. *In* (3.7)*, suppose that the* T_i *are invertible. Then, for any one of the cyclic products* $T_iT_{i-1}T_{i-2}$ *appearing in* (3.7)*, the equation* $T_iT_{i-1}T_{i-2} = -1$ *(with indices taken modulo* 3*) is equivalent to the equation of each of the other two cyclic products with* -1*.*

Proof. Note e.g. $T_3T_2T_1 = -1 \Rightarrow T_3T_2 = -T_1^{-1} \Rightarrow T_1T_3T_2 = -1$, using right multiplication by T_1^{-1} and then left multiplication by T_1. $\square$

3.3. **Faithfulness of the homogeneous representation.** It will now be shown that the free versions of the homogeneous representations discussed in the previous section are faithful. Equating the coefficients from the top row of Figure 2 with the coefficients from (3.1)–(3.3) yields

$$T_1 = L^{-1}\,, \quad T_2 = R\,, \quad T_3 = -LR^{-1}\,, \tag{3.8}$$

where

$$T_3T_2T_1 = -LR^{-1}RL^{-1} = -1\,. \tag{3.9}$$

By (3.8), the T_i here are invertible. By Lemma 3.6, the relation (3.9) implies all the equations of (3.7). In other words, the homogeneous representation furnished by the coefficient variables models the faithful representation from §3.1. These observations may be summarized in the following reformulation of Proposition 3.1.

Theorem 3.7. [25, Th. 7.4] *Suppose that S is a commutative, unital ring. Consider the algebra*

$$\mathfrak{H}_S = S\,\langle T_1, T_2, T_3 \mid 1 + T_3T_2T_1, 1 + T_2T_1T_3, 1 + T_1T_3T_2\rangle \tag{3.10}$$

of polynomials, with coefficients taken from S, in the invertible non-commuting variables T_1, T_2, T_3 and their inverses, subject to relations that annihilate the indicated polynomials.

(a) *The actions*

$$(1\ 2)\colon T_1 \mapsto T_2^{-1},\ T_2 \mapsto T_1^{-1},\ T_3 \mapsto T_3^{-1}$$

and

$$(2\ 3)\colon T_1 \mapsto T_1^{-1},\ T_2 \mapsto T_3^{-1},\ T_3 \mapsto T_2^{-1}\,.$$

generate a group S_3 of automorphisms of $\mathfrak{H}_S$.

(b) *Linear S-piques are equivalent to right $\mathfrak{H}_{\mathbb{S}}$-modules, with*

$$x \circ y = xT_1^{-1} + yT_2\,, \tag{3.11}$$

$$x/y = xT_2^{-1} + yT_3\,, \tag{3.12}$$

$$x\backslash y = xT_3^{-1} + yT_1\,, \tag{3.13}$$

$$e = 0 \tag{3.14}$$

as the linear representations of the opposite multiplication, right division, left division, and pointed idempotent respectively.

(c) *The action of S_3 on $\mathfrak{H}_S$ from* (a) *induces a triality action on central S-piques as represented in* (b).

Definition 3.8. The algebra $\mathfrak{H}_S$ of (3.10) is called the *homogeneous algebra* over the ring S.

Remark 3.9. (a) It is often convenient (in §4.2 below, for example) to use

$$\mathfrak{H}_S = S\,\langle T_i \mid 1 + T_iT_{i-1}T_{i-2}\rangle_{i\in\mathbb{Z}/_3} \tag{3.15}$$

as a more compact form of the presentation (3.10) of the homogeneous algebra. Here and elsewhere, representatives for the residue classes constituting $\mathbb{Z}/_3$ are usually taken as $1, 2, 3$.

(b) Abstractly, with $S = \mathbb{Z}$, (3.10) or (3.15) provide an alternative presentation of the integral group algebra $\mathbb{Z}\langle R, L\rangle$ of the free group on two generators, as it appears in Proposition 3.1.

(c) The arrow notation

$$X \xrightarrow{x*y} Y \tag{3.16}$$

is useful for the situation where the quasigroup operation $x * y$ appears in the $\mathfrak{H}_S$-module structure as the linear combination $xX + yY$. Thus, the operations (3.11)–(3.13) and their opposites may be presented as

$$\begin{tikzcd} T_1^{-1} \arrow[d, bend left, "x\circ y"] & T_2^{-1} \arrow[d, bend left, "x/y"] & T_3^{-1} \arrow[d, bend left, "x\backslash y"] \\ T_2 \arrow[u, bend left, "x\cdot y"] & T_3 \arrow[u, bend left, "x//y"] & T_1 \arrow[u, bend left, "x\backslash\backslash y"] \end{tikzcd} \tag{3.17}$$

These arrow pairs are permuted by triality. In subsequent paragraphs, the discrete arrow pairs (3.17) will appear as features embedded in geometries.

(d) In comparison with Proposition 3.1, it should be noted that the integral coefficients of the monomials appearing in central quasigroup words represented by Theorem 3.7 are all nonnegative. This simple observation implies an alternative formulation of the *index polynomials* of [13] — compare [21, §11.1, Exercise 5].

3.4. **Involutive antiautomorphisms.** The involution

$$\mathbb{Z}\langle R, L\rangle \to \mathbb{Z}\langle R, L\rangle; R \mapsto R^{-1}, L \mapsto L^{-1}$$

of inversion is an antiautomorphism of $\mathbb{Z}\langle R, L\rangle$. The relations (3.8) transform to

$$T_1^{-1} = L\,, \quad T_2^{-1} = R^{-1}\,, \quad T_3^{-1} = -RL^{-1} \tag{3.18}$$

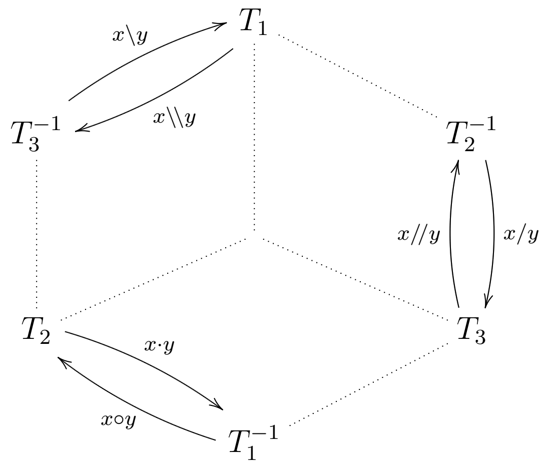
under the action of this antiautomorphism, while the defining relations of (3.15) transform to

$$1 + T_{i-2}^{-1}T_{i-1}^{-1}T_i^{-1}$$

for $1 \leq i \leq 3$. Thus, inversion of the generators T_i for $1 \leq i \leq 3$ induces an involutive antiautomorphism $\mathfrak{t}$ of $\mathfrak{H}_S$.

3.5. **The Euclidean representation.** With the goal of providing a geometrical interpretation of Figure 2, and a satisfactory replacement for (3.5), the following diagram summarizes the triality of central piques as a configuration in the Euclidean space spanned by unit vectors T_1, T_2, T_3.

(3.19)



Note the use of the arrow notation from (3.17). Making reference back to Theorem 3.7(a), the triality group S_3 acts as follows:

- The permutation (1 2) rotates the unit cube by π about the axis through the respective midpoints of the edges $T_1(T_1 + T_3)$ and $T_2(T_2 + T_3)$;
- The permutation (2 3) rotates the unit cube by π about the axis through the respective midpoints of the edges $T_2(T_2 + T_1)$ and $T_3(T_3 + T_1)$.

Thus, the triality group S_3 is the stabilizer of $0(T_1 + T_2 + T_3)$ in the full rotation group S_4 of the cube which permutes all four

$$T_1(T_2 + T_3),\ T_2(T_3 + T_1),\ T_3(T_1 + T_2),\ 0(T_1 + T_2 + T_3)$$

of its diagonals. This representation will be reinterpreted in Section 4.4 below.

4. Quaternion models

4.1. **Quaternion algebras.** Let S be a commutative, unital ring. For the following definition, compare [3], [24, Exercise II.2.4S]. (Note that the more general quaternion algebras considered in [10, §I.2.C], [12, §III.1] for the case of a field S do not guarantee satisfaction of (4.1) below.)

Definition 4.1. The *quaternion algebra* $\mathbb{H}_S$ over S is defined as the subalgebra of the algebra S_4^4 of 4×4-matrices over S that the matrices

$$I=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix},\ J=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{bmatrix},\ K=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix}$$

generate.

Proposition 4.2. *Let S be a commutative, unital ring. Consider the quaternion algebra $\mathbb{H}_S$ over S.*

(a) *The equations $I^2=J^2=K^2=-1$ (writing 1 for the identity element of the unital matrix ring S_4^4), along with the equations $IJ=K, JK=I$, and $KI=J$, hold in $\mathbb{H}_S$.*

(b) *The equations*

$$1+IJK=1+JKI=1+KIJ=0 \tag{4.1}$$

and

$$1+I^2=1+J^2=1+K^2=0 \tag{4.2}$$

hold in $\mathbb{H}_S$.

(c) *The linear structure of the quaternion algebra $\mathbb{H}_S$ is the free S-module over the set $\{1,I,J,K\}$.*

(d) *The subset $Q=\{\pm1,\pm I,\pm J,\pm K\}$ of the quaternion algebra $\mathbb{H}_S$ is closed under multiplication.*

(e) *If S is not of characteristic 2, the set Q forms a copy of the quaternion group Q_8.*

(f) *For elements t,x,y,z of S, consider the matrix*

$$H=t\cdot 1+xI+yJ+zK$$

in $\mathbb{H}_S$. Then

$$H^T=t\cdot 1-xI-yJ-zK\in\mathbb{H}_S\,, \tag{4.3}$$

while

$$HH^T=t^2+x^2+y^2+z^2\,, \tag{4.4}$$

an element of S described as the norm *of H.*

Definition 4.3. (a) Within the free S-module $\mathbb{H}_S$ on $\{1,I,J,K\}$ given by Proposition 4.2(c), the submodule generated by $\{I,J,K\}$ is known as the space of *pure quaternions* over S.

(b) The restriction of the transposition antiautomorphism of S_4^4 to $\mathbb{H}_S$ assured by (4.3) is known as *conjugation*.

Example 4.4. The usual real quaternion algebra $\mathbb{H}$, a noncommutative division ring, is $\mathbb{H}_{\mathbb{R}}$. Here, the conjugation of Definition 4.3(b) is the usual quaternion conjugation (cf. [2, p.17], say).

Example 4.5. The quaternion algebra $\mathbb{H}_{\mathsf{GF}(2)}$ over the two-element field is the group algebra over $\mathbb{Z}/_2$ of the Klein 4-group V_4. Indeed, in this case, (4.1) and (4.2) show that the set $\{1, I, J, K\}$ is closed under multiplication. Here, the conjugation is trivial.

Note that this interpretation of a quaternion algebra in characteristic 2 does not match the definition of a quaternion algebra over a field of characteristic 2 given in [10, §I.2.C].

Example 4.6. As a vector space, the complex quaternion algebra $\mathbb{H}_{\mathbb{C}}$ is four-dimensional, isomorphic to the ring $\mathbb{C}_2^2$ of 2×2 complex matrices. Further details of this important example are taken up in Section 4.3 below.

4.2. **The projection to the quaternion algebra.** The following theorem is motivated by consideration of Proposition 4.2.

Theorem 4.7. *Let S be a unital, commutative ring. Then*

$$S\,\langle T_i \mid 1 + T_i T_{i-1} T_{i-2}, 1 + T_i^2\rangle_{i\in\mathbb{Z}/_3}\ , \tag{4.5}$$

with non-commuting invertible variables T_1, T_2, T_3, is a presentation of the quaternion algebra $\mathbb{H}_S$.

Proof. Consider the free S-module H_S over the set $\{1, T_1, T_2, T_3\}$, and the relations expressed in (4.5). If $1 + T_i^2 = 0$, then $T_i^{-1} = -T_i$ for $i \in \mathbb{Z}/_3$. Multiplying $T_{i+1}T_iT_{i-1} = -1$ on the right by $-T_{i-1}$ yields $T_{i+1}T_i = T_{i-1}$. Thus, $T_{i-1}T_i = T_{i-1}^{-1}T_i^{-1} = (T_iT_{i-1})^{-1} = T_{i+1}^{-1} = -T_{i+1}$. In other words, the assignments

$$R(T_i)\colon T_{i-1} \mapsto -T_{i+1}\,,\ T_i \mapsto -1\,,\ T_{i+1} \mapsto T_{i-1}$$

hold. Then, referring to the ordered basis $\{1 < T_1 < T_3 < T_2\}$ of H_S, the respective right multiplications by T_1, T_3, T_2 are given in matrix form as

$$R(T_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = I, \tag{4.6}$$

$$R(T_3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = J, \tag{4.7}$$

$$R(T_2) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = K \tag{4.8}$$

using the notation of Definition 4.1. It follows that the module H_S is closed under multiplication, and thereby implements the quaternion algebra $\mathbb{H}_S$. □

Corollary 4.8. *Imposition of the power relations $T_i^2 = -1$ induces a surjective homomorphism $\pi\colon \mathfrak{H}_S \to \mathbb{H}_S$ onto the quaternion algebra, such that the diagram*

$$\begin{tikzcd} \mathfrak{H}_S \arrow[r, "\pi"] \arrow[d, "\mathfrak{t}"'] & \mathbb{H}_S \arrow[d, "T"] \\ \mathfrak{H}_S \arrow[r, "\pi"'] & \mathbb{H}_S , \end{tikzcd}$$

involving the involution $\mathfrak{t}$ of Section 3.4 and the restricted transposition T of Definition 4.3(b), commutes.

4.3. **The Pauli group.** This section provides more detail about the complex quaternion algebra introduced in Example 4.6, and puts the apparent order-reversal of (4.7) and (4.8) into context.[2]

In quantum information theory, the *Pauli group* is the $\mathbb{R}$-spanning subset

$$G_1 = \{ \pm 1, \pm i, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ \} \tag{4.9}$$

of the Hilbert space $\mathbb{C}_2^2$ [15, (10.81)], with *Pauli matrices*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.10}$$

[2]The alphabetical order of (4.10) matches the numerical order of (4.11).

[15, §2.1.3]. The suffix 1 in G_1 refers to the fact that (4.9) is the Pauli group for a single qubit. Following the convention of Proposition 4.2(a), the first two pairs of elements in the list (4.10) refer to scalar matrices.

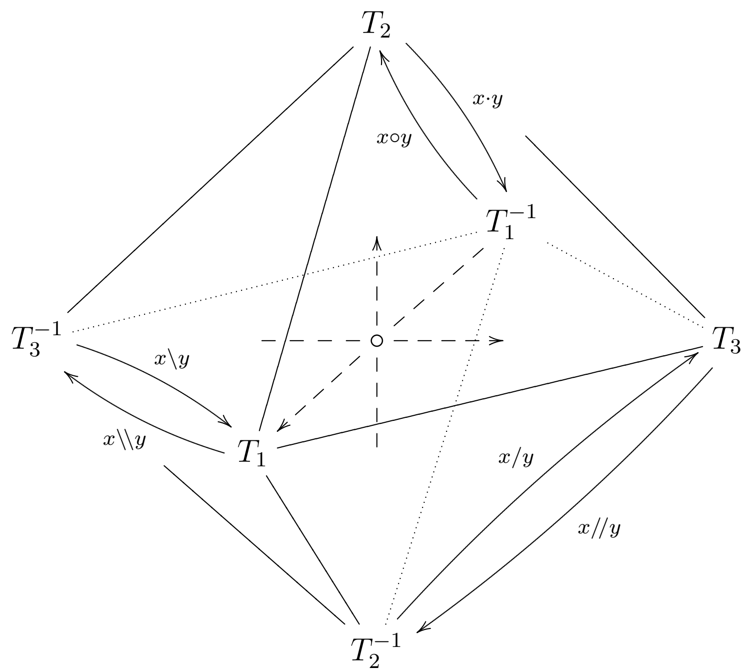
Now, within the Pauli group G_1, set

$$T_1 = iX\,, \quad T_2 = iY\,, \quad \text{and} \quad T_3 = iZ\,. \tag{4.11}$$

The relations of (4.5) then hold within $\mathbb{C}_2^2$. By Theorem 4.7 and the fact that $\dim_{\mathbb{C}} \mathbb{C}_2^2 = 4 = \dim_{\mathbb{C}} \mathbb{H}_{\mathbb{C}}$, the latter equation holding by virtue of Proposition 4.2(c), it follows that $\mathbb{C}_2^2 = \mathbb{H}_{\mathbb{C}}$, as claimed in Example 4.6. Since the matrices (4.11) are unitary, the conjugation on $\mathbb{C}_2^2$ is implemented by taking the complex conjugate transpose $\dagger$.

4.4. **The quaternion representation.** Over a commutative, unital ring S, the space of pure quaternions introduced in Definition 4.3 gives a natural representation of triality, thanks to the invertibility of the respective pure quaternions (4.6)–(4.8) that may be identified with T_1, T_3, T_2:

(4.12)



The group of symmetries of an octahedron in 3-dimensional space is S_4, permuting the four pairs of opposite faces of the octahedron. In (4.12), the faces of just one of these pairs (namely $\{\, T_3T_2T_1, T_1^{-1}T_2^{-1}T_3^{-1}\,\}$) do not contain an edge which is associated with quasigroup operations. The group S_3 of triality thus appears as the stabilizer of this opposite pair within the symmetry group S_4 of the octahedron. The Euclidean representation of triality presented in Section 3.5 may now be seen as being based on the cube which is the dual of the octahedron, for $S = \mathbb{R}$.

4.5. **Quaternionic structures.** Let S be a commutative, unital ring. As before, write $\underline{\underline{S}}$ for the category of S-modules.

Definition 4.9. A *quaternionic structure* on an S-module P is given as a homomorphism

$$\mathbb{H}_S \to \underline{\underline{S}}(P,P); H \mapsto \big[p \mapsto pH\big] \tag{4.13}$$

of S-algebras from the quaternion algebra $\mathbb{H}_S$ to the endomorphism ring of the S-module P.

For the following example, compare Example 4.4 and [11].

Example 4.10. Consider $P = \mathbb{H}^n$ for a positive integer n, as a real vector space of dimension $4n$. Then

$$\mathbb{H} \to \underline{\underline{\mathbb{R}}}(P,P); q \mapsto \big[(p_1, \dots, p_n) \mapsto (p_1 q, \dots, p_n q)\big]$$

endows P with a quaternionic structure.

Example 4.11. For $S = \mathbb{Z}/_2$, Example 4.5 interprets $\mathbb{H}_{Z/_2}$ as the group algebra $\mathbb{Z}/_2 V_4$ of the Klein 4-group $V_4 = \{1, I, J, K\}$. Thus, a Boolean group W (elementary abelian group of exponent 2) forms a quaternionic structure over $\mathbb{Z}/_2$ precisely when it affords a 2-modular representation of V_4. If W is finite, it forms a *bisymmetric linear quantum quasigroup* [8, Th. 4.31].

Example 4.12. The 2-dimensional complex space $\mathbb{C}^2$ is the Hilbert space of a 2-state quantum system — a qubit [15, §1.2]. The right action of the matrix ring $\mathbb{C}^2_2 = \mathbb{H}_{\mathbb{C}}$ from Section 4.3 then provides a complex quaternionic structure on $\mathbb{C}^2$.

4.6. **Quaternionic piques.**

Definition 4.13. Let S be a commutative, unital ring. A *quaternionic pique* over S is defined as a quaternionic structure P *over* S equipped with the pique operations

$$x \circ y = xT_1^{-1} + yT_2 = -xI + yK\,, \tag{4.14}$$

$$x / y = xT_2^{-1} + yT_3 = -xK + yJ\,, \tag{4.15}$$

$$x \backslash y = xT_3^{-1} + yT_1 = -xJ + yI \tag{4.16}$$

of (3.11)–(3.13), along with $e = 0$. More generically, a *quaternionic pique* is a quaternionic pique over $\mathbb{Z}$.

Example 4.14. Consider the quaternionic structure of Example 4.12 on the qubit space $\mathbb{C}^2$ equipped with the *computational basis*

$$\{ \mid 0\rangle = (1,0), \mid 1\rangle = (0,1) \}$$

[15, §1.2]. The opposite multiplication structure of the quaternionic pique structure is given as follows with respect to the computational basis:

$\circ$	$\lvert 0\rangle$	$\lvert 1\rangle$
$\lvert 0\rangle$	$(1-i)\lvert 1\rangle$	$-\lvert 0\rangle - i\lvert 1\rangle$
$\lvert 1\rangle$	$-i\lvert 0\rangle + \lvert 1\rangle$	$(1+i)\lvert 0\rangle$

Note that neither the rows, nor the columns, are mutually orthogonal under the Hilbert space product. However, the unnormalized quasigroup products presented in the body of the multiplication table all have the same norm of $\sqrt{2}$, since the coefficients $T_1^{-1} = -iX$ and $T_2 = iY$ of the opposite quasigroup multiplication are unitary.

Theorem 4.15. *Within the class of central piques, the full set of quasigroup equations*

$$x = (((x \circ y) \circ y) \circ y) \circ y\,, \tag{4.17}$$

$$x = (((x/y)/y)/y)/y\,, \tag{4.18}$$

$$x = (((x\backslash y)\backslash y)\backslash y)\backslash y \tag{4.19}$$

distinguishes the quaternionic piques.

Proof. Suppose that P is a central pique that satisfies the quasigroup equations (4.17)–(4.19). Consider P as an $\mathfrak{H}_{\mathbb{Z}}$-module according to Theorem 3.7(b). Then (3.11) implies

$$x \circ y \;=\; xT_1^{-1} + yT_2\,, \tag{4.20}$$

$$\begin{aligned}(x \circ y) \circ y &\overset{(4.20)}{=} (xT_1^{-1} + yT_2)T_1^{-1} + yT_2 \\ &= xT_1^{-2} + yT_2(T_1^{-1} + 1)\,, \text{ and}\end{aligned} \tag{4.21}$$

$$\begin{aligned}(((x \circ y) \circ y) \circ y) \circ y &\overset{(4.21)}{=} ([xT_1^{-2} + yT_2(T_1^{-1} + 1)] \circ y) \circ y \\ &\overset{(4.21)}{=} [xT_1^{-2} + yT_2(T_1^{-1} + 1)]T_1^{-2} + yT_2(T_1^{-1} + 1) \\ &= xT_1^{-4} + yT_2(T_1^{-3} + T_1^{-2} + T_1^{-1} + 1)\,.\end{aligned} \tag{4.22}$$

Thus, (4.17) may be rewritten according to (4.22) in the linear form

$$x = xT_1^{-4} + yT_2(T_1^{-3} + T_1^{-2} + T_1^{-1} + 1)\,.$$

More generally (or by triality), the equation

$$x = xT_i^{-4} + yT_{i+1}(T_i^{-3} + T_i^{-2} + T_i^{-1} + 1) \tag{4.23}$$

holds for $1 \leq i \leq 3$.

Recall that central quasigroups, and therefore also central piques, form varieties in the sense of universal algebra — classes of models of a set of identities [21, Th. 3.2]. Thus, take P as the free pique on the set $\{x, y\}$ in the variety of central piques defined by the identities (4.17)–(4.19). Setting $x = 1$ and $y = 0$ in (4.23) shows that each T_i for $1 \leq i \leq 3$ satisfies the equation $0 = X^4 - 1 = (X^2+1)(X+1)(X-1)$ in $\mathbb{Z}[X]$. The freeness of P excludes the possibilities $T_i = \pm 1$. Thus, $T_i^2 = -1$ for $1 \leq i \leq 3$. Corollary 4.8 shows that on each central pique P that satisfies the quasigroup equations (4.17)–(4.19), the $\mathfrak{H}_{\mathbb{Z}}$-module structure descends to a $\mathbb{H}_{\mathbb{Z}}$-module structure, rendering P quaternionic.

Conversely, suppose that P is a quaternionic pique. By (4.22),

$$(((x \circ y) \circ y) \circ y) \circ y = x + yT_2(-T_1^{-1} - 1 + T_1^{-1} + 1) = x\,.$$

The derivation of the identities (4.18) and (4.19) is similar. $\square$

Corollary 4.16. *Quaternionic piques form a variety in the sense of universal algebra.*

4.7. **Quaternionic quasigroup identities.** This section examines the quasigroup equations (4.17)–(4.19) in more detail. Collectively, along with equivalent identities to be derived in this section, they will be described as *quaternionic* (*quasigroup*) *identities*, and quasigroups that satisfy them are said to be *quaternionic*.

In their original form (4.17)–(4.19), the three identities appear in terms of the set $\{\circ, /, \backslash\}$ of three operations which form an orbit under the subgroup C_3 of the triality group S_3 (compare §2.2). However, since an equational quasigroup is usually given as $(Q, \cdot, /, \backslash)$, it is helpful to consider the quaternionic identities in terms of these operations.

Proposition 4.17. *The quaternionic quasigroup identities belong to the three equivalence classes*

$$x = y \cdot (y \cdot (y \cdot (y \cdot x))) \ \Leftrightarrow\ y\backslash(y\backslash(y\backslash(y\backslash x))) = x\,; \tag{4.24}$$

$$x = (((x \cdot y) \cdot y) \cdot y) \cdot y \ \Leftrightarrow\ (((x/y)/y)/y)/y = x\,; \tag{4.25}$$

$$x = (((x\backslash y)\backslash y)\backslash y)\backslash y) \ \Leftrightarrow\ y/(y/(y/(y/x))) = x\,; \tag{4.26}$$

when written in terms of the basic operations $\cdot, /, \backslash$ on a quasigroup.

Proof. Use the notation of (2.2) and (2.3). For (4.24), we have the equivalence of $x = xL_\bullet(y)^4$ with $xL_\bullet(y)^{-4} = x$. For (4.25), we have the equivalence of $x = xR_\bullet(y)^4$ with $xR_\bullet(y)^{-4} = x$.

Now, the quasigroup identities (2.6) express the inverse relationship $R_\backslash(x)^{-1} = L_/(x)$. The equivalence of $x = xR_\backslash(y)^4$ with $xR_\backslash(y)^{-4} = x$ then establishes (4.26). □

Figure 3. Triality of the quaternionic quasigroup identities.

$$\begin{array}{ccccc}
\boxed{xR_\bullet(y)^4 = x} & \Longleftrightarrow & \boxed{xR_\backslash(y)^4 = x} & \longleftrightarrow & \boxed{xR_{//}(y)^4 = x} \\
\updownarrow & & & & \Updownarrow \\
\boxed{xR_\circ(y)^4 = x} & \Longleftrightarrow & \boxed{xR_{\backslash\backslash}(y)^4 = x} & \longleftrightarrow & \boxed{xR_/(y)^4 = x}
\end{array}$$

Corollary 4.18. *The identities of Proposition 4.17 appear respectively as:*

$$x = xR_\circ(y)^4 \;\Leftrightarrow\; x = xR_{\backslash\backslash}(y)^4\,; \tag{4.27}$$

$$x = xR_\bullet(y)^4 \;\Leftrightarrow\; x = xR_/(y)^4\,; \tag{4.28}$$

$$x = xR_\backslash(y)^4 \;\Leftrightarrow\; x = xR_{//}(y)^4\,. \tag{4.29}$$

Thus, triality action on the six quasigroup operations induces a triality action on the quaternionic quasigroup identities.

The triality action from Corollary 4.18 is displayed in Figure 3. The equivalent pairs constitute orbits of the left action of $\langle(2\ 3)\rangle$ on the Cayley diagram.

Remark 4.19. Replacing the exponent 4 in Corollary 4.18 with 2 yields identities that define the variety of *totally symmetric* quasigroups — compare [21, §1.6]. Indeed, the identity $x = xR_\circ(y)^2$ defines the class of *left symmetric* quasigroups [21, (1.38)], and similarly the dual identity $x = xR_\cdot(y)^2$ defines the class of *right symmetric* quasigroups [21, (1.39)]. By [21, Proposition 1.9] and the fact that the triality group S_3 is generated by any pair of distinct transpositions, the intersection of these two classes is the class of totally symmetric quasigroups (which, in particular, includes the classes of Steiner triple systems and Boolean groups). Thus, totally symmetric quasigroups are quaternionic.

4.8. **Independence of the quaternionic quasigroup identities.** In this section, it will be shown that the three equivalent pairs of quaternionic quasigroup identities appearing as (4.24)–(4.26) in the statement of Proposition 4.17, or as (4.27)–(4.29) in the statement of Corollary 4.18, are independent: None of them is a consequence of the other two. In fact, the independence will be shown to hold even within the class of central or linear piques, and not just at the general quasigroup level.

Theorem 4.20. *Within the class of linear piques, the three equivalent pairs of quaternionic quasigroup identities appearing as* (4.24)–(4.26) *are logically independent.*

Proof. Take two copies $\langle X_1 \rangle$ and $\langle X_2 \rangle$ of the cyclic group C_2, so that $X_1^2 = X_2^2 = 1$. Consider the group algebra

$$H = \mathbb{Z}[i]\left(\langle X_1 \rangle * \langle X_2 \rangle\right),$$

over the ring $\mathbb{Z}[i]$ of Gaussian integers, of the free product $\langle X_1 \rangle * \langle X_2 \rangle$ (compare [17, Ex. 1.5.1]). In H, take elements $T_1 = iX_1$ and $T_2 = iX_2$. Thus, the equations

$$0 = 1 + T_1^2 = 1 + T_2^2$$

hold in H.

Now, in H, set $T_3 = -T_1T_2 = X_1X_2$, so that

$$1 + T_3T_2T_1 = 0\,. \tag{4.30}$$

By the known structure of free products, say as given in [17, p.2], the *reduced words* (normal forms) for elements of $\langle X_1 \rangle * \langle X_2 \rangle$ are w, where w is a word in the alphabet $\{X_1, X_2\}$ which does not contain X_1X_1 or X_2X_2 as a subword. Each element of $\langle X_1 \rangle * \langle X_2 \rangle$ then has a unique expression as a reduced word w [17, Th. I.1]. The reduced form of T_3^4 is $X_1X_2X_1X_2X_1X_2X_1X_2$, so T_3 is not of order 4 or less.

By (4.30) and Lemma 3.6. the equations (3.7) hold. Thus, we may use the equations (3.6) to define a linear pique structure P on the underlying abelian group of H. By Lemma 3.5, we have

$$T_1^{-1} = R_\circ(0)\,,\ T_2 = L_\circ(0)\,,\ T_3 = L_/(0)\,.$$

Recall $T_1^4 = T_2^4 = 1$, but $T_3^4 \neq 1$. By Lemma 3.5 again, we have $R_\backslash(0)^4 \neq 1$ and $R_{//}(0)^4 \neq 1$. Corollary 4.18 therefore shows that while the pairs of equivalent identities (4.27) and (4.28) are satisfied in the linear pique P, the remaining pair (4.29) is not. Finally, application of triality shows that each of the pairs of equivalent quaternion quasigroup identities (4.27)–(4.29) is independent of the other two. The statement of the theorem follows. $\square$

Corollary 4.21. *The three equivalent pairs of quaternionic quasigroup identities are logically independent.*

Remark 4.22. Corollary 4.21 contrasts with the situation discussed in Remark 4.19, considering the identities obtained by replacing the exponent 4 in Corollary 4.18 with 2. In that situation, any two of the equivalent pairs of identities imply the third.

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