SYMMETRY CLASSES OF QUANTUM QUASIGROUPS

BOKHEE IM[1], ALEX W. NOWAK[2], AND JONATHAN D. H. SMITH[3]

Abstract. The theory of groups has a twofold symmetry, sending a group to its opposite. Groups invariant under the symmetry are abelian. The theory of quasigroups has a richer, sixfold symmetry, obtained by permuting the multiplication with its two divisions. The Sixfold Way identifies the various classes of quasigroups which are invariant under the respective subgroups of the symmetry group of the theory.

Quantum quasigroups provide a self-dual framework to unify the study of quasigroups and Hopf algebras. The goal of this paper is to classify the symmetry classes of quantum quasigroups. Corresponding to the Sixfold Way for classical quasigroups, we are able to identify a Sevenfold Way for general classes exhibiting a symmetry, and initiate a study of a fuller symmetry which holds for linear quantum quasigroups.

1. Introduction

Quasigroups and Hopf algebras represent two distinct extensions of the concept of a group. Like groups, quasigroups are set-theoretical objects, consisting of a set on which a cancellative multiplication has been defined. Unlike group multiplications, however, quasigroup multiplications are not required to be associative. On the other hand, Hopf algebras extend the group concept to a linear setting, say to a vector space A, with a linear *multiplication* $\nabla\colon A\otimes A \to A; x\otimes y \mapsto x\cdot y$ that is required to be associative, and a linear *antipode* mapping $S\colon A \to A$ (which we do not require to be invertible *a priori*) that plays the role of the inversion in a group. The concept of a Hopf algebra is self-dual, so the multiplication is accompanied

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by a *comultiplication* $\Delta\colon A \to A \otimes A$ that is coassociative and compatible (mutually homomorphic) with the multiplication.

Initial extensions of the concept of a Hopf algebra to comprise a non-associative multiplication, such as [1, 4, 12, 13, 18] for example, took what are now regarded as *semi-classical* approaches. Restricting themselves to the linearization of certain equationally defined classes of quasigroups, such as inverse-property loops [25, §I.4.1], these approaches impose a linearized version of the defining equations on the non-associative multiplication. For example, *Hopf quasigroups* of Majid *et al.* linearize the inverse property [12, Prop. 4.2(1)], while the *Moufang-Hopf algebras* of Benkhart *et al.* linearize Moufang properties [1, Def'n 1.2]. These various semi-classical approaches lack the self-duality that is characteristic of Hopf algebras.

Quantum quasigroups were introduced [23] as a self-dual framework for the unification of quasigroups and Hopf algebras, within the general setting of any symmetric monoidal category. Viewed from the quasigroup side, they linearize an elegant characterization of quasigroups that was presented by the topologist I.M. James in the nineteen-sixties [10]. Viewed from the Hopf algebra side, they abstract the property (well-known to experts, but usually obscured under cohomological conditions) that each Hopf algebra A is an A-A-bi-Galois object [2, Ex. 1.2].

The cancellativity of a classical quasigroup $(Q,\cdot)$ means that, along with its (potentially) non-associative multiplication $(x,y)\mapsto x\cdot y$, it also carries a left division structure $(Q,\backslash)$ and a right division structure $(Q,/)$, connected by the identities (2.8) and (2.9). If $(Q,\cdot)$ is a group, for example, then $x/y = x\cdot y^{-1}$ and $x\backslash y = x^{-1}\cdot y$. To avoid clashing with the antipode notation, we write Σ_n in this paper for the symmetric group on n letters. The group Σ_3 acts on the language of quasigroups, permuting the basic operations of multiplication and division, together with their opposites. The quasigroup identities form a set that is invariant under the action of Σ_3, which thus becomes a group of symmetries of the theory of quasigroups.

In contrast, the theory of groups is only invariant under a comparable Σ_2-action which permutes the multiplication with its opposite. Among all the classes of groups that are defined by identities (*varieties* in the sense of [17]), the class of abelian groups plays a special role, since it forms the class of all groups which are invariant under the symmetry Σ_2 of the theory of groups. The comparable class of Hopf algebras consists of those which are commutative and cocommutative. We may speak of *symmetry classes* of groups and Hopf algebras.

The richer symmetry of the theory of quasigroups yields a richer palette of symmetry classes [21, §1.8], described in this paper as the *Sixfold Way* (§2.3), in (dual) Galois correspondence with the six subgroups of Σ_3. In addition to their characterization in terms of invariance under subgroups of

Σ_3, these six classes may also be defined by identities, as in Proposition 2.6, for example. The problem then arises of identifying corresponding classes of quantum quasigroups. In a predecessor paper [9], the authors took a semi-classical approach to the problem, translating defining identities of the classical symmetry classes into the language of symmetric monoidal categories to whatever extent possible.

The goal of the current paper is to take a more intrinsically quantum approach to the problem, as much as possible working with symmetries of quantum quasigroups that track the symmetries of the theory of classical quasigroups. As with Hopf algebras, commutativity causes little trouble, leading in straightforward fashion to a notion of *quantum commutativity* (§4.1). Other symmetries of classical quasigroups, however, transfer to an approximate symmetry of quantum quasigroups, which requires much more delicate handling. A particular feature of quantum quasigroups is that an additional symmetry class (*bisymmetry*) emerges from the combination of left and right symmetry (the two non-self-dual classes), not necessarily collapsing to total symmetry as in the classical case. The upshot, the main topic of the paper, is the classification of quantum quasigroups into seven symmetry classes, forming the so-called *Sevenfold Way* (§4.6).

1.1. **Plan of the paper.** Chapter 2 presents those aspects of the theory of classical quasigroups which form the foundation for the study of symmetry classes of quantum quasigroups. It begins with a definition of quasigroups that is given purely in terms of symmetry (Definition 2.1), and then shown in the subsequent section to coincide with the usual equational definition of quasigroups (Proposition 2.3). Definition 2.4 introduces the six *conjugates* of a classical quasigroup which are generated by the action of the symmetry group Σ_3 of the theory of quasigroups. Section 2.3 then shows how the Galois theory of the action creates the symmetry classes of classical quasigroups which constitute the Sixfold Way (2.12). The final section of the chapter presents the classical notion of idempotence, and the version of the Sixfold Way pertaining to idempotent quasigroups. Two of the idempotent symmetry classes are described by the combinatorial structures (Steiner and Mendelsohn triple systems) that they determine (Definition 2.11).

Chapter 3 recalls the definition of a quantum quasigroup (A, ∇, Δ) in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ with twist morphism τ, in terms of the invertibility of the *left composite* morphism $\mathsf{G}\colon A \otimes A \to A \otimes A$ (3.3) and *right composite* morphism $\partial\colon A \otimes A \to A \otimes A$ (3.4). Section 3.2 then gives a summary of the special features of quantum quasigroups in the linear setting of the Cartesian monoidal category $(\underline{\underline{S}}, \oplus, \{\,0\,\})$ of modules over a commutative, unital ring S. The following section deals with the quantum analogues of the conjugates of a classical quasigroup, building on the earlier treatment from [9, §4]. While that treatment restricted itself to quantum quasigroups, Section 3.3 expands to the context of *bimagmas* (A, ∇, Δ), for which the only requirement is the mutual homomorphism between the multiplication ∇ and the comultiplication Δ. The expansion is necessary, since while the quantum conjugates of a quantum quasigroup are always bimagmas, they need not themselves form quantum quasigroups.

The conjugates of a classical quasigroup have an exact Σ_3-symmetry, as displayed in the *triality* diagram (2.10). On the other hand, the *gauge groups* of linear bimagmas introduced in §3.3.3 show that the left and right conjugates of a quantum quasigroup, even when they do form quantum quasigroups, need not be uniquely determined. The conjugate triality of quantum quasigroups introduced in Section 3.4 therefore constitutes only an approximate Σ_3-symmetry, as displayed in the quantum analogue (3.12) of the classical triality diagram (2.10). Example 3.20 illustrates how this approximate Σ_3-symmetry applies to a linear quantum quasigroup under the action of its gauge group. Section 3.5 then initiates a study of a larger symmetry which applies specifically to linear quantum quasigroups.

Chapter 4 presents the various symmetry classes of quantum quasigroups, as summarized by the general Sevenfold Way diagram (4.18) or the linear Sevenfold Way diagram (4.20) that are quantum counterparts of the classical Sixfold Way diagram (2.12). The first five sections of Chapter 4 are devoted to detailed examinations of the individual classes, giving specifications for them as they apply to general bimagmas or quantum quasigroups in terms of commuting diagrams or equations involving the composites and the twist morphism, and as they apply to linear bimagmas or quantum quasigroups in terms of structural module endomorphisms.

The new quantum version of semisymmetry from Section 4.3 may be contrasted with the semi-classical version of [9, Def'n 5.3(c)], which only applied to bimagmas having an augmentation to the unit of the underlying monoidal category. Like the classical version, under which a semisymmetric magma becomes a quasigroup, the new quantum version is strong enough to imply that a quantum semisymmetric bimagma is a quantum quasigroup (Proposition 4.13). On the other hand, the semi-classical versions were not strong enough to make bimagmas into quantum quasigroups [9, Th, 5.9(b)].

The quantum version of total symmetry from Section 4.4 has an intriguing connection to the idempotent version of classical total symmetry, namely to Steiner triple systems. According to Theorem 4.18, it transpires that the two composites of a coassociative and totally symmetric quantum quasigroup, in conjunction with the twist morphism of the underlying symmetric monoidal category, carry a natural totally symmetric quasigroup structure. As recorded in Example 4.29, instances of Theorem 4.18 are ubiquitous, being associated with every abelian group. Quantum totally symmetric

Hopf algebras are examined in §4.4.2. Just as totally symmetric groups are Boolean (i.e., elementary abelian of exponent two), Theorem 4.23 shows that over an algebraically closed field of characteristic zero, the quantum totally symmetric Hopf algebras coincide with the group algebras of Boolean groups.

Section 4.5 examines the "seventh" symmetry class that distinguishes the classical Sixfold Way from the quantum Sevenfold Way — bisymmetry — contrasting it with quantum total symmetry. Theorem 4.30 classifies finite-dimensional bisymmetric quantum quasigroups in the Cartesian monoidal category of vector spaces over a field which is not of characteristic two, while Theorem 4.33 handles the totally symmetric case. On an n-dimensional space, there are $\binom{n+3}{n}$ isomorphism classes of bisymmetric quantum quasigroups, $n+1$ of which are totally symmetric. Example 4.32, on the other hand, shows that no such classification of bisymmetric quantum quasigroups is possible in characteristic two. Nevertheless, Theorem 4.34 does classify the $1+\lfloor\frac{n}{2}\rfloor$ isomorphism classes of n-dimensional totally symmetric quantum quasigroups in the Cartesian category over a field of characteristic two.

The final Chapter 5 deals with the idempotent version of the Sevenfold Way, identifying the symmetry classes of bimagmas and quantum quasigroups that satisfy the condition of quantum idempotence as recalled in Section 5.1. The main focus is on linear idempotent quantum quasigroups. Left and right symmetric idempotent quantum quasigroups are exemplified by the Fibonacci quantum quasigroups presented in §5.3.1, and are classified by Theorem 5.8 in Cartesian categories of vector spaces over fields whose characteristic is not two. The quantum versions of Mendelsohn and Steiner triple systems, respectively combining idempotence with semisymmetry and total symmetry, are examined in the final two sections. Here, Theorem 5.16 classifies Mendelsohn quantum systems on n-dimensional vector spaces in the Cartesian category over a field with a sixth root of unity. There are $\binom{n+11}{n}$ isomorphism classes. Theorem 5.23 performs a similar classification for Steiner quantum systems, where the field must be of characteristic three. In this case, quantum idempotence is automatic, and the situation reduces to that of Theorem 4.33.

1.2. **Notational conventions.** As the default options, this paper follows the notational conventions of [25]. In order to minimize the occurrence of parentheses in our non-associative contexts, we adopt the "algebraic" or "diagrammatic" convention which composes functions in the natural reading order from left to right. Thus functions are placed to the right of their arguments, either on the line or as a superfix (as in $n!$ or x^2, for example). The group of permutations (self-bijections) of a set X is written as $X!$, and K_n^n will denote the ring of $n \times n$-matrices over a commutative ring K.

2. Quasigroups and the Sixfold Way

2.1. **Symmetry of combinatorial quasigroups.** A *magma* $(Q, \cdot)$ is a set Q with a binary *multiplication*

$$Q \times Q \to Q; (x, y) \mapsto x \cdot y \tag{2.1}$$

[20, Pt. I, §IV.4.1]. Often, it is convenient to denote the product $x \cdot y$ of two magma elements x, y merely by their juxtapostion xy. The juxtaposition will then bind more tightly than the product with the explicit multiplication operation. The associative law, for example, may take the form $x{\cdot}yz = xy{\cdot}z$.

The *multiplication table* of a magma $(Q, \cdot)$ is defined to be the graph of its multiplication (2.1), namely the ternary relation

$$\mathcal{T}(Q, \cdot) = \{ (x_1, x_2, x_3) \in Q^3 \mid x_1 \cdot x_2 = x_3 \}$$

[21, §1.9]. The group homomorphism

$$\rho\colon \Sigma_3 \to Q^3!; g \mapsto [(x_1, x_2, x_3) \mapsto (x_{1g}, x_{2g}, x_{3g})] \tag{2.2}$$

from the symmetric group $\Sigma_3 = \{ 1, 2, 3 \}!$ on the symbols $1, 2, 3$ defines a right permutation action of Σ_3 on Q^3. This symmetry defines quasigroups.

Definition 2.1. A magma $(Q, \cdot)$ is a (*combinatorial*) *quasigroup* if $\mathcal{T}(Q, \cdot)^g$ is the multiplication table of a magma $(Q, \cdot)^g$ for each $g \in \Sigma_3$.

The requirement for a ternary relation U on Q to be the multiplication table of a magma means that for each pair x_1, x_2 of elements of Q, there exists a unique element x_3 of Q with $(x_1, x_2, x_3) \in U$.

Example 2.2. Consider the cancellative magma $(\mathbb{Z}^+, +)$ of addition on the set $\mathbb{Z}^+$ of positive integers, with its multiplication table

$$\mathcal{T}_{(\mathbb{Z}^+,+)} = \{ (m, n, m + n) \mid m, n \in \mathbb{Z}^+ \} .$$

Note that

$$\mathcal{T}^{(2\ 3)}_{(\mathbb{Z}^+,+)} = \{ (m, m + n, n) \mid m, n \in \mathbb{Z}^+ \}$$

is not the multiplication table of a magma, since it does not contain $(1, 1, n)$ for any positive integer n.

If a magma $(Q, \cdot)$ is a combinatorial quasigroup, it follows by symmetry that each of the magmas $(Q, \cdot)^g$, for $g \in \Sigma_3$, is a combinatorial quasigroup in its own right. In relation to the original magma $(Q, \cdot)^{(1)}$ with its own *multiplication* $x_1 \cdot x_2 = x_3$, the binary operations of the magmas $(Q, \cdot)^g$ have special notations and names as follows:

$$(Q,\cdot)^{(1\ 2)} = (Q,\circ), \tag{2.3}$$

so $x_2 \circ x_1 = x_3$ with the *opposite multiplication*;

$$(Q,\cdot)^{(2\ 3)} = (Q,\backslash), \tag{2.4}$$

so $x_1 \backslash x_3 = x_2$ with the *left division*;

$$(Q,\cdot)^{(3\ 1)} = (Q,/), \tag{2.5}$$

so $x_3 / x_2 = x_1$ with the *right division*;

$$(Q,\cdot)^{(1\ 2\ 3)} = (Q,//), \tag{2.6}$$

so $x_2 // x_3 = x_1$ with the *opposite right division*;

$$(Q,\cdot)^{(1\ 3\ 2)} = (Q,\backslash\backslash), \tag{2.7}$$

so $x_3 \backslash\backslash x_1 = x_2$ with the *opposite left division*.

Here, as usual, we are using cycle representations for permutations, reading their action from left to right (as should be apparent from (2.6) and (2.7) above). The left division $x\backslash y$ may be vocalized as "x dividing y", while the right division x/y is vocalized as "x divided by y". In a group, we have $x\backslash y = x^{-1}y$ and $x/y = xy^{-1}$.

2.2. **Symmetry of equational quasigroups.** An (*equational*) *quasigroup* $(Q,\cdot,/,\backslash)$ is a set equipped with three binary operations, multiplication $\cdot$, right division $/$, and left division $\backslash$ such that for all $x,y\in Q$, the following identities are satisfied:

$$\begin{aligned} &\text{(SL)}\quad x\cdot(x\backslash y) = y\,; &&\text{(SR)}\quad y = (y/x)\cdot x\,;\\ &\text{(IL)}\quad x\backslash(x\cdot y) = y\,; &&\text{(IR)}\quad y = (y\cdot x)/x\,. \end{aligned} \tag{2.8}$$

The labels of the identities record that (SL), (SR) give the surjectivity of the left multiplication

$$L(x)\colon Q\to Q; y\mapsto xy$$

and right multiplication

$$R(x)\colon Q\to Q; y\mapsto yx$$

respectively for each $x\in Q$, while (IL), (IR) give their injectivity. By virtue of these injectivities,

$$[x/(y\backslash x)]R(y\backslash x) \overset{(SR)}{=} x \overset{(SL)}{=} yR(y\backslash x)$$

and

$$yL(x/y) \overset{(SR)}{=} x \overset{(SL)}{=} [(x/y)\backslash x]L(x/y)$$

imply the identities

$$\text{(DL)}\quad x/(y\backslash x) = y\,; \qquad \text{(DR)}\quad y = (x/y)\backslash x \tag{2.9}$$

involving the divisions.

Proposition 2.3. *Consider a set Q.*

(a) *Let $(Q,\cdot)$ be a combinatorial quasigroup, with right division* (2.5) *and left division* (2.4). *Then $(Q,\cdot,/,\backslash)$ is an equational quasigroup.*
(b) *Suppose that $(Q,\cdot,/,\backslash)$ is an equational quasigroup. Then $(Q,\cdot)$ is a combinatorial quasigroup.*

Proof. (a) By symmetry, it suffices to check any one of the identities (2.8), (2.9). Consider an element (x_1,x_2,x_3) of $\mathcal{T}_{(Q,\cdot)}$, so that $x_3 = x_1 \cdot x_2$. Note that such an element exists for any $x_1, x_2 \in Q$. Then $(x_3,x_2,x_1) \in \mathcal{T}^{(3\ 1)}_{(Q,\cdot)}$. In the notation of (2.5), this means that $x_3/x_2 = x_1$, so we have $(x_1 \cdot x_2)/x_2 = x_1$ as required.

(b) By definition, $(Q,\cdot)$, $(Q,/)$ and $(Q,\backslash)$ are magmas, along with their respective opposites $(Q,\circ)$, $(Q,//)$ and $(Q,\backslash\backslash)$. The argument of (a) shows that $\mathcal{T}_{(Q,/)} = \mathcal{T}^{(3\ 1)}_{(Q,\cdot)}$. The remaining four multiplication table images $T^g_{(Q,\cdot)}$ for $g \in \Sigma_3 \smallsetminus \{\,(1),(3\ 1)\,\}$ are similarly seen to be magma tables. □

On the basis of Proposition 2.3, which exhibits the equivalence between the concepts of a combinatorial quasigroup and an equational quasigroup, it will suffice merely to refer to *quasigroups*, augmenting a combinatorial quasigroup to a full equational quasigroup as occasion demands.

Definition 2.4. Let $(Q,\cdot)$ be a quasigroup. Then the elements $(Q,\cdot)^g$ of the orbit of $(Q,\cdot)$ under $g \in \Sigma_3$ are called the *conjugates* or "parastrophes" of $(Q,\cdot)$. In particular, $(Q,\cdot)^g$ is called the *g-conjugate* of $(Q,\cdot)$.

Typically, the $(3\ 1)$-conjugate $(Q,/)$ of a quasigroup $(Q,\cdot)$ is called its *right conjugate*, as in (2.5), while the $(2\ 3)$-conjugate $(Q,\backslash)$ is called its *left conjugate*, as in (2.4). The $(1\ 2)$-conjugate is the *opposite* quasigroup $(Q,\circ)$, as in (2.3).

Consider the Cayley diagram for the presentation

$$\langle (1\ 2),(2\ 3) \mid (1\ 2)^2 = (2\ 3)^2 = 1,\ (1\ 2)(2\ 3)(1\ 2) = (2\ 3)(1\ 2)(2\ 3)\rangle$$

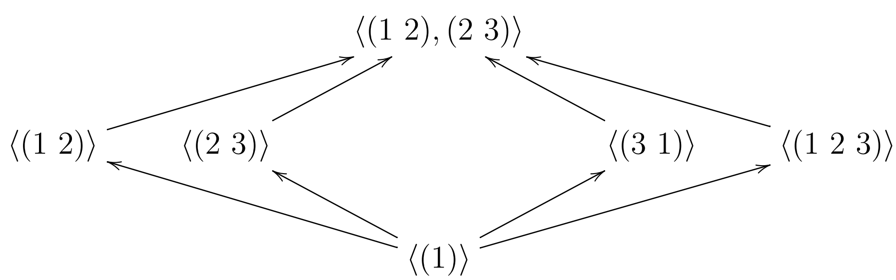
of Σ_3, where single-shafted arrows correspond to left multiplication by $(1\ 2)$, and double-shafted arrows to left multiplication by $(2\ 3)$. The display

$$\begin{array}{ccccc}
\boxed{\begin{array}{c}(Q,\cdot)^{(1)} = \\ (Q,\cdot,/,\backslash)\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}(Q,\cdot)^{(1\ 2)} = \\ (Q,\circ,\backslash\backslash,//)\end{array}} & \Longleftrightarrow & \boxed{\begin{array}{c}(Q,\cdot)^{(1\ 2\ 3)} = \\ (Q,//,\backslash,\circ)\end{array}} \\
\Updownarrow & & & & \updownarrow \\
\boxed{\begin{array}{c}(Q,\cdot)^{(2\ 3)} = \\ (Q,\backslash,//,\cdot)\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}(Q,\cdot)^{(1\ 3\ 2)} = \\ (Q,\backslash\backslash,\circ,/)\end{array}} & \Longleftrightarrow & \boxed{\begin{array}{c}(Q,\cdot)^{(1\ 3)} = \\ (Q,/,\cdot,\backslash\backslash)\end{array}}
\end{array} \tag{2.10}$$

integrates that Cayley diagram, in the permutations $g \in \Sigma_3$ that act on a combinatorial quasigroup $(Q, \cdot)$, into a corresponding record of the six conjugates $(Q, \cdot)^g$ of $(Q, \cdot)$, exhibited as full equational quasigroups. It is convenient to refer to (2.10) as the *triality diagram* for quasigroups [21, §1.8]. Note that Figure 1.4 in the latter reference was based on the Cayley diagram for right multiplications by the generating transpositions. Here, we switch to left multiplications, since that is the convention which will directly match our subsequent treatment of quantum triality in Section 3.4.

2.3. **The Sixfold Way.** Consider the six subgroups

(2.11)



of Σ_3, arranged as a lattice with their covering containment relations (Hasse diagram [25, Ex. III.1.7]). The Galois theory of the Σ_3-action displayed in (2.10) leads to a certain classification of quasigroups, the *Sixfold Way.*

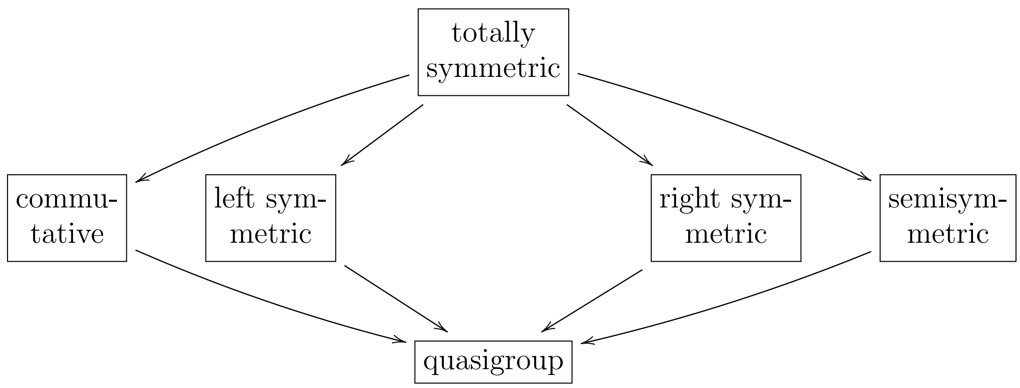
Definition 2.5. For each subgroup H of Σ_3, a quasigroup $(Q, \cdot)$ is said to be H*-symmetric* if $(Q, \cdot)^h = (Q, \cdot)$ for all $h \in H$.

Terminology for the individual classes goes as follows. A quasigroup that is . . .

(a) . . . $\langle(1\ 2)\rangle$-symmetric is *commutative*,
(b) . . . $\langle(2\ 3)\rangle$-symmetric is *left symmetric*,
(c) . . . $\langle(3\ 1)\rangle$-symmetric is *right symmetric*,
(d) . . . $\langle(1\ 2\ 3)\rangle$-symmetric is *semisymmetric*,
(e) . . . Σ_3-symmetric is *totally symmetric*.

As usual in Galois theory, the containment relations between the invariant classes are displayed by the dual

(2.12)



of the subgroup lattice (2.11) of the Galois group Σ_3.

The following propositions are direct consequences of the Galois theory, including the structure of Σ_3.

Proposition 2.6. *A quasigroup* $(Q,\cdot)$ *is ...*

(a) *...commutative iff* $xy = yx$ *holds in* $(Q,\cdot)$,
(b) *...right symmetric iff* $xy \cdot y = x$ *holds in* $(Q,\cdot)$,
(c) *...left symmetric iff* $y \cdot yx = x$ *holds in* $(Q,\cdot)$,
(d) *...semisymmetric iff* $xy \cdot x = y$ *holds in* $(Q,\cdot)$, *and*
(e) *...totally symmetric iff any two of* (a)–(d) *hold in* $(Q,\cdot)$.

Proposition 2.7. *A quasigroup is ...*

(a) *...commutative*
iff its right conjugate is left symmetric
iff its left conjugate is right symmetric;
(b) *...right symmetric*
iff its opposite is left symmetric
iff its left conjugate is commutative;
(c) *...left symmetric*
iff its opposite is right symmetric
iff its right conjugate is commutative;
(d) *...semisymmetric iff its opposite is semisymmetric;*
(e) *...totally symmetric iff any of its conjugates is totally symmetric.*

Example 2.8. (a) A Boolean group, i.e., an elementary abelian group of exponent 2, is totally symmetric. Indeed, for any abelian group $(A,+,0)$, the quasigroup $(A,*)$ with multiplication $x*y = -x-y$ is totally symmetric.

(b) For an abelian group $(A,+,0)$, the quasigroup $(A,*)$ with multiplication $x*y = -x+y$ is right symmetric.

(c) For an abelian group $(A, +, 0)$, the quasigroup $(A, *)$ with multiplication $x * y = -y + x$ is left symmetric.

(d) Let ζ be a primitive sixth root of unity. Then the quasigroup $(\mathbb{C}, *)$ with multiplication $x * y = \zeta x + \overline{\zeta} y$ is semisymmetric, but not commutative, and therefore not totally symmetric.

2.4. **Idempotent quasigroups.** A second version of the Sixfold Way (2.12) is obtained in the context of quasigroups which are idempotent, according to the following definition.

Definition 2.9. Let $(Q, \cdot)$ be a quasigroup.

(a) An element x of Q is *idempotent* if $x \cdot x = x$, so that $\{ x \}$ is a singleton subquasigroup of Q.
(b) The quasigroup Q, or sometimes more specifically, its multiplication, are said to be *idempotent* if each element of Q is idempotent.

Lemma 2.10. *Let $(Q, \cdot)$ be an idempotent quasigroup. Then if $x = y$, each of the equations from Proposition 2.6(a)–(d) holds.*

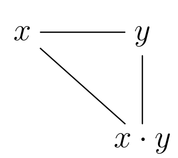
Proof. In universal algebra (cf. [21, App. B], say), singleton algebras satisfy all possible identities. □

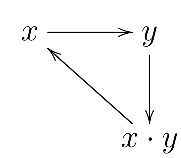
Because of their relevance to constructions in combinatorics and statistics (experimental designs) [5, 6, 16], two of the symmetry classes of idempotent quasigroups have special names.

Definition 2.11. Let $(Q, \cdot)$ be an idempotent quasigroup.

(a) If Q is totally symmetric, it is said to form a *Steiner triple system* (STS).
(b) If Q is semisymmetric, it is said to form a *Mendelsohn triple system* (MTS).

As experimental designs, triple systems consist of a set Q with a set $\mathcal{B}$ of *blocks* or *triples*, special 3-element subsets. In a Steiner triple system, each pair x, y of distinct elements of Q is required to lie in a unique block $\{ x, y, x \cdot y \}$, while in a Mendelsohn triple system, each ordered pair x, y of distinct elements of Q is required to lie in a unique block $\{ x, y, x \cdot y \}$:

STS block: 

MTS block: 

For $x = y$, Lemma 2.10 always guarantees the satisfaction of the equalities from Proposition 2.6. For distinct elements x, y of Q, the validity of the semisymmetric identity $xy \cdot x = y$ from Proposition 2.6(d) is apparent from the form of each block. For Steiner triple systems, the commutativity is also apparent, whence total symmetry follows by Proposition 2.6(e).

3. Quantum quasigroups and their conjugates

3.1. **Quantum quasigroups.** Quantum quasigroups provide a self-dual unification of quasigroups and Hopf algebras [23].

Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ with twist morphism $\tau \colon A \otimes A \to A \otimes A$. A *bimagma* (A, ∇, Δ) is a $\mathbf{V}$-object A, equipped with *multiplication* $\nabla \colon A \otimes A \to A$ and *comultiplication* $\Delta \colon A \to A \otimes A$ that are mutually homomorphic. Mutual homomorphism between the multiplication and comultiplication is expressed by the *bimagma diagram* [9, (2.1)] [23, (2.4)] written as

$$\begin{array}{ccccc} a \otimes b & \xmapsto{\nabla} & a \cdot b & \xmapsto{\Delta} & (a \cdot b)^L \otimes (a \cdot b)^R \\ {\scriptstyle \Delta\otimes\Delta}\downarrow & & & & \uparrow{\scriptstyle \nabla\otimes\nabla} \\ a^L \otimes a^R \otimes b^L \otimes b^R & & \xmapsto[1_A\otimes\tau\otimes 1_A]{} & & a^L \otimes b^L \otimes a^R \otimes b^R \end{array} \tag{3.1}$$

or equationally as

$$x^L \cdot y^L = (x \cdot y)^L \quad \text{and} \quad x^R \cdot y^R = (x \cdot y)^R \tag{3.2}$$

in elementary form with $\nabla \colon a \otimes b \mapsto a \cdot b$ and our "non-coassociative Sweedler notation" $\Delta \colon a \mapsto a^L \otimes a^R$ (cf. [11],[9, 23, Rem. 2.2(b)]).

Definition 3.1. Let (A, ∇, Δ) be a bimagma in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) The *left composite* is

$$\mathsf{G} \colon A \otimes A \xrightarrow{\Delta\otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A\otimes\nabla} A \otimes A\,; \tag{3.3}$$

$$x \otimes y \longmapsto x^L \otimes x^R \otimes y \longmapsto x^L \otimes x^R y$$

("G" for "Gauche").

(b) The *right composite* is

$$\reflectbox{G} \colon A \otimes A \xrightarrow{1_A\otimes\Delta} A \otimes A \otimes A \xrightarrow{\nabla\otimes 1_A} A \otimes A\,; \tag{3.4}$$

$$x \otimes y \longmapsto x \otimes y^L \otimes y^R \longmapsto x y^L \otimes y^R$$

("$\reflectbox{$\mathsf{G}$}$" for "Droite"), the dual of the left composite.

(c) The bimagma (A, ∇, Δ) is a *quantum quasigroup* if the left composite and right composite are invertible.

On the one hand, quasigroups taken with a diagonal comultiplication $\Delta\colon x \mapsto x \otimes x$ are quantum quasigroups within the Cartesian category of sets with direct products (in which context an ordered pair (x, y) is written as $x \otimes y$) [10]. On the other hand, any Hopf algebra $(A, \nabla, \eta, \Delta, \varepsilon, S)$ reduces to a quantum quasigroup (A, ∇, Δ). Most previously studied nonassociative generalizations of Hopf algebras, including the *Hopf quasigroups* of Majid *et al.*, also reduce to quantum quasigroups [1, 4, 12, 13, 18]. However, these earlier concepts are not self-dual. With the term "Hopf quasigroup" already taken, the term "quantum quasigroup" has been adopted for the general concept, in line with one of the many senses of the term "quantum group" (compare [15], for example). In this context, it is often convenient to refer to "ordinary" quasigroups as *classical quasigroups*.

3.2. **Linear quantum quasigroups.** Suppose that S is a commutative unital ring. Let $(\underline{\underline{S}}, \oplus, \{0\})$ denote the Cartesian monoidal category of modules over S. Let (A, ∇, Δ) be a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$, equipped with multiplication

$$\nabla\colon A^2 \to A; \begin{bmatrix} x & y \end{bmatrix} \mapsto \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \rho \\ \lambda \end{bmatrix}$$

and comultiplication

$$\Delta\colon A \to A^2; \begin{bmatrix} x \end{bmatrix} \mapsto \begin{bmatrix} x \end{bmatrix} \begin{bmatrix} L & R \end{bmatrix}$$

for endomorphisms ρ, λ, L, R of A. The bimagma condition (3.2) amounts to the mutual commutativity of the subalgebras $S(\lambda, \rho)$ and $S(L, R)$ of the endomorphism ring $\underline{\underline{S}}(A, A)$ of A [9, Prop. 3.7][23, Prop. 3.39]. It is convenient to write $A(\rho, \lambda, L, R)$ for such a bimagma.

Definition 3.2. A *homomorphism* $P\colon A(\rho_1, \lambda_1, L_1, R_1) \to B(\rho_2, \lambda_2, L_2, R_2)$ of bimagmas in $(\underline{\underline{S}}, \oplus, \{0\})$ is an element P of $\underline{\underline{S}}(A, B)$ such that $f_1 P = P f_2$ for $f_i \in \{\rho_i, \lambda_i, L_i, R_i\}$. In particular, two bimagmas $A(\rho_1, \lambda_1, L_1, R_1)$ and $B(\rho_2, \lambda_2, L_2, R_2)$ in the category $(\underline{\underline{S}}, \oplus, \{0\})$ are isomorphic if and only if there is an "intertwining" $\underline{\underline{S}}$-isomorphism $P\colon A \to B$ such that $f_1 P = P f_2$ for $f_i \in \{\rho_i, \lambda_i, L_i, R_i\}$.

Definition 3.3. Let A be an object in $\underline{\underline{S}}$. Let

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$$

be an endomorphism of A^2 (compare [14, §III.5, §VIII.2]).

(a) The endomorphism Θ is *upper triangular* if $\theta_{21} = 0$.
(b) The endomorphism Θ is *lower triangular* if $\theta_{12} = 0$.

If $A(\lambda,\rho,L,R)$ is a bimagma in $(\underline{\underline{S}},\oplus,\{0\})$, then the respective left and right composites of $A(\lambda,\rho,L,R)$ are

$$\mathsf{G} = \left(\begin{bmatrix} L & R\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix}\right)\left(\begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}\rho\\ \lambda\end{bmatrix}\right) = \begin{bmatrix} L & R\rho \\ 0 & \lambda\end{bmatrix} \tag{3.5}$$

and

$$\reflectbox{G} = \left(\begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix} L & R\end{bmatrix}\right)\left(\begin{bmatrix}\rho\\ \lambda\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix}\right) = \begin{bmatrix} \rho & 0 \\ L\lambda & R\end{bmatrix} \tag{3.6}$$

[9, Lemma 3.9]. Thus G is upper triangular, and $\reflectbox{$\mathsf{G}$}$ is lower triangular.

We write $\underline{\underline{S}}(A,A)^*$ for the automorphism group of A, the group of units of the endomorphism monoid $\underline{\underline{S}}(A,A)$ of A. If the underlying module A of a quantum quasigroup $A(\lambda,\rho,L,R)$ in $(\underline{\underline{S}},\oplus,\{0\})$ is finitely generated as an S-module, then $\lambda,\rho,L,R \in \underline{\underline{S}}(A,A)^*$ [9, Th. 3.14].

Proposition 3.4. *Consider a bimagma $A(\lambda,\rho,L,R)$ in $(\underline{\underline{S}},\oplus,\{0\})$.*

(a) *If $\rho,\lambda,L,R \in \underline{\underline{S}}(A,A)^*$, then $A(\rho,\lambda,L,R)$ is a quantum quasigroup.*
(b) *Let $A(\rho,\lambda,L,R)$ be a quantum quasigroup in $(\underline{\underline{S}},\oplus,\{0\})$.*
  (i) *If G^{-1} is upper triangular, then $\lambda,L \in \underline{\underline{S}}(A,A)^*$.*
  (ii) *If $\reflectbox{$\mathsf{G}$}^{-1}$ is lower triangular, then $\rho,R \in \underline{\underline{S}}(A,A)^*$.*

Proof. (a) See [9, Prop. 3.13], or compare (3.9) below, and its dual.

(b) Suppose

$$\begin{bmatrix}\alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22}\end{bmatrix}\begin{bmatrix} L & R\rho \\ 0 & \lambda\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix} L & R\rho \\ 0 & \lambda\end{bmatrix}\begin{bmatrix}\alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22}\end{bmatrix} \tag{3.7}$$

for endomorphisms α_{ij} of the module A with $1 \le i \le j \le 2$. Then the $(1,1)$-components of (3.7) yield $\alpha_{11}L = 1 = L\alpha_{11}$, so $L \in \underline{\underline{S}}(A,A)^*$. Again, the $(2,2)$-components of (3.7) yield $\alpha_{22}\lambda = 1 = \lambda\alpha_{22}$, so $\lambda \in \underline{\underline{S}}(A,A)^*$. Dually, $R,\rho \in \underline{\underline{S}}(A,A)^*$. □

Bimagmas $A(\rho,\lambda,L,R)$ in $(\underline{\underline{S}},\oplus,\{0\})$ are described as *linear quantum quasigroups* if the endomorphisms ρ,λ,L,R of A lie in the automorphism group $\underline{\underline{S}}(A,A)^*$ [9, Def'n. 3.11]. These objects, studied in some detail in [9, §3.4], embrace a number of important constructions in various areas of mathematics, and provide a rich source of examples and counterexamples for the general theory of quantum quasigroups. A first illustration will be provided in §3.3.3 below.

3.3. **Quantum conjugates.** Quantum conjugates of quantum quasigroups were introduced in [9, §4]. Here, we take a rather more basic approach, working with quantum conjugates of bimagmas whenever possible. Let $(\mathbf{V},\otimes,\mathbf{1})$ denote a symmetric monoidal category with twist morphism τ.

3.3.1. *Opposites.*

Definition 3.5. Let A be an object of $\mathbf{V}$.

(a) Suppose that $\nabla\colon A\otimes A\to A$ is a multiplication on A, and suppose that $\Delta\colon A\to A\otimes A$ is a comultiplication on A. Then the structure $\mathcal{A}=(A,\nabla,\Delta)$ is described as a *weak bimagma* in $(\mathbf{V},\otimes,\mathbf{1})$.

(b) Let $\mathcal{A}=(A,\nabla,\Delta)$ be a *weak bimagma* in $(\mathbf{V},\otimes,\mathbf{1})$. Define the *opposite multiplication* $\nabla_t := \tau\nabla$ and the *opposite comultiplication* $\Delta_t := \Delta\tau$. Then $\mathcal{A}_t=(A,\nabla_t,\Delta_t)$ is described as the *opposite* weak bimagma of $\mathcal{A}$.

Remark 3.6. Suppose that $\mathcal{A}=(A,\nabla,\Delta)$ is a weak bimagma in $(\mathbf{V},\otimes,\mathbf{1})$.

(a) We have

$$\nabla_t\colon a\otimes b\mapsto b\cdot a \text{ and } \Delta_t\colon a\mapsto a^R\otimes a^L \tag{3.8}$$

in our elementary notation.

(b) Definition 3.5 specifies a unique opposite $\mathcal{A}_t=(A,\nabla_t,\Delta_t)$.

(c) Since $\nabla_{tt}=\tau\tau\nabla=\nabla$ and $\Delta_{tt}=\Delta\tau\tau=\Delta$, we have $\mathcal{A}_{tt}=\mathcal{A}$.

Lemma 3.7. *Suppose that* $\mathcal{A}=(A,\nabla,\Delta)$ *is a bimagma in* $(\mathbf{V},\otimes,\mathbf{1})$*. Then* $\mathcal{A}_t=(A,\nabla_t,\Delta_t)$ *is a bimagma, the* opposite bimagma *of* $\mathcal{A}$*.*

Proof. Using (3.8), we have the following diagram chase in (3.1)

$$\begin{array}{ccccc}
a\otimes b & \xmapsto{\nabla_t} & b\cdot a & \xmapsto{\Delta_t} & (b\cdot a)^R\otimes(b\cdot a)^L \\
{\scriptstyle \Delta_t\otimes\Delta_t}\downarrow & & & & \uparrow{\scriptstyle \nabla_t\otimes\nabla_t} \\
a^R\otimes a^L\otimes b^R\otimes b^L & & \xmapsto[1_A\otimes\tau\otimes 1_A]{} & & a^R\otimes b^R\otimes a^L\otimes b^L
\end{array}$$

with $b^L\cdot a^L=(b\cdot a)^L$ and $b^R\cdot a^R=(b\cdot a)^R$ by (3.2). □

The following result, formulated for quantum quasigroups, appeared as [9, Lemma 4.4].

Lemma 3.8. *Let* $\mathcal{A}=(A,\nabla,\Delta)$ *be a weak bimagma with opposite* $\mathcal{A}_t$*. Then the left composite of* $\mathcal{A}_t$ *is* $\tau\reflectbox{$\mathsf{G}$}\tau$*, while the right composite of* $\mathcal{A}_t$ *is* $\tau\mathsf{G}\tau$*.*

Proof. The first assertion is verified by the elementary diagram chase

$$\begin{array}{ccccc}
a\otimes b & \xmapsto{\Delta_t\otimes 1_A} & a^R\otimes a^L\otimes b & \xmapsto{1_A\otimes\nabla_t} & a^R\otimes b\cdot a^L \\
{\scriptstyle\tau}\downarrow & & & & \uparrow{\scriptstyle\tau} \\
b\otimes a & \xmapsto[1_A\otimes\Delta]{} & b\otimes a^L\otimes a^R & \xmapsto[\nabla\otimes 1_A]{} & b\cdot a^L\otimes a^R\,,
\end{array}$$

while the second assertion is dual. □

Proposition 3.9. [9, Prop. 4.5] *Suppose that $\mathcal{A} = (A, \nabla, \Delta)$ is a quantum quasigroup in $(\mathbf{V}, \otimes, \mathbf{1})$. Then the opposite $\mathcal{A}_t = (A, \nabla_t, \Delta_t)$ of $\mathcal{A}$ is a quantum quasigroup.*

Proof. By Lemma 3.7, $\mathcal{A}_t$ is a bimagma. Its left and right composites are invertible by Lemma 3.8. □

3.3.2. *Left and right conjugates.*

Definition 3.10. Let $\mathcal{A} = (A, \nabla, \Delta)$ be a bimagma with left composite G and right composite $\reflectbox{\textsf{G}}$.

(a) The bimagma $\mathcal{A}$ is *quantum left conjugable* if G is invertible, and there is a bimagma $\mathcal{A}_l = (A, \nabla_l, \Delta_l)$ whose left composite G_l is G^{-1}. In this case, $\mathcal{A}_l$ is said to be a *quantum left conjugate* of $\mathcal{A}$.

(b) The bimagma $\mathcal{A}$ is *quantum right conjugable* if $\reflectbox{\textsf{G}}$ is invertible, and there is a bimagma $\mathcal{A}_r = (A, \nabla_r, \Delta_r)$ whose right composite $\reflectbox{\textsf{G}}_r$ is $\reflectbox{\textsf{G}}^{-1}$. In this case, $\mathcal{A}_r$ is said to be a *quantum right conjugate* of $\mathcal{A}$.

Remark 3.11. If $\mathcal{A}$ in Definition 3.10 happens to be a quantum quasigroup, then of course both its left composite G and right composite $\reflectbox{\textsf{G}}$ are invertible. However, in Definition 3.10(a), only the invertibility of the left composite of the quantum left conjugate is required. Indeed, as witnessed later by Example 4.10, the right composite of a quantum left conjugate may fail to be invertible. A dual situation pertains to Definition 3.10(b).

The respective inverse relationships $\mathsf{G}_l = \mathsf{G}^{-1}$ and $\reflectbox{\textsf{G}}_r = \reflectbox{\textsf{G}}^{-1}$ expressed in Definition 3.10 immediately imply the following.

Lemma 3.12. *Let $\mathcal{A} = (A, \nabla, \Delta)$ be a bimagma.*

(a) *If $\mathcal{A}$ is quantum left conjugable, with a quantum left conjugate $\mathcal{A}_l$, then $\mathcal{A}_l$ is quantum left conjugable, with $\mathcal{A}$ as one of its quantum left conjugates.*

(b) *If $\mathcal{A}$ is quantum right conjugable, with a quantum right conjugate $\mathcal{A}_l$, then $\mathcal{A}_l$ is quantum right conjugable, with $\mathcal{A}$ as one of its quantum right conjugates.*

Proposition 3.13. *Let $\mathcal{A} = A(\rho, \lambda, L, R)$ be a quantum quasigroup in the Cartesian monoidal category $(\underline{\underline{S}}, \oplus, \{\,0\,\})$ of monoids over a commutative, unital ring S.*

(a) *If $\mathcal{A}$ is quantum left conjugable, then λ and L are invertible.*

(b) *If $\mathcal{A}$ is quantum right conjugable, then ρ and R are invertible.*

Proof. (a) Consider the left composite G of $\mathcal{A}$. Its inverse G^{-1}, as the left composite G_l of a left conjugate bimagma $\mathcal{A}_l$, is upper triangular according to (3.5). Proposition 3.4(a)(i) then shows that λ and L are invertible.

(b) is dual to (a). □

3.3.3. *Gauge groups.* In Definition 3.10(a), where a bimagma $\mathcal{A} = (A, \nabla, \Delta)$ has a certain quantum left conjugate $\mathcal{A}_l$, that quantum left conjugate need not be uniquely determined by $\mathcal{A}$. A dual result applies for quantum right conjugates $\mathcal{A}_r$ in the situation of Definition 3.10(b). The gauge groups introduced in this paragraph provide examples, as seen in Proposition 3.16 below. Contrast with Remark 3.6(b), which noted that the opposite $\mathcal{A}_t$ of a weak bimagma $\mathcal{A}$ is uniquely specified.

As in Section 3.2, consider the Cartesian monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of monoids over a commutative, unital ring S. Recall that if X is a subset of a ring, then the *commutant* $C(X)$ is the subring consisting of all those ring elements which commute with each element x of X. In the following definition, the commutants are taken in the endomorphism ring $\underline{\underline{S}}(A, A)$ of an S-module A.

Definition 3.14. Let $\mathcal{A} = A(\rho, \lambda, L, R)$ be a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$.

(a) The *left gauge group* of $\mathcal{A}$ is the subgroup $C(\{\lambda, L, R\rho\}) \cap \underline{\underline{S}}(A, A)^*$ of $\underline{\underline{S}}(A, A)^*$.
(b) The *right gauge group* of $\mathcal{A}$ is the subgroup $C(\{\lambda L, R, \rho\}) \cap \underline{\underline{S}}(A, A)^*$ of $\underline{\underline{S}}(A, A)^*$.
(b) The *gauge group* of $\mathcal{A}$ is the subgroup $C(\{\rho, \lambda, L, R\}) \cap \underline{\underline{S}}(A, A)^*$ of $\underline{\underline{S}}(A, A)^*$.

Note that the gauge group is contained in the intersection of the left and right gauge groups. It always contains the multiples of the identity morphism 1_A by invertible scalars from S.

Lemma 3.15. *Let $\mathcal{A} = A(\rho, \lambda, L, R)$ be a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$.*

(a) *Suppose that α is an element of the left gauge group of $\mathcal{A}$. Then the subalgebras $S(\rho\alpha, \lambda)$ and $S(L, \alpha^{-1}R)$ of $\underline{\underline{S}}(A, A)$ commute. Thus $A(\lambda, \rho\alpha, L, \alpha^{-1}R)$ is a bimagma.*
(b) *Suppose that α is an element of the right gauge group of $\mathcal{A}$. Then the subalgebras $S(\rho, \lambda\alpha)$ and $S(\alpha^{-1}L, R)$ of $\underline{\underline{S}}(A, A)$ commute. Thus $A(\rho, \lambda\alpha, \alpha^{-1}L, R)$ is a bimagma.*

Proof. (a) By Definition 3.14(a) and (3.2), we have

$$\begin{gathered}\rho\alpha \cdot \alpha^{-1}R = \rho R = R\rho = \alpha^{-1}R \cdot \rho\alpha\,, \\ \rho\alpha L = \rho L\alpha = L\rho\alpha\,, \text{ and} \\ \lambda \cdot \alpha^{-1}R = \alpha^{-1}\lambda R = \alpha^{-1}R \cdot \lambda\end{gathered}$$

as required.

(b) is dual to (a). □

Let $\mathcal{A} = A(\rho, \lambda, L, R)$ be a bimagma in $(\underline{\underline{S}}, \oplus, \{0\})$. Suppose that the endomorphisms λ and L are automorphisms of A. Then the left composite

$$\mathsf{G}_l = \begin{bmatrix} L^{-1} & -L^{-1}R\rho\lambda^{-1} \\ 0 & \lambda^{-1} \end{bmatrix} \tag{3.9}$$

of $A(-\rho\lambda^{-1}, \lambda^{-1}, L^{-1}, L^{-1}R)$ is the inverse of the left composite (3.5) of $\mathcal{A}$.

Now let α be an element of the left gauge group of the bimagma $\mathcal{A} = A(\rho, \lambda, L, R)$. Then the left composite of the bimagma $A(\rho\alpha, \lambda, L, \alpha^{-1}R)$ of Lemma 3.15(a) reduces to the left composite (3.5) of $A(\lambda, \rho, L, R)$. Similarly, if the endomorphisms λ and L are automorphisms of the module A, then the left composite of the bimagma $A(-\rho\alpha\lambda^{-1}, \lambda^{-1}, L^{-1}, L^{-1}\alpha^{-1}R)$ is the inverse (3.9) of the left composite (3.5) of $\mathcal{A}$. Summarizing and dualizing, we have the following.

Proposition 3.16. *Consider a bimagma $\mathcal{A} = A(\rho, \lambda, L, R)$ in $(\underline{\underline{S}}, \oplus, \{0\})$.*

(a) *Suppose that λ and L are invertible. Then for each element α of the left gauge group of $\mathcal{A}$, the bimagma $A(-\rho\alpha\lambda^{-1}, \lambda^{-1}, L^{-1}, L^{-1}\alpha^{-1}R)$ is a quantum left conjugate $\mathcal{A}_l$ of $\mathcal{A}$.*

(b) *Suppose that ρ and R are invertible. Then for each element α of the right gauge group of $\mathcal{A}$, the bimagma $A(\rho^{-1}, -\lambda\alpha\rho^{-1}, R^{-1}\alpha^{-1}L, R^{-1})$ is a quantum right conjugate $\mathcal{A}_r$ of $\mathcal{A}$.*

3.4. **Conjugate triality of quantum quasigroups.** It is natural to ask for a quantum quasigroup analog of the triality diagram (2.10) for quasigroups. That diagram was based on a Cayley diagram for the degree 3 symmetric group Σ_3, in which single-shafted arrows corresponded to left multiplication by (1 2), and double-shafted arrows to left multiplication by (2 3). The single-shafted arrows served to connect quasigroups with their opposites, while double-shafted arrows connected quasigroups with their left conjugates.

Each quantum quasigroup has a uniquely defined opposite, which is itself a quantum quasigroup by Proposition 3.9. The opposite of the opposite is the original quantum quasigroup. Thus, within the setting of quantum quasigroups, we may take a double-headed single-shafted arrow

$$\mathcal{A} \leftrightarrow \mathcal{A}_t \tag{3.10}$$

to denote that $\mathcal{A}$ and $\mathcal{A}_t$ are mutually opposite quantum quasigroups, each uniquely determining the other.

The potential failure of uniqueness for quantum left conjugates, not to mention their conditional existence, is more of a challenge. We take a single-headed double-shafted arrow

$$\mathcal{A} \Rightarrow \mathcal{A}_l \tag{3.11}$$

to denote that both $\mathcal{A}$ and $\mathcal{A}_l$ are quantum quasigroups, where the latter is one particular quantum left conjugate of $\mathcal{A}$. In particular, (3.11) records that $\mathcal{A}$ is indeed quantum left conjugable. Although $\mathcal{A}$ is one of the quantum left conjugates of $\mathcal{A}_l$ in the situation of (3.11) according to Lemma 3.12, we refrain from using a double-headed double-shafted arrow in order to help avoid creating any erroneous impression that $\mathcal{A}$ might be uniquely determined by $\mathcal{A}_l$.

The display

$$
\begin{array}{ccccc}
\boxed{\begin{array}{c}\mathcal{A} = \\ (A, \mathsf{G}, \rotatebox[origin=c]{180}{G})\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}\mathcal{A}_t = \\ (A, \tau\rotatebox[origin=c]{180}{G}\tau, \tau\mathsf{G}\tau)\end{array}} & \begin{array}{c}\Longleftarrow \\ \Longrightarrow\end{array} & \boxed{\begin{array}{c}\mathcal{A}_{tl} = \\ (A, \tau\rotatebox[origin=c]{180}{G}^{-1}\tau, \mathsf{X}^{-1})\end{array}} \\
\Downarrow\Uparrow & & & & \updownarrow \\
\boxed{\begin{array}{c}\mathcal{A}_l = \\ (A, \mathsf{G}^{-1}, \mathsf{X})\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}\mathcal{A}_{lt} = \\ (A, \tau\mathsf{X}\tau, \tau\mathsf{G}^{-1}\tau)\end{array}} & \begin{array}{c}\Longleftarrow \\ \Longrightarrow\end{array} & \boxed{\begin{array}{c}\mathcal{A}_{ltl} = \\ (A, \tau\mathsf{X}^{-1}\tau, \rotatebox[origin=c]{180}{G}^{-1})\end{array}}
\end{array}
\tag{3.12}
$$

presents the most immediate translation of the triality diagram (2.10) to the quantum quasigroup setting, recording left and right composites rather than multiplications and comultiplications. Here, X will denote any suitable automorphism of $A \otimes A$, in our symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ with twist morphism τ, that is compatible with the requirement to appear as the right composite in a certain choice of quantum left conjugate quasigroup for $(A, \mathsf{G}, \rotatebox[origin=c]{180}{G})$. Note that the actions of the single-shafted arrows are determined by Lemma 3.8.

Definition 3.17. Let $\mathcal{A} = (A, \nabla, \Delta)$ be a quantum quasigroup in $(\mathbf{V}, \otimes, \mathbf{1})$ with twist morphism τ. Then $\mathcal{A}$ is said to exhibit (*quantum*) (*conjugate*) *triality* if there is an automorphism X of $A \otimes A$ such that all the relations depicted in the diagram (3.12) according to (3.10) and (3.11) actually hold.

Inspection of the diagram (3.12) yields the following results.

Proposition 3.18. *Consider a quantum quasigroup $\mathcal{A} = (A, \nabla, \Delta)$ that exhibits conjugate triality. Then:*

(a) *$\mathcal{A}$ has a quantum left conjugate $\mathcal{A}_l$;*
(b) *$\mathcal{A}$ has a quantum right conjugate $\mathcal{A}_r = \mathcal{A}_{tlt} = \mathcal{A}_{ltl}$;*
(c) *$\mathcal{A}_{lt}$ is a quantum left conjugate for $\mathcal{A}_r$; and*
(d) *$\mathcal{A}_{tl}$ is a quantum right conjugate for $\mathcal{A}_l$.*

At first glance, it might appear that duality was broken (in the sense of a broken symmetry) by the emphasis on quantum left conjugates in the construction of the diagram (3.12). However, we could equally well have used quantum right conjugates, starting with a particular quantum right conjugate $\mathcal{A}_r = (A, \mathsf{Y}, \reflectbox{∂}^{-1})$ of $\mathcal{A}$, involving an unknown automorphism Y of $A \otimes A$. Production of the corresponding diagram would then have yielded $\mathcal{A}_l = (A, \mathsf{G}^{-1}, \tau\mathsf{Y}^{-1}\tau)$ as the chosen quantum left conjugate of $\mathcal{A}$. The two approaches are connected by the relationship $\mathsf{X}\tau\mathsf{Y} = \tau$ or

$$\begin{array}{ccc} A \otimes A & \overset{\tau}{\longleftrightarrow} & A \otimes A \\ {\scriptstyle \mathsf{X}}\big\downarrow & & \big\uparrow{\scriptstyle \mathsf{Y}} \\ A \otimes A & \underset{\tau}{\longleftrightarrow} & A \otimes A \end{array}$$

which, like the conjugate triality diagram (3.12) itself, is seen to be self-dual.

Example 3.19 (Classical quasigroups)**.** Take a (classical) quasigroup $(Q, \cdot)$, with diagonal comultiplication, as a quantum quasigroup in the Cartesian monoidal category $(\mathbf{Set}, \times, \top)$ of sets. (Here, the singleton monoidal unit is recorded as a terminal object.) Then with $\mathsf{X} = 1_{Q\times Q}$, the quantum quasigroup $(Q, \cdot)$ exhibits quantum conjugate triality as follows:

$$\begin{array}{ccccc} \boxed{\begin{matrix}(Q,\cdot) = \\ (Q, \cdot, /, \backslash)\end{matrix}} & \longleftrightarrow & \boxed{\begin{matrix}(Q,\cdot)_t = \\ (Q, \circ, \backslash\backslash, //)\end{matrix}} & \begin{matrix}\Longleftarrow \\ \Longrightarrow\end{matrix} & \boxed{\begin{matrix}(Q,\cdot)_{tl} = \\ (Q, //, \backslash, \circ)\end{matrix}} \\ \Downarrow\Uparrow & & & & \updownarrow \\ \boxed{\begin{matrix}(Q,\cdot)_l = \\ (Q, \backslash, //, \cdot)\end{matrix}} & \longleftrightarrow & \boxed{\begin{matrix}(Q,\cdot)_{lt} = \\ (Q, \backslash\backslash, \circ, /)\end{matrix}} & \begin{matrix}\Longleftarrow \\ \Longrightarrow\end{matrix} & \boxed{\begin{matrix}(Q,\cdot)_r = \\ (Q, /, \cdot, \backslash\backslash)\end{matrix}} \end{array}.$$

Thus, in this case, the diagram (3.12) immediately recovers (2.10).

Example 3.20 (Linear quantum quasigroups)**.** Let $\mathcal{A} = A(\rho, \lambda, L, R)$ be a linear quantum quasigroup within the Cartesian category $\big(\underline{\underline{S}}, \oplus, \{\, 0 \,\}\big)$ of modules over a commutative, unital ring S. Let α be an element of the gauge group of $\mathcal{A}$, the centralizer of $\{\, \rho, \lambda, L, R \,\}$ in $\underline{\underline{S}}(A, A)^*$. Then $\mathcal{A}$

exhibits conjugate triality as follows:
(3.13)

$$\begin{array}{ccc}
\boxed{\begin{matrix}\mathcal{A} = \\ A(\rho, \lambda, L, R)\end{matrix}} & \longleftrightarrow & \boxed{\begin{matrix}\mathcal{A}_t = \\ A(\lambda, \rho, R, L)\end{matrix}} \\
\Downarrow\Uparrow & & \Downarrow\Uparrow \\
\boxed{\begin{matrix}\mathcal{A}_l = \\ A(-\rho\alpha\lambda^{-1}, \lambda^{-1}, L^{-1}, L^{-1}\alpha^{-1}R)\end{matrix}} & & \boxed{\begin{matrix}\mathcal{A}_{tl} = \mathcal{A}_{rt} = \\ A(-\lambda\alpha\rho^{-1}, \rho^{-1}, R^{-1}, R^{-1}\alpha^{-1}L)\end{matrix}} \\
\updownarrow & & \updownarrow \\
\boxed{\begin{matrix}\mathcal{A}_{lt} = \\ A(\lambda^{-1}, -\rho\alpha\lambda^{-1}, L^{-1}\alpha^{-1}R, L^{-1})\end{matrix}} & \begin{matrix}\Longleftarrow \\ \Longrightarrow\end{matrix} & \boxed{\begin{matrix}\mathcal{A}_r = \\ A(\rho^{-1}, -\lambda\alpha\rho^{-1}, R^{-1}\alpha^{-1}L, R^{-1})\end{matrix}}
\end{array}$$

(compare [9, Th. 4.9], which corresponds to the case $\alpha = 1_A$). Here, we have

$$\mathsf{X} = \begin{bmatrix} -\rho\alpha\lambda^{-1} & 0 \\ L^{-1}\lambda^{-1} & L^{-1}\alpha^{-1}R \end{bmatrix}$$

in the notation of (3.12). Indeed, the list

$$\begin{aligned}
\mathcal{A} &= \left(A, \begin{bmatrix} L & R\rho \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \rho & 0 \\ L\lambda & R \end{bmatrix}\right), \\
\mathcal{A}_t &= \left(A, \begin{bmatrix} R & L\lambda \\ 0 & \rho \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ R\rho & L \end{bmatrix}\right), \\
\mathcal{A}_l &= \left(A, \begin{bmatrix} L^{-1} & -L^{-1}R\rho\lambda^{-1} \\ 0 & \lambda^{-1} \end{bmatrix}, \begin{bmatrix} -\rho\alpha\lambda^{-1} & 0 \\ L^{-1}\lambda^{-1} & L^{-1}\alpha^{-1}R \end{bmatrix}\right), \\
\mathcal{A}_{tl} = \mathcal{A}_{rt} &= \left(A, \begin{bmatrix} R^{-1} & -R^{-1}L\lambda\rho^{-1} \\ 0 & \rho^{-1} \end{bmatrix}, \begin{bmatrix} -\lambda\alpha\rho^{-1} & 0 \\ R^{-1}\rho^{-1} & R^{-1}\alpha^{-1}L \end{bmatrix}\right), \\
\mathcal{A}_{lt} &= \left(A, \begin{bmatrix} L^{-1}\alpha^{-1}R & L^{-1}\lambda^{-1} \\ 0 & -\rho\alpha\lambda^{-1} \end{bmatrix}, \begin{bmatrix} \lambda^{-1} & 0 \\ -L^{-1}R\rho\lambda^{-1} & L^{-1} \end{bmatrix}\right), \\
\mathcal{A}_{ltl} = \mathcal{A}_{tlt} = \mathcal{A}_r &= \left(A, \begin{bmatrix} R^{-1}\alpha L & R^{-1}\rho^{-1} \\ 0 & -\lambda\alpha^{-1}\rho^{-1} \end{bmatrix}, \begin{bmatrix} \rho^{-1} & 0 \\ -R^{-1}L\lambda\rho^{-1} & R^{-1} \end{bmatrix}\right)
\end{aligned}$$

serves to identify the left and right composites of these quantum conjugates of $\mathcal{A}$.

3.5. **Generalized conjugates of linear quantum quasigroups.** In the theory of Hopf algebras for which the antipode is required to be invertible, there is a C_2^2- or V_4-symmetry. Namely, if $(H, \nabla, \Delta, \eta, \varepsilon, S)$ is such a Hopf algebra, then the structure at each vertex of the square

$$\begin{array}{ccc} H=(H,\nabla,\Delta,S) & \overset{t_\nabla}{\longleftrightarrow} & H^{\mathsf{op}}=(H,\tau\nabla,\Delta,S^{-1}) \\ {\scriptstyle t_\Delta}\updownarrow & & \updownarrow{\scriptstyle t_\Delta} \\ H^{\mathsf{coop}}=(H,\nabla,\Delta\tau,S^{-1}) & \underset{t_\nabla}{\longleftrightarrow} & H^{\mathsf{op\text{-}coop}}=(H,\tau\nabla,\Delta\tau,S) \end{array}$$

is a Hopf algebra [15, Ex. 1.3.3]. (Here, $S\colon H \to H^{\mathsf{op\text{-}coop}}$ is actually an isomorphism.)

Linear quantum quasigroups exhibit a comparable C_2^2- or V_4-symmetry. Indeed, given $A(\rho,\lambda,L,R)$, define $\mathrm{Aut}(A)^4$-permutations

$$t_\nabla : (\rho,\lambda,L,R) \mapsto (\lambda,\rho,L,R), \tag{3.14}$$

$$t_\Delta : (\rho,\lambda,L,R) \mapsto (\rho,\lambda,R,L), \tag{3.15}$$

realizing the opposite and co-opposite quantum quasigroups of $A(\rho,\lambda,L,R)$, respectively. Recall from the previous section that a linear quantum quasigroup may have multiple quantum left conjugates. However, $A(\rho,\lambda,L,R)$ does have only one quantum left conjugate whose multiplication structure coincides with the classical left conjugate, corresponding to the choice of the identity element of the gauge group. Let

$$s_{\nabla\Delta} : (\rho,\lambda,L,R) \mapsto (-\rho\lambda^{-1},\lambda^{-1},L^{-1},L^{-1}R) \tag{3.16}$$

denote the permutation assigning this unique conjugate to $A(\rho,\lambda,L,R)$.

We may now interpret (3.13) as a Σ_3-Cayley diagram where single-shafted arrows denote action by $t_{\nabla\Delta} := t_\nabla t_\Delta = t_\Delta t_\nabla$, and the action of $s_{\nabla\Delta}$ is denoted by double-shafted arrows. If we bring t_∇ and t_Δ into the fold as separate maps, then we realize a symmetry of $\Sigma_3^2 \cong$

$$\langle t_\nabla, t_\Delta, s_{\nabla\Delta} | t_\nabla^2 = t_\Delta^2 = t_{\nabla\Delta}^2 = s_{\nabla\Delta}^2 = (s_{\nabla\Delta}t_{\nabla\Delta})^3 = (s_{\nabla\Delta}t_\nabla s_{\nabla\Delta}t_\Delta)^2 = 1\rangle\ ,$$

and then the interpreted Σ_3-Cayley diagram (3.13) represents the diagonal subgroup with respect to this action. Indeed, we have

$$s_\nabla := (t_\nabla s_{\nabla\Delta})^2 t_\nabla : (\rho,\lambda,L,R) \mapsto (-\rho\lambda^{-1},\lambda^{-1},L,R),$$

and

$$s_\Delta := (t_\Delta s_{\nabla\Delta})^2 t_\Delta : (\rho,\lambda,L,R) \mapsto (\rho,\lambda,L^{-1},L^{-1}R),$$

with which we may construct the standard direct product presentation for Σ_3^2 :

$$\langle s_\nabla, t_\nabla, s_\Delta, t_\Delta | s_i^2 = t_i^2 = (s_i t_i)^3 = (s_i s_j)^2 = (s_i t_j)^2 = (t_i t_j)^2 = 1\rangle_{i\neq j\in\{\nabla,\Delta\}}\ .$$

Consider the commutation graph

$$\begin{array}{ccc} \rho & \text{——} & L \\ | & & | \\ R & \text{——} & \lambda \end{array} \tag{3.17}$$

where an edge between two of the automorphisms serves to denote their mutual commutativity under the bimagma condition. The symmetry group of the square (3.17) is D_4. Thus the theory of linear bimagmas is equipped with a D_4-symmetry. We use p to denote the 4-cycle

$$p : \rho \mapsto R \mapsto \lambda \mapsto L \mapsto \rho \tag{3.18}$$

corresponding to a counterclockwise rotation of (3.17) by $\pi/2$. Note

$$p^2 = t_{\nabla\Delta}, \tag{3.19}$$

and

$$(ps_i)^6 = (s_i p)^6 = 1,\ \text{ for } i \in \{\nabla, \Delta\}. \tag{3.20}$$

Definition 3.21. Let $A(\rho, \lambda, L, R)$ be a linear quantum quasigroup. A quantum quasigroup $A(\rho', \lambda', L', R')$ is a *generalized quantum conjugate* of $A(\rho, \lambda, L, R)$ if there is a group word g in $\{s_\nabla, t_\nabla, s_\Delta, t_\Delta, p\}$ such that

$$(\rho, \lambda, L, R) \overset{g}{\mapsto} (\rho', \lambda', L', R')$$

under the actions specified by (3.14)–(3.16) and (3.18).

Writing F_2 for the free group on two generators, let A denote the $\mathbb{Z}$-linear quantum quasigroup on $\mathbb{Z}[F_2 \times F_2] \cong \mathbb{Z}\langle \rho, \lambda\rangle \otimes \mathbb{Z}\langle L, R\rangle$, under the operations

$$\nabla : a \otimes b \mapsto a\rho + b\lambda \text{ and}$$
$$\Delta : a \mapsto aL \otimes aR.$$

Let $\mathcal{C}_A$ denote the set of generalized quantum conjugates of A.

Problem 3.22. Determine a presentation of a group that acts fully and faithfully on $\mathcal{C}_A$.

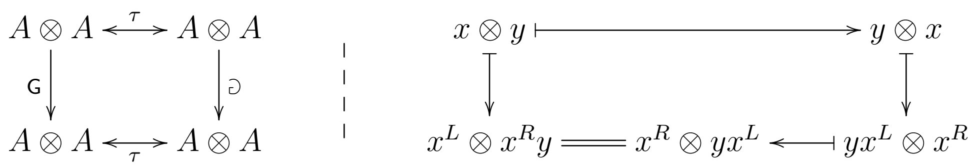
A solution to Problem 3.22 would fulfill the role that Σ_3 plays in the theory of classical quasigroup conjugates.

4. The Sevenfold Way

Section 2.3 used classical quasigroup triality to identify classes of quasigroups that were defined by symmetry properties. This chapter is devoted to the analogous identification of various classes of quantum quasigroups, generally taken within a symmetric, monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ with twist morphism τ.

4.1. **Quantum commutativity.**

Definition 4.1. A bimagma (A, ∇, Δ) in $(\mathbf{V}, \otimes, \mathbf{1})$ is said to be *quantum commutative* if the self-dual diagram



commutes, i.e.

$$x^L \otimes x^R y = x^R \otimes y x^L \tag{4.1}$$

at the elementary level.

Proposition 4.2. *Consider a bimagma $\mathcal{A} = (A, \nabla, \Delta)$ in $(\mathbf{V}, \otimes, \mathbf{1})$.*

(a) *If the bimagma is commutative and cocommutative, it is quantum commutative.*
(b) *If the bimagma is unital, counital and quantum commutative, then it is commutative and cocommutative.*
(c) *The bimagma $\mathcal{A}$ is quantum commutative iff $\mathcal{A} = \mathcal{A}_t$.*

Proof. (a) If the bimagma is commutative and cocommutative, (4.1) holds.

(b) The self-duality of the claimed implication shows that it suffices to verify the cocommutativity on the basis of (4.1):

$$x^L \otimes x^R = (x \otimes 1)\mathsf{G} = (x \otimes 1)\tau ⅁ \tau = x^R \otimes x^L$$

with 1 as the multiplicative unit in elementary notation.

(c) is immediate from Definition 4.1. □

Example 4.3. (a) Since the diagonal comultiplication is cocommutative, a classical quasigroup enriched with the diagonal comultiplication is quantum commutative in $(\mathbf{Set}, \times, \top)$ iff its multiplication is commutative.

(b) Commutative Moufang loops that are enriched with certain nondiagonal cocommutative comultiplications have left composites that yield new set-theoretical solutions to the quantum Yang-Baxter equation [24, Cor. 6.8]. These quantum quasigroups in $(\mathbf{Set}, \times, \top)$ are quantum commutative by Proposition 4.2(a).

Lemma 4.4. *Suppose that $\mathcal{A} = A(\rho, \lambda, L, R)$ is a bimagma within the Cartesian category $\left(\underline{\underline{S}}, \oplus, \{ 0 \}\right)$ of modules over a commutative, unital ring S. Then $\mathcal{A}$ is quantum commutative iff $\rho = \lambda$ and $L = R$.*

4.2. Quantum left and right symmetry.

Definition 4.5. Consider a bimagma (A, ∇, Δ) in $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) The bimagma is (*quantum*) *left symmetric* if $\mathsf{G}^2 = 1_{A\otimes A}$:

$$\begin{array}{ccccc}
x^L \otimes x^R \otimes y & & \xmapsto{1_A\otimes\nabla} & & x^L \otimes x^R y \\
\uparrow{\scriptstyle \Delta\otimes 1_A} & & & & \downarrow{\scriptstyle \Delta\otimes 1_A} \\
x \otimes y & = & x^{LL} \otimes x^{LR}\left(x^R y\right) & \xleftarrow[1_A\otimes\nabla]{} & x^{LL} \otimes x^{LR} \otimes x^R y
\end{array}$$

so that

$$x^{LL} \otimes x^{LR}(x^R y) = x \otimes y \tag{4.2}$$

in elementary terms.

(b) Dually, the bimagma is (*quantum*) *right symmetric* if $\reflectbox{$\mathsf{G}$}^2 = 1_{A\otimes A}$:

$$\begin{array}{ccccc}
yx^L \otimes x^{RL} \otimes x^{RR} & & \xleftarrow{1_A\otimes\Delta} & & yx^L \otimes x^R \\
\downarrow{\scriptstyle \nabla\otimes 1_A} & & & & \uparrow{\scriptstyle \nabla\otimes 1_A} \\
\left(yx^L\right) x^{RL} \otimes x^{RR} & = & y \otimes x & \xmapsto[1_A\otimes\Delta]{} & y \otimes x^L \otimes x^R
\end{array}$$

so that

$$y \otimes x = \left(yx^L\right) x^{RL} \otimes x^{RR} \tag{4.3}$$

in elementary terms.

Remark 4.6. (a) Quantum left and right symmetry, as in Definition 4.5, should be distinguished from the corresponding concepts of semi-classical left and right symmetry presented in the predecessor paper [9, §5.2]. While it is sometimes convenient to suppress the qualifying "quantum" in the current context, it may always be reinstated to disambiguate from the earlier notions.

(b) The mutual duality between quantum left and right symmetry is seen nicely in the tensor category $\left(\underline{\underline{K}}, \otimes, K\right)$ of vector spaces over a field K. Let Q be a finite classically left symmetric quasigroup, for instance an abelian group under subtraction as in Example 2.8(c). The quasigroup algebra KQ, with multiplication extended linearly from Q, and equipped with diagonal comultiplication, is quantum left symmetric. The dual space KQ^* of linear functionals on the finite-dimensional vector space KQ then forms a quantum right symmetric quantum quasigroup.

Proposition 4.7. *Consider a bimagma* $\mathcal{A} = (A, \nabla, \Delta)$ *in* $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) *If* $\mathcal{A}$ *is quantum left symmetric, then its left composite is invertible.*

(b) *If $\mathcal{A}$ is quantum right symmetric, its right composite is invertible.*
(c) *The bimagma $\mathcal{A}$ is quantum left symmetric iff it is quantum left conjugable, with $\mathcal{A} = \mathcal{A}_l$.*
(d) *Dually, the bimagma $\mathcal{A}$ is quantum right symmetric iff it is quantum right conjugable, with $\mathcal{A} = \mathcal{A}_r$.*

Remark 4.8. In Proposition 4.7(c), the nature of the right composite in the chosen quantum left conjugate $\mathcal{A}_l$ has no impact on the quantum left symmetry of $\mathcal{A}$. In particular, as witnessed by Example 4.10, there is no need for the right composite to be invertible. A dual remark applies to Proposition 4.7(d).

Proposition 4.9. *Suppose that $\mathcal{A} = A(\rho, \lambda, L, R)$ is a bimagma within the Cartesian category $\big(\underline{\underline{S}}, \oplus, \{ 0 \}\big)$ of modules over a commutative, unital ring S.*

(a) *Then the bimagma $\mathcal{A}$ is quantum left symmetric iff $\lambda^2 = L^2 = 1_A$ and $\rho(L + \lambda)R = 0_A$.*
(b) *Dually, the bimagma $\mathcal{A}$ is quantum right symmetric iff $\rho^2 = R^2 = 1_A$ and $\lambda(R + \rho)L$.*
(c) *If $\mathcal{A}$ is a linear quantum quasigroup, it is quantum left symmetric iff $L = -\lambda$ and $\lambda^2 = 1_A$.*
(d) *Dually, if $\mathcal{A}$ is a linear quantum quasigroup, it is quantum right symmetric iff $R = -\rho$ and $\rho^2 = 1_A$.*

Proof. (a) Note

$$\mathsf{G}^2 \overset{(3.5)}{=} \begin{bmatrix} L & R\rho \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} L & R\rho \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} L^2 & LR\rho + R\rho\lambda \\ 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} 1_A & 0_A \\ 0_A & 1_A \end{bmatrix},$$

so $L^2 = \lambda^2 = 1_A$ and $0_A = LR\rho + R\rho\lambda = \rho LR + \rho\lambda R = \rho(L + \lambda)R$.

(c) If ρ and R are invertible, the conditions of (a) are equivalent to $L = -\lambda$ and $\lambda^2 = 1$. □

Example 4.10. (a) Let S be a commutative, unital ring S. Then bimagmas in $\big(\underline{\underline{S}}, \oplus, \{ 0 \}\big)$ of the form $A(0_A, \lambda, L, 0_A)$ are quantum left symmetric when $\lambda^2 = L^2 = 1_A$.

(b) Consider the Klein 4-group A in the Cartesian category $\big(\underline{\underline{\mathbb{Z}}}, \oplus, \{ 0 \}\big)$ of abelian groups. Recall that the automorphism group of the group A is the symmetric group Σ_3 as represented by the actions of automorphisms on the set of the three non-zero 2-dimensional binary vectors. Then the smallest examples of non-trivial quantum left symmetric linear quantum quasigroups are given by $A(\rho, \tau, \tau, R)$, where τ is a transposition and $\rho, R \in \{ 1_A, \tau \}$.

4.3. **Quantum semisymmetry.** A concept of *semisymmetry* was defined semiclassically in [9, Def'n 5.3(c)]. In this section, we use an analogue of the definition of classical semisymmetry as a part of the Sixfold Way to present a new notion of quantum semisymmetry. The analogy is best seen from Proposition 4.13(a)(b).

Definition 4.11. A bimagma (A, ∇, Δ) in $(\mathbf{V}, \otimes, \mathbf{1})$ is said to be (*quantum*) *semisymmetric* if the self-dual diagram

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\mathsf{G}} & A\otimes A \\ \tau\updownarrow & & \updownarrow\tau \\ A\otimes A & \xleftarrow[\Game]{} & A\otimes A \end{array} \qquad \Bigg| \qquad \begin{array}{ccccc} y\otimes x & \longmapsto & & & y^L\otimes y^R x \\ \uparrow & & & & \downarrow \\ x\otimes y & = & \left(y^R x\right) y^{LL}\otimes y^{LR} & \longleftarrow & y^R x\otimes y^L \end{array}$$

commutes, i.e.,

$$x\otimes y = \left(y^R x\right) y^{LL}\otimes y^{LR} \tag{4.4}$$

at the elementary level.

Remark 4.12. As discussed in Remark 4.6 in connection with quantum left and right symmetry, the word "quantum" qualifying "semisymmetric" in Definition 4.11 will often be omitted, e.g., in the diagram (4.18). It may always be reinstated to disambiguate from the earlier semiclassical notion of [9, Def'n 5.3(c)].

Proposition 4.13. *Consider a bimagma* $\mathcal{A} = (A, \nabla, \Delta)$ *in* $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) *The bimagma* $\mathcal{A}$ *is quantum semisymmetric iff its opposite is a left conjugate, i.e., if* $\mathcal{A}_t = \mathcal{A}_l$.
(b) *The bimagma* $\mathcal{A}$ *is quantum semisymmetric iff its opposite is a right conjugate, i.e., if* $\mathcal{A}_t = \mathcal{A}_r$.
(c) *If the bimagma* $\mathcal{A}$ *is quantum semisymmetric, then it is a quantum quasigroup.*

Proof. (a) Construe quantum semisymmetry as saying that the respective left composites $\tau\Game\tau$ and G^{-1} of $\mathcal{A}_t$ and $\mathcal{A}_l$ agree.

(b) Dually, construe quantum semisymmetry as saying that the respective right composites $\tau\mathsf{G}\tau$ and $\Game^{-1}$ of $\mathcal{A}_t$ and $\mathcal{A}_r$ agree.

(c) If $\mathcal{A}$ is quantum semisymmetric, we have $\tau\Game\tau$ as the inverse of G and $\tau\mathsf{G}\tau$ as the inverse of $\Game$. □

The elementary expression (4.4) of quantum semisymmetry was obtained by starting the diagram chase from the representative element $x \otimes y$ of the copy of $A \otimes A$ in the lower left-hand corner of the commuting square. The full set

$$x \otimes y = \begin{cases} \left(y^R x\right) y^{LL} \otimes y^{LR} \\ y^{RR}\left(xy^L\right) \otimes y^{RL} \\ x^{RL} \otimes x^{RR}\left(yx^L\right) \\ x^{LR} \otimes \left(x^R y\right) x^{LL} \end{cases} \tag{4.5}$$

of equivalent elementary characterizations is obtained by starting the chase from each of the corners of the commuting square.

Problem 4.14. For what conditions, weaker than quantum semisymmetry, could any two or more of the four words on the right hand side of (4.5) be identically equal?

Theorem 4.15. *Let $\mathcal{A} = A(\rho, \lambda, L, R)$ be a bimagma within the Cartesian category $\left(\underline{S}, \oplus, \{0\}\right)$ of modules over a commutative, unital ring S.*

(a) *If the bimagma $\mathcal{A}$ is quantum semisymmetric, then it is a linear quantum quasigroup.*
(b) *The bimagma $\mathcal{A}$ is quantum semisymmetric if and only if it has the form $\mathcal{A} = A(\rho, \rho^{-1}, L, L^{-1})$ with $L^3 = -\rho^3$.*

Proof. (a) As in the proof of Proposition 4.13(c), if the bimagma is quantum semisymmetric, we have the upper triangular endomorphism $\tau \reflectbox{G} \tau$ as the inverse of G and the lower triangular endomorphism $\tau \mathsf{G} \tau$ as the inverse of $\reflectbox{G}$. Proposition 3.4(b) then shows that ρ, λ, L and R are invertible.

(b) The bimagma is quantum symmetric iff

$$\mathsf{G}(\tau \reflectbox{G} \tau) = \begin{bmatrix} L & R\rho \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} R & L\lambda \\ 0 & \rho \end{bmatrix} = \begin{bmatrix} LR & L^2\lambda + R\rho^2 \\ 0 & \lambda\rho \end{bmatrix} = \begin{bmatrix} 1_A & 0_A \\ 0_A & 1_A \end{bmatrix},$$

which holds precisely when

$$LR = \lambda\rho = 1_A \quad \text{and} \tag{4.6}$$

$$L^2\lambda + R\rho^2 = 0_A\,. \tag{4.7}$$

If ρ, λ, L and R are invertible, then (4.6) is equivalent to $\lambda = \rho^{-1}$ and $R = L^{-1}$. The equation (4.7) is then equivalent to $0 = L(L^2\lambda + R\rho^2)\rho = L(L^2\rho^{-1} + L^{-1}\rho^2)\rho = L^3 + \rho^3$. □

4.4. **Total quantum symmetry.**

Definition 4.16. A bimagma (A, ∇, Δ) in $(\mathbf{V}, \otimes, \mathbf{1})$ is *quantum totally symmetric* if it is quantum commutative and quantum semisymmetric.

Note that Definition 4.16 is self-dual. By Proposition 4.13(c), a quantum totally symmetric bimagma is a quantum quasigroup.

Lemma 4.17. *The following combinations of symmetries are sufficient for total symmetry in a quantum quasigroup* (A, ∇, Δ)*:*

(a) (A, ∇, Δ) *is quantum commutative and quantum left symmetric;*
(b) (A, ∇, Δ) *is quantum commutative and quantum right symmetric;*
(c) (A, ∇, Δ) *is quantum commutative and quantum semisymmetric;*
(d) (A, ∇, Δ) *is quantum semisymmetric and quantum left symmetric;*
(e) (A, ∇, Δ) *is quantum semisymmetric and quantum right symmetric.*

Proof. All of these results are consequences of the fact that the order of an isomorphism in $(\mathbf{V}, \otimes, \mathbf{1})$ is invariant under conjugation by τ. □

4.4.1. *Quantum and classical total symmetry.* The main theorem of this paragraph will serve to connect coassociative, totally symmetric quantum quasigroups with idempotent, totally symmetric classical quasigroups — Steiner triple systems. In the presence of coassociativity, the proof uses the traditional Sweedler notation $\Delta\colon x \mapsto x_{(1)} \otimes x_{(2)}$. Note that our "non-coassociative Sweedler notation" $\Delta\colon a \mapsto a^L \otimes a^R$ reduces down to the traditional notation in the coassociative case by numbering our superfices, words in the ordered alphabet $L < R$, in lexicographic order. For example, the equivalent versions of quantum semisymmetry from (4.5) become

$$x \otimes y = \begin{cases} \big(y_{(3)}x\big)\, y_{(1)} \otimes y_{(2)} & \text{since } LL < LR < R\,, \\ y_{(3)}\,\big(xy_{(1)}\big) \otimes y_{(2)} & \text{since } L < RL < RR\,, \\ x_{(2)} \otimes x_{(3)}\,\big(yx_{(1)}\big) & \text{since } L < RL < RR\,, \text{ and} \\ x_{(2)} \otimes \big(x_{(3)}y\big)\, x_{(1)} & \text{since } LL < LR < R \end{cases} \tag{4.8}$$

in traditional Sweeder notation when the comultiplication is coassociative.

Theorem 4.18. *Consider a coassociative and totally symmetric quantum quasigroup* (A, ∇, Δ) *within a symmetric monoidal category* $(\mathbf{V}, \otimes, \mathbf{1})$ *with symmetry morphism* τ*. Then with*

$$f \odot g := fgf\,, \tag{4.9}$$

the subset $\{\, \mathsf{G}, \reflectbox{G}, \tau \,\}$ *of the automorphism group* $\mathbf{V}(A \otimes A, A \otimes A)^*$ *forms a Steiner triple system* $(\{\, \mathsf{G}, \reflectbox{G}, \tau \,\}, \odot)$*.*

Proof. We prove that

$\odot$	τ	G	$\reflectbox{$\mathsf{G}$}$
τ	τ	$\reflectbox{$\mathsf{G}$}$	G
G	$\reflectbox{$\mathsf{G}$}$	G	τ
$\reflectbox{$\mathsf{G}$}$	G	τ	$\reflectbox{$\mathsf{G}$}$

is the multiplication table associated with (4.9).The left and right symmetry of (A, ∇, Δ) mean that, along with τ, the morphisms G and $\reflectbox{\partial}$ are involutions in $\mathbf{V}$, so $\odot$ is idempotent. The identities $\tau \odot \mathsf{G} = \reflectbox{\partial}$ and $\tau \odot \reflectbox{\partial} = \mathsf{G}$ come directly from the semisymmetry.

Now, $\reflectbox{\partial} \odot \tau = \reflectbox{\partial}\tau\reflectbox{\partial} = \mathsf{G}$ if and only $\reflectbox{\partial}(\tau\reflectbox{\partial}\tau) = \mathsf{G}\tau$ if and only if $\reflectbox{\partial}\mathsf{G}^{-1} = \mathsf{G}\tau$ (by semisymmetry) if and only if $\mathsf{G} \odot \tau = \mathsf{G}\tau\mathsf{G} = \reflectbox{\partial}$. Moreover, $\mathsf{G} \odot \reflectbox{\partial} = \mathsf{G}\reflectbox{\partial}\mathsf{G} = \tau$ if and only if $\mathsf{G} \odot \tau = \mathsf{G}\tau\mathsf{G} = \reflectbox{\partial}$ by left symmetry, and $\reflectbox{\partial} \odot \mathsf{G} = \reflectbox{\partial}\mathsf{G}\reflectbox{\partial} = \tau$ if and only if $\reflectbox{\partial} \odot \tau = \reflectbox{\partial}\tau\reflectbox{\partial} = \mathsf{G}$ by right symmetry. Thus, if we can show $\reflectbox{\partial} \odot \tau = \mathsf{G}$, we are done. Using (4.8), and for any $a, b \in A$, we have

$$
\begin{aligned}
& & b_{(3)}(a_{(1)}b_{(1)}) \otimes b_{(2)} \otimes a_{(2)} &= a_{(1)} \otimes b \otimes a_{(2)} \\
&\Rightarrow & b_{(3)}(a_{(1)}b_{(1)}) \otimes a_{(2)} \otimes b_{(2)} &= a_{(1)} \otimes a_{(2)} \otimes b \\
&\Rightarrow & (a \otimes b)\reflectbox{\partial}\tau\reflectbox{\partial} = b_{(3)}(a_{(1)}b_{(1)}) \otimes a_{(2)}b_{(2)} &= a_{(1)} \otimes a_{(2)}b = (a \otimes b)\mathsf{G}\,,
\end{aligned}
$$

and so $\reflectbox{\partial} \odot \tau = \mathsf{G}$ as desired. □

Ubiquitous elementary illustrations of Theorem 4.18, associated with any abelian group, are presented in Example 4.29 below.

4.4.2. *Totally symmetric Hopf algebras.* The purpose of this paragraph is to extend, to Hopf algebras over an algebraically closed field of characteristic zero, two standard results from group theory:

- Each of left symmetry, right symmetry, or semisymmetry alone is a sufficient condition for a group multiplication to become totally symmetric; and
- All totally symmetric groups are Boolean — compare Example 2.8(a).

In fact, the sufficiency of quantum left-/right-/semi-symmetry for total quantum symmetry will extend to a Hopf algebra over any field. As in the previous paragraph, we continue to use the standard Sweedler notation for comultiplications. We work over an underlying field K, initally without any implicit restriction on the characteristic.

Given a Hopf algebra $(H, \nabla, \Delta, \eta, \varepsilon, S)$, we let

$$G(H) = \left\{ g \in H \;\middle|\; g^{\varepsilon} \neq 0\,,\ g^{\Delta} = g \otimes g \right\} \tag{4.10}$$

denote the set of *grouplike* elements and

$$P(H) = \left\{ x \in H \;\middle|\; x^{\Delta} = x \otimes 1 + 1 \otimes x \right\} \tag{4.11}$$

the set of *primitive elements*.

Lemma 4.19. *A totally symmetric Hopf algebra has a Boolean group of grouplike elements.*

Proof. For any $x \in G(H)$, $x \otimes 1 = (x \otimes 1)^{\mathsf{G}^2} = x \otimes x^2$, so $x^2 = 1 \in G(H)$. □

Lemma 4.20. *Let L be a Lie algebra over K, where* $\mathrm{char}(K) \neq 2$. *If the universal enveloping Hopf algebra $U(L)$ is totally symmetric, then $L = 0$.*

Proof. Consider a primitive element $x \in U(L)$. Then left symmetry implies $x \otimes 1 = (x \otimes 1)^{\mathsf{G}^2} = x \otimes 1 + 2 \otimes x$, so either $2 = 0$, or $x = 0$. □

Lemma 4.21. *If a Hopf algebra $(H, \nabla, \Delta, \eta, \varepsilon, S)$ is left symmetric, right symmetric, or semisymmetric, then $S = 1_H$.*

Proof. We wish to show $x_{(1)}x_{(2)} = x^\varepsilon \cdot 1$ for all $x \in H$. Under the assumption of left symmetry, we have $x_{(1)}x_{(2)} = (x^\varepsilon_{(1)}x_{(2)})x_{(3)} = (x \otimes 1)\mathsf{G}^2(\varepsilon \otimes 1_H) = (x \otimes 1)(\varepsilon \otimes 1_H) = x^\varepsilon \cdot 1$. Dually, assuming right symmetry yields $x_{(1)}x_{(2)} = x_{(1)}(x_{(2)}x^\varepsilon_{(3)}) = (1 \otimes x)\reflectbox{G}^2(1_H \otimes \varepsilon) = (1 \otimes x)(1_H \otimes \varepsilon) = x^\varepsilon \cdot 1$.

To prove that semisymmetry implies $S = 1_H$, we need to make use of the coopposite bialgebra $H^{\mathrm{cop}} := (H, \nabla, \eta, \Delta\tau, \varepsilon)$. If H^{cop} is a Hopf algebra with antipode S', then S' is invertible, and $S = S'^{-1}$ [19, Prop. 7.1.10]. Hence, it suffices to show that, if H is semisymmetric, then H^{cop} is a Hopf algebra with antipode equal to the identity, which is equivalent to $x_{(2)}x_{(1)} = x^\varepsilon \cdot 1$ for all $x \in H$. Indeed, under the assumption of semisymmetry, we have $x_{(2)}x_{(1)} = (x^\varepsilon_{(2)}x_{(3)})x_{(1)} = (x \otimes 1)\tau\reflectbox{G}\tau\mathsf{G}(\varepsilon \otimes 1) = (x \otimes 1)(\varepsilon \otimes 1_H) = x^\varepsilon \cdot 1$. □

Theorem 4.22. *For a Hopf algebra $(H, \nabla, \Delta, \eta, \varepsilon, S)$, the following are equivalent:*

(a) *H is quantum semisymmetric;*
(b) *H is quantum left symmetric;*
(c) *H is quantum right symmetric;*
(d) *H is quantum totally symmetric.*

Proof. Certainly, (d) implies any one of (a)–(c). We complete the proof by showing that any one of (a)–(c) implies (d). Assume that H belongs to one of the symmetry classes of (a)–(c). By Lemma 4.21, $S = 1_H$ in any of these cases. But the antipode of a Hopf algebra is an antihomomorphism of the algebra and coalgebra structures [19, Prop. 7.1.9]. So if 1_H is a (co)algebra antihomomorphism, then H must be bicommutative, and thus quantum commutative. Finally, by Lemma 4.17, we conclude that H is totally symmetric. □

Theorem 4.23. *Let $(H, \nabla, \Delta, \eta, \varepsilon, S)$ be a Hopf algebra over an algebraically closed field K of characteristic 0. Then the the following are equivalent:*

(a) *H is the group Hopf algebra of a Boolean group;*
(b) *H is quantum semisymmetric;*
(c) *H is quantum left symmetric;*
(d) *H is quantum right symmetric;*
(e) *H is quantum totally symmetric.*

Proof. By Theorem 4.22, it suffices to show that (a) and (e) are equivalent. If (a) holds, then the classical total symmetry of the basis group $G(H)$ of grouplike elements implies directly that H is quantum totally symmetric.

Conversely, assume that H is totally symmetric. By Proposition 4.2(b), H is cocommutative. The Cartier-Kostant-Milnor-Moore Theorem states that a cocommutative K-Hopf algebra is a semidirect product of the form $U(P(H))\#KG(H)$ [19, Th. 15.3.4]. However, we know $P(H) = 0$ from Lemma 4.20. By Lemma 4.19, $G(H)$ is a Boolean group. □

4.5. **Quantum bisymmetry.** Taken together, left and right symmetry of classical quasigroups imply total symmetry, as seen in the lattice diagram (2.12) that typifies the Sixfold Way. It will be witnessed by Examples 4.27 and 4.32, however, that the conjunction of quantum left and right symmetry does not imply quantum total symmetry. This phenomenon distinguishes quantum quasigroup symmetry from its classical counterpart, introducing the extra symmetry class that creates the Sevenfold Way.

The extra symmetry class forms the topic of this section, together with its relationship to quantum total symmetry and a description of various examples of quantum totally symmetric quantum quasigroups.

Definition 4.24. A bimagma (A, ∇, Δ) is (*quantum*) *bisymmetric* if it is both left and right symmetric, so $\mathsf{G}^2 = 1_{A\otimes A} = \eth^2$. In particular, a quantum quasigroup $\mathcal{A}$ with conjugate triality is quantum bisymmetric if and only if $\mathcal{A}_l = \mathcal{A} = \mathcal{A}_r$.

Lemma 4.25. *A bisymmetric bimagma is a quantum quasigroup.*

Proposition 4.26. *Suppose that* $\mathcal{A} = A(\rho, \lambda, L, R)$ *is a bimagma within the Cartesian category* $\big(\underline{S}, \oplus, \{ 0 \}\big)$ *of modules over a commutative, unital ring* S*.*

(a) *The bimagma* $\mathcal{A}$ *is bisymmetric iff it is of the form* $A(\rho, \lambda, -\lambda, -\rho)$ *with* $\rho^2 = \lambda^2 = 1_A$*.*
(b) *If* $\mathcal{A}$ *is bisymmetric, it forms a linear quantum quasigroup.*
(c) *The bimagma* $\mathcal{A}$ *is totally quantum symmetric iff it is of the form* $A(\rho, \rho, -\rho, -\rho)$ *with* $\rho^2 = 1_A$*.*

Proof. By Proposition 4.9(a),(b), $\mathcal{A}$ is bisymmetric iff

$$1_A = \rho^2 = R^2 = L^2 = \lambda^2 \quad \text{and} \tag{4.12}$$

$$0_A = \rho(L + \lambda)R = \lambda(R + \rho)L\,. \tag{4.13}$$

If (4.12) holds, then (4.13) is equivalent to the conjunction of $L = -\lambda$ and $R = -\rho$. Thus, if $\mathcal{A}$ is bisymmetric, it must be of the form $A(\rho, \lambda, -\lambda, -\rho)$ with $\rho^2 = \lambda^2 = 1_A$, and in particular is a linear quantum quasigroup.

Conversely, if $\mathcal{A}$ is of the form $A(\rho, \lambda, -\lambda, -\rho)$ with $\rho^2 = \lambda^2 = 1_A$, then (4.12) and (4.13) hold, so that $\mathcal{A}$ is bisymmetric.

Finally, the statement (c) follows since total quantum symmetry is the conjunction of bisymmetry with commutativity, by Lemma 4.17(a)(b). □

Example 4.27. Let A be an S-module over a commutative, unital ring S. Then by Proposition 4.26, the quasigroup operation of subtraction on the underlying abelian group A augments to the structure $A\left(1_A, -1_A, 1_A, -1_A\right)$ of a bisymmetric linear quantum quasigroup $\mathcal{A}$ in the Cartesian category $\left(\underline{\underline{S}}, \oplus, \{0\}\right)$. If S is not of characteristic 2, then $\mathcal{A}$ is not commutative, and thus is not quantum totally symmetric.

Example 4.28. In the Cartesian category $\left(\underline{\underline{\mathbb{Z}/_2}}, \oplus, \{0\}\right)$ of Boolean groups, consider the free module $A = \mathbb{Z}/_2 \oplus \mathbb{Z}/_2$. The linear quantum quasigroup $A\left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\right)$ is bisymmetric, but neither quantum totally symmetric nor commutative.

Example 4.29. Consider the Cartesian category $\left(\underline{\underline{\mathbb{Z}}}, \oplus, \{0\}\right)$ of abelian groups. Let A be an abelian group.

(a) As noted in Example 2.8(a), the operation $x * y = -x - y$ on A yields a classical totally symmetric quasigroup. By Proposition 4.26(c), this classical structure augments to a coassociative quantum totally symmetric quantum quasigroup $A(-1_A, -1_A, 1_A, 1_A)$. It exemplifies Theorem 4.18, with

$$\{\,\mathsf{G}, \eth, \tau\,\} = \left\{ \begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}, \begin{bmatrix}-1 & 0\\-1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \right\}. \tag{4.14}$$

(b) Dually, the addition of A augments to a coassociative quantum totally symmetric quantum quasigroup $A(1_A, 1_A, -1_A, -1_A)$. Here,

$$\{\,\mathsf{G}, \eth, \tau\,\} = \left\{ \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \right\}. \tag{4.15}$$

exemplifies Theorem 4.18

Under matrix multiplication, the sets (4.14) and (4.15) generate Σ_3, so the STS operation $\odot$ on those sets of transpositions corresponds to conjugation.

4.5.1. *Bisymmetry of vector spaces.*

Theorem 4.30. *Let K be a field whose characteristic is not 2. Consider the category $\left(\underline{\underline{K}}, \oplus, \{0\}\right)$. Let n be a natural number.*

(a) *Each finite-dimensional, bisymmetric quantum quasigroup $\mathcal{A}$ in the category is isomorphic to a direct sum of subquasigroups*

$$K(k_\rho, k_\lambda, -k_\lambda, -k_\rho) \tag{4.16}$$

where k_ρ, k_λ lie in the zero-dimensional sphere $S^0 = \{\pm 1\}$.

(b) *For each natural number n, there are $\binom{n+3}{n}$ isomorphism classes of n-dimensional bisymmetric quantum quasigroups in $(\underline{\underline{K}}, \oplus, \{0\})$.*

Proof. (a) By Proposition 4.26(a), $\mathcal{A}$ has the form $A(\rho, \lambda, -\lambda, -\rho)$ with commuting elements ρ, λ satisfying $\rho^2 = \lambda^2 = 1_A$. These automorphisms are simultaneously diagonalizable, with eigenvalues restricted to ± 1. Thus, there are four possibilities for the 1-dimensional subquasigroups constituted by the common eigenspaces, each of which corresponds to an ordered pair $(k_\rho, k_\lambda) \in S^0 \times S^0$.

(b) If the vector space A has dimension n, then each isomorphism class corresponds to a choice of an n-element multiset whose elements may be any of the 4 possible types (4.16) of 1-dimensional subquasigroup. □

In the characteristic 2 case, Proposition 4.26(a) leads to the following theorem. For the recovery of group representations from linear quasigroups, compare [22].

Theorem 4.31. *Suppose that K is a field of characteristic 2. Consider the category $(\underline{\underline{K}}, \oplus, \{0\})$. Then finite-dimensional bisymmetric linear quantum quasigroups in the category correspond to the modular representations of the Klein four-group $C_2 \times C_2$.*

Example 4.32 (Compare [26, Ex. 1.4.9])**.** Suppose that K is a field of characteristic 2. On $A = K^2$, define

$$\Delta_k \colon A \to A \otimes A; \begin{bmatrix} x_1 & x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + kx_2 & x_2 & x_1 + x_2 & x_2 \end{bmatrix} \quad \text{and}$$
$$\nabla_k \colon A \otimes A \to A; \begin{bmatrix} x_1 & x_2 & y_1 & y_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + x_2 + y_1 + ky_2 & x_2 + y_2 \end{bmatrix}$$

for $k \in K$. Then for $k \neq k' \in K$, the bisymmetric quantum quasigroups (A, ∇_k, Δ_k) and $(A, \nabla_{k'}, \Delta_{k'})$ in $(\underline{\underline{K}}, \oplus, \{0\})$ are not isomorphic. Thus if K is infinite, there are infinitely many isomorphism classes of bisymmetric linear quantum quasigroups in $(\underline{\underline{K}}, \oplus, \{0\})$ of dimension 2. By contrast, Theorem 4.34 will show that there are only 2 isomorphism classes of totally symmetric linear quantum quasigroups in $(\underline{\underline{K}}, \oplus, \{0\})$ of dimension 2. In particular, while Example 4.27 served to witness the distinction between bisymmetry and quantum total symmetry in odd or zero characteristics, the current example witnesses the distinction in characteristic 2.

4.5.2. *Total quantum symmetry of vector spaces.* This paragraph presents two theorems that classify quantum totally symmetric quantum quasigroup structures in $(\underline{\underline{K}}, \oplus, \{ 0 \})$ on finite-dimensional vector spaces over a field K.

Theorem 4.33. *Let K be a field whose characteristic is not 2. Consider the category $(\underline{\underline{K}}, \oplus, \{ 0 \})$.*

(a) *Each finite-dimensional, totally symmetric quantum quasigroup $\mathcal{A}$ is isomorphic to a direct sum of subquasigroups $K(1, 1, -1, -1)$ and $K(-1, -1, 1, 1)$.*
(b) *For each natural number n, there are $n + 1$ isomorphism classes of n-dimensional quantum totally symmetric quantum quasigroups.*

Proof. By Proposition 4.26(c), $\mathcal{A}$ has the form $A(\rho, \rho, -\rho, -\rho)$ with $\rho^2 = 1_A$.

(a) The direct sum decomposition of (a) is determined by the eigenspaces of ρ for the respective eigenvalues ± 1.

(b) For each $0 \leq m \leq n$, an isomorphism class is determined uniquely by an automorphism ρ_m for which the multiplicity of the eigenvalue 1 is m. □

Theorem 4.34. *Let K be a field of characteristic 2. Consider the category $(\underline{\underline{K}}, \oplus, \{ 0 \})$. Let n be a natural number.*

(a) *Each n-dimensional quantum totally symmetric quantum quasigroup $\mathcal{A}$ is isomorphic to one of the form $(K^n, \rho_m, \rho_m, \rho_m, \rho_m)$, where*

$$\rho_m = I_{n-2m} \oplus \bigoplus_{i=1}^{m} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.17}$$

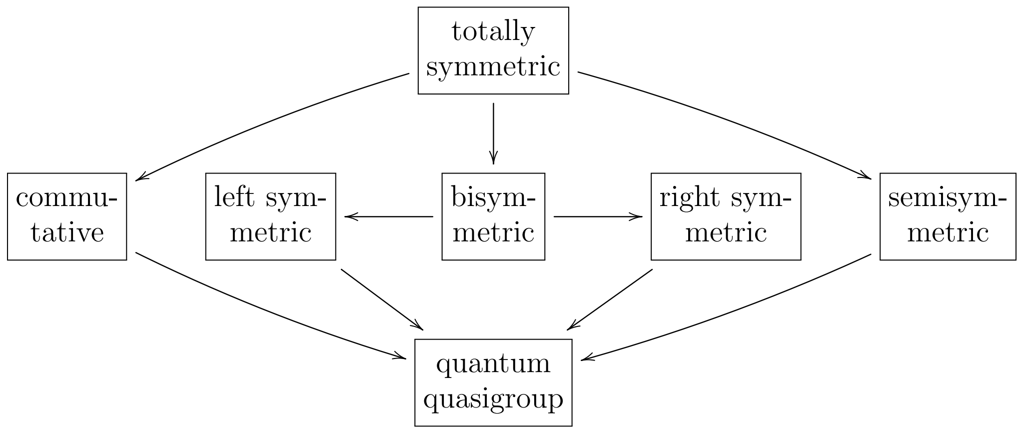
and $0 \leq 2m \leq n$.
(b) *There are $1 + \left\lfloor \frac{n}{2} \right\rfloor$ isomorphism classes of quantum totally symmetric quantum quasigroups of dimension n.*

Proof. (a) By Proposition 4.26(c), $\mathcal{A}$ has the form $K^n(\rho, \rho, \rho, \rho)$ with $\rho^2 = 1_{K^n}$. Hence, totally symmetric K-linear structures are classified by rational canonical forms (RCF) for matrices annihilated by $X^2 + 1 = (X + 1)^2$. The blocks of the RCF of ρ can either be $[1]$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) For each $0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor$, a distinct isomorphism class is determined by an automorphism ρ_m as in (4.17). □

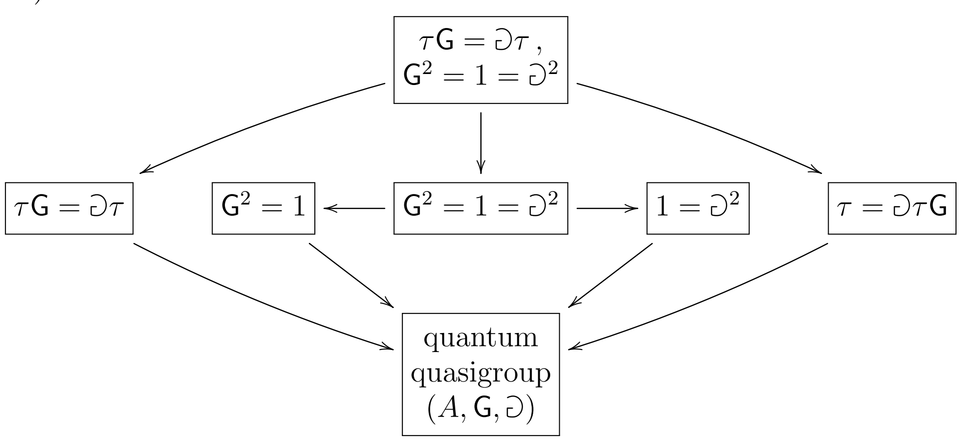
4.6. **The Sevenfold Way.** The symmetry classes of quantum quasigroups defined in this chapter arrange themselves into the following lattice under containment:

(4.18)



This arrangement, the quantum quasigroup analogue of the Sixfold Way (2.12), is described as the *Sevenfold Way*. Note that, while the lattice of the Sixfold Way is modular, the lattice of the Sevenfold Way is not. The two leftmost maximal chains, for example, form the nonmodular lattice N_5 [3]. The diagram

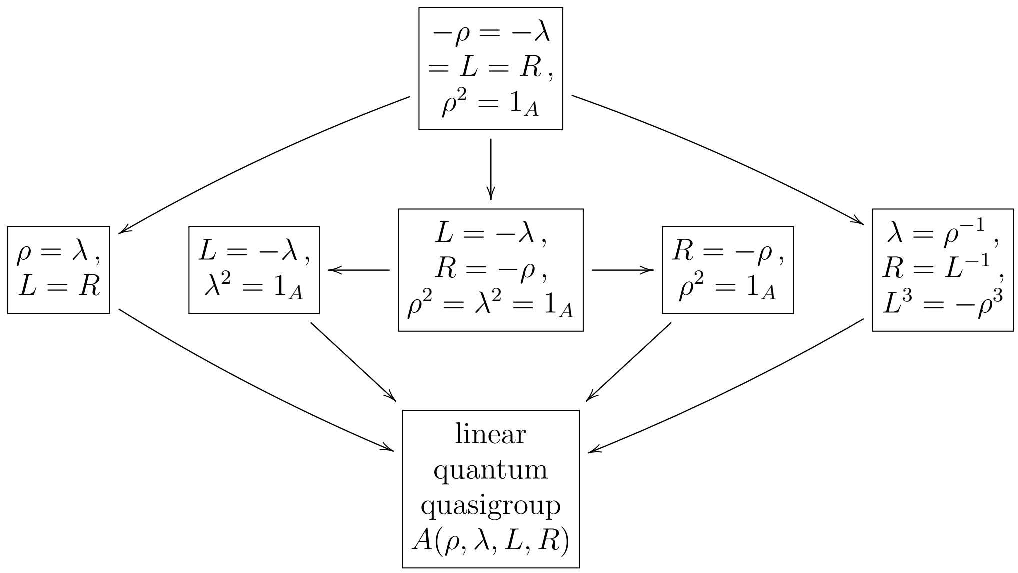
(4.19)



summarizes the respective conditions on the composites of a quantum quasigroup within the classes of the Sevenfold Way, making clear that, with the exception of the mutually dual classes of left and right quantum symmetry, all the classes are self-dual. The identity morphisms in (4.19), of course, are all on $A \otimes A$.

4.6.1. *The Sevenfold Way for linear quantum quasigroups.* In analogy with (4.19), it is also convenient to identify the respective classes of the Sevenfold Way in terms of conditions on the automorphisms ρ, λ, L, R that define a linear quantum quasigroup structure $\mathcal{A} = A(\rho, \lambda, L, R)$ within the Cartesian

category $\left(\underline{\underline{S}}, \oplus, \{ 0 \}\right)$ of modules over a commutative, unital ring S:
(4.20)

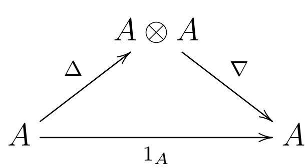


Here, we do not record the bimagma condition that $\{ \rho, \lambda \}$ should commute with $\{ L, R \}$.

5. The idempotent version of the Sevenfold Way

5.1. **Quantum idempotence.** Section 2.4 examined the version of the Sixfold Way which describes the symmetry classes of idempotent quasigroups. This chapter studies the corresponding version of the Sevenfold Way that analyses the symmetry classes of quantum quasigroups which are quantum idempotent, where the comultiplication is a section for the multiplication. The main emphasis is on linear idempotent quantum quasigroups.

Definition 5.1. [8],[9, Def'n. 6.1(a)],[24, Def'n. 5.1] Suppose that (A, ∇, Δ) is a bimagma in a symmetric, monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. If the diagram



commutes in $\mathbf{V}$, then the bimagma is said to satisfy the (self-dual) condition of (*quantum*) *idempotence*.

Lemma 5.2. [9, Lemma 6.3(c)] *Suppose that* $(\underline{\underline{S}}, \oplus, \{ 0 \})$ *is the symmetric monoidal category of modules over a commutative, unital ring* S*. Suppose* *that $\mathcal{A} = A(\rho, \lambda, R, L)$ is a bimagma in $(\underline{\underline{S}}, \oplus, \{ 0 \})$. Then the quantum idempotence property for $\mathcal{A}$ amounts to*

$$L\rho + R\lambda = 1_A \tag{5.1}$$

in the endomorphism algebra $\underline{\underline{S}}(A, A)$.

5.2. **Quantum commutative idempotent quantum quasigroups.**

Proposition 5.3. *Let K be a field. Then a linear quantum quasigroup $\mathcal{A} = (A, \rho, \lambda, L, R)$ is quantum commutative and idempotent in $(\underline{\underline{K}}, \oplus, \{ 0 \})$ if and only if 2 is invertible in K and $\mathcal{A} = A(\rho, \rho, \rho^{-1}/2, \rho^{-1}/2)$.*

Proof. According to Lemma 4.4, $\mathcal{A}$ is commutative iff $\rho = \lambda$ and $L = R$. Thus the commutative structure $A(\rho, \rho, L, L)$ is idempotent if and only if $\rho L + \rho L = 2\rho L = 1$. This is possible if and only if 2 is invertible in K and $L = (2\rho)^{-1}$. □

Remark 5.4. Suppose that a linear quantum quasigroup $\mathcal{A} = A(\rho, \lambda, L, R)$ has the diagonal comultiplication $L = R = 1_A$. In this situation, Lemma 4.4 and Proposition 5.3 show that $\mathcal{A}$ is quantum commutative (idempotent) iff its multiplication is classically commutative (idempotent).

Example 5.5. The unique commutative, idempotent classical linear quasigroup $\mathbb{Z}/_5$ of order 5 has multiplication

$$\nabla \colon \mathbb{Z}/_5 \oplus \mathbb{Z}/_5 \to \mathbb{Z}/_5; x \oplus y \mapsto 3(x + y)\,.$$

By contrast, there are 4 mutually nonisomorphic quantum commutative, idempotent linear quantum quasigroups of order 5. By Proposition 5.3, they correspond to $\rho \in \{ 1, 2, 3, 4 \}$.

5.3. **Left and right symmetric idempotent quantum quasigroups.**

Proposition 5.6. *Let $\mathcal{A} = A(\rho, \lambda, L, R)$ be a bimagma in the Cartesian category $\big(\underline{\underline{S}}, \oplus, \{ 0 \}\big)$ of modules over a commutative, unital ring S.*

(a) *Then the bimagma $\mathcal{A}$ is quantum left symmetric idempotent if and only if $\lambda^2 = L^2 = 1_A$, $L\rho + R\lambda = 1_A$ and $\rho(L + \lambda)R = 0_A$.*
(b) *Dually, the bimagma $\mathcal{A}$ is quantum right symmetric idempotent if and only if $\rho^2 = R^2 = 1_A$, $L\rho + R\lambda = 1_A$ and $\lambda(R + \rho)L$.*
(c) *If $\mathcal{A}$ is a linear quantum quasigroup, it is quantum left symmetric idempotent iff $L = -\lambda = \rho - R$ and $(\rho - R)^2 = 1_A$.*
(d) *Dually, if $\mathcal{A}$ is a linear quantum quasigroup, it is quantum right symmetric iff $R = -\rho = \lambda - L$ and $(\lambda - L)^2 = 1_A$.*

Proof. (a) simply melds Proposition 4.9(a) and Lemma 5.2.

(c) If ρ and R are invertible, $L + \lambda = 0_A$. Thus $L\rho + R\lambda = 1$ if and only if $-\lambda\rho + \lambda R = 1$ if and only if $R - \rho = \lambda$. □

5.3.1. *Fibonacci quantum quasigroups.*

Example 5.7. We construct an infinite family of idempotent left symmetric $\mathbb{Z}$-linear quantum quasigroups from the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$. In particular, we will show that for $k \in \mathbb{N}$, with $n = F_{2k+2}$ and $\rho, R \in (\mathbb{Z}/_n)^*$, the linear bimagmas

$$\mathbb{Z}/_n(\rho, F_{2k}, F_{2k+1}, R) \quad \text{and} \quad \mathbb{Z}/_n(\rho, F_{2k+1}, F_{2k}, R)$$

are left symmetric quantum quasigroups, and, so long as $\rho \pm F_{2k} \in (\mathbb{Z}/_n)^*$,

$$\mathbb{Z}/_n(\rho, F_{2k}, F_{2k+1}, \rho + F_{2k}) \quad \text{and} \quad \mathbb{Z}/_n(\rho, F_{2k+1}, F_{2k}, \rho + F_{2k+1}) \tag{5.2}$$

are left symmetric idempotent quantum quasigroups.

By the definition of $\{F_n\}_{n=0}^{\infty}$, we have $F_{2k} \equiv -F_{2k+1} \bmod F_{2k+2}$, so it now suffices to show that $F_{2k}^2 \equiv F_{2k+1}^2 \equiv 1 \bmod F_{2k+2}$. To this end, recall that $F_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$, where $\varphi = \frac{1+\sqrt{5}}{2}$, and $\psi = -\varphi^{-1}$. Then

$$\begin{aligned}
F_{2k}^2 &= \frac{\varphi^{4k} + \psi^{4k}}{5} - \frac{2}{5} \\
&= \frac{\varphi^{4k} + \psi^{4k}}{5} - \left(\frac{\varphi^4 + \psi^4}{5} - 1\right) \\
&= \frac{\varphi^{4k} - (\varphi\psi)^{2k-2}\varphi^4 - (\varphi\psi)^{2k-2}\psi^4 + \psi^{4k}}{5} + 1 \\
&= \frac{(\varphi^{2k-2} - \psi^{2k-2})(\varphi^{2k+2} - \psi^{2k+2})}{5} + 1 \\
&= F_{2k-2}F_{2k+2} + 1.
\end{aligned}$$

A similar argument shows $F_{2k+1}^2 = F_{2k}F_{2k+2} + 1$ (or see [7, p.102]).

5.3.2. *Idempotent left symmetric quantum quasigroups over vector spaces.*

Theorem 5.8. *Let K be a field which is not of characteristic 2. Then for each natural number n, each n-dimensional left symmetric idempotent quantum quasigroup in $(\underline{\underline{K}}, \oplus, \{\, 0 \,\})$ is, up to isomorphism, a direct sum*

$$K^k(\rho^+, I_k, -I_k, \rho^+ + I_k) \oplus K^l(\rho^-, -I_l, I_l, \rho^- - I_l), \tag{5.3}$$

where k and l are natural numbers with $k + l = n$, while $\rho^+ \in K_k^k$ and $\rho^- \in K_l^l$ are invertible matrices with $-1 \notin \operatorname{Spec} \rho^+$ and $1 \notin \operatorname{Spec} \rho^-$.

Proof. According to Definition 3.2 and Proposition 5.6(c), an n-dimensional quantum left symmetric idempotent quantum quasigroup in $(\underline{\underline{K}}, \oplus, \{\, 0 \,\})$ is isomorphic to a bimagma $K^n(\rho, \lambda, L, R)$ with $L = -\lambda = \rho - R$ and $(\rho - R)^2 = I_n$. Since K is not of characteristic 2, the polynomial $X^2 - 1$ splits over K as $(X-1)(X+1)$. Thus $\lambda = R - \rho$ diagonalizes with eigenvalues in $\{\, 1, -1 \,\}$, and may be chosen as $I_k \oplus -I_l$ for some k, l as described.

The bimagma condition implies

$$\rho[(I_k \oplus -I_l) + \rho] = \rho R = R\rho = [(I_k \oplus -I_l) + \rho]\rho\,. \tag{5.4}$$

Take $\rho = [\rho_{ij}]_{1\leq i,j\leq n}$. For $i \leq k$ and $j > k$, the ij-entry of (5.4) is

$$-\rho_{ij} + \sum_{l=1}^{n} \rho_{il}\rho_{lj} = \rho_{ij} + \sum_{l=1}^{n} \rho_{il}\rho_{lj}\,,$$

so $\rho_{ij} = 0$ since K is not of characteristic 2. Similarly, $\rho_{ji} = 0$. Thus $\rho = \rho^+ \oplus \rho^-$ with $\rho^+ \in K_k^k$ and $\rho^- \in K_l^l$. Then, $R = (\rho^+ + I_k) \oplus (\rho^- - I_l)$ is invertible if and only if $-1 \notin \operatorname{Spec}\rho^+$ and $1 \notin \operatorname{Spec}\rho^-$. (By convention, the spectrum of I_0 is empty.) □

Corollary 5.9. *Let K be a field which is not of characteristic 2. Then for each natural number n, each n-dimensional right symmetric idempotent quantum quasigroup in $(\underline{\underline{K}}, \oplus, \{0\})$ is, up to isomorphism, a direct sum*

$$K^k(I_k, \lambda^+, \lambda^+ + I_k, -I_k) \oplus K^l(-I_l, \lambda^-, \lambda^- - I_l, I_l), \tag{5.5}$$

where k and l are natural numbers with $k + l = n$, while $\lambda^+ \in K_k^k$ and $\lambda^- \in K_l^l$ are invertible matrices with $-1 \notin \operatorname{Spec}\lambda^+$ and $1 \notin \operatorname{Spec}\lambda^-$.

Example 5.10. For $K = \mathbb{Z}/_3$, it follows from Theorem 4.33(a) that

$$K(1, 1, -1, -1) \quad \text{and} \quad K(-1, -1, 1, 1) \tag{5.6}$$

are bisymmetric. By Lemma 5.2, they are idempotent. Then according to Theorem 5.8, we have $\rho^+ = 1 \in K_1^1$ for $K(1, 1, -1, -1)$, and $\rho^- = -1 \in K_1^1$ for $K(-1, -1, 1, 1)$. According to Corollary 5.8, we have $\lambda^+ = 1 \in K_1^1$ for $K(1, 1, -1, -1)$, and $\lambda^- = -1 \in K_1^1$ for $K(-1, -1, 1, 1)$.

It is also worth noting that the idempotent left quantum quasigroups (5.6) are exactly the Fibonacci quantum quasigroups (5.2) for $k = 1$, with $\rho = 1$ and $\rho = -1$ respectively.

5.4. **Quantum Mendelsohn systems.**

Definition 5.11. A *quantum Mendelsohn system* (QMS) is an idempotent, quantum semisymmetric quantum quasigroup within a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

Remark 5.12. Recall that the term "Mendelsohn quantum quasigroup" was used for the corresponding semi-classically defined concept of [9, §6].

Given a bimagma $A(\rho, \lambda, L, R)$ in the Cartesian category $\big(\underline{\underline{S}}, \oplus, \{0\}\big)$ of modules over a commutative, unital ring S, we set $\mu = L\rho$ throughout this section.

Proposition 5.13. *A linear quantum quasigroup* $\mathcal{A} = A(\rho, \lambda, L, R)$ *in* $\big(\underline{\underline{S}}, \oplus, \{\,0\,\}\big)$ *is a quantum Mendelsohn system iff* $\mathcal{A} = A(\rho, \rho^{-1}, L, L^{-1})$, $L^3 = -\rho^3$, *and* $\mu^2 - \mu + 1_A = 0$.

Proof. According to Theorem 4.15(b), it will be sufficent to show that a linear quantum quasigroup of the form $A(\rho, \rho^{-1}, L, L^{-1})$ with $L^3 = -\rho^3$ is quantum idempotent if and only if $\mu^2 - \mu + 1_A = 0$.

Recalling $\mu^{-1} = (L\rho)^{-1} = L^{-1}\rho^{-1}$ from the bimagma condition, we have

$$\begin{aligned} \mu^2 - \mu + 1_A = 0 \quad &\Leftrightarrow \quad \mu^2 + 1_A = \mu \\ &\Leftrightarrow \quad \mu + \mu^{-1} = 1_A \quad \Leftrightarrow \quad L\rho + L^{-1}\rho^{-1} = 1_A\,. \end{aligned}$$

The final condition coincides with the idempotence expression (5.1) in this case, since $R = L^{-1}$ and $\lambda = \rho^{-1}$. □

Corollary 5.14. *A linear quantum quasigroup* $\mathcal{A} = A(\rho, \lambda, L, R)$ *within the Cartesian categpry* $\big(\underline{\underline{S}}, \oplus, \{\,0\,\}\big)$ *is a quantum Mendelsohn system iff* $\mathcal{A} = A(\rho, \rho^{-1}, L, L^{-1})$, *where* ρ *and* L *are commuting sixth roots of unity in* $\underline{\underline{S}}(A, A)^*$ *such that* $\mu^2 - \mu + 1_A = 0$.

Proof. By Proposition 5.13, it suffices to show that, with respect to a linear quantum quasigroup of the form $A(\rho, \rho^{-1}, L, L^{-1})$ in which $\mu^2 - \mu + 1_A = 0$, we have $L^3 = -\rho^3$ if and only if the commuting automorphisms ρ, L from $\underline{\underline{S}}(A, A)^*$ are sixth roots of unity. Note that, in such a quantum quasigroup, since $\rho^3 L^3 + 1_A = \mu^3 + 1_A = (\mu + 1_A)(\mu^2 - \mu + 1_A) = 0$, we always have $\rho^3 L^3 = -1_A$. Then

$$\begin{aligned} L^3 = -\rho^3 \quad &\Leftrightarrow \quad L^3 = (\rho^3 L^3)\rho^3 = L^3\rho^6 \text{ and } \rho^3 = (\rho^3 L^3)L^3 = \rho^3 L^6 \\ &\Leftrightarrow \quad \rho^6 = L^6 = 1_A \end{aligned}$$

as required. □

Lemma 5.15. *Let* K *be a field with a primitive sixth root* ζ *of unity. Then in the category* $(\underline{\underline{K}}, \oplus, \{\,0\,\})$, *there are* 12 *distinct isomorphism classes of* 1*-dimensional quantum Mendelsohn systems. They take the form*

$$(K, \zeta^a, \zeta^{-a}, \zeta^b, \zeta^{-b}), \tag{5.7}$$

where $a + b \equiv \pm 1 \bmod 6$.

Proof. Suppose that $(K, \rho, \rho^{-1}, L, L^{-1})$ is a quantum Mendelsohn system. By Corollary 5.14, ρ and L are taken from the roots of the polynomial $X^6 - 1 = \prod_{d=0}^{5}(X - \zeta^d)$. Suppose $\rho = \zeta^a$ and $L = \zeta^b$. Then $\mu = \zeta^{a+b}$ is annihilated by $X^2 - X + 1$ if and only if $a + b \equiv \pm 1 \bmod 6$. There are twelve such pairs (a, b). Each represents a distinct isomorphism class, because $\underline{\underline{K}}(K, K)^* \cong K^*$ is commutative (cf. Definition 3.2). □

Theorem 5.16. *Let* K *be a field with a primitive sixth root of unity.*

(a) *Within the category* $(\underline{\underline{K}}, \oplus, \{ 0 \})$, *each finite dimensional quantum Mendelsohn system is isomorphic to a direct sum of* 1*-dimensional quantum quasigroups of the form* (5.7).

(b) *For each* $n \in \mathbb{N}$, *there are* $\binom{n+11}{n}$ *distinct isomorphism classes of* n*-dimensional quantum Mendelsohn systems.*

Proof. (a) To show that $K^n(\rho, \rho^{-1}, L, L^{-1})$ is isomorphic to a direct sum of 1-dimensional structures, it suffices by Definition 3.2 to show that ρ and L are simultaneously diagonalizable. The bimagma condition implies that ρ and L commute, so we just need to show that they are each diagonalizable. But this follows from the fact that their minimal polynomials divide $X^6 - 1$, which splits into distinct linear factors over K.

(b) follows from (a), as the pairs of $n \times n$ diagonal matrices meeting the conditions of Corollary 5.14 correspond to multisets of size n with elements drawn from the 12 structures of Lemma 5.15. □

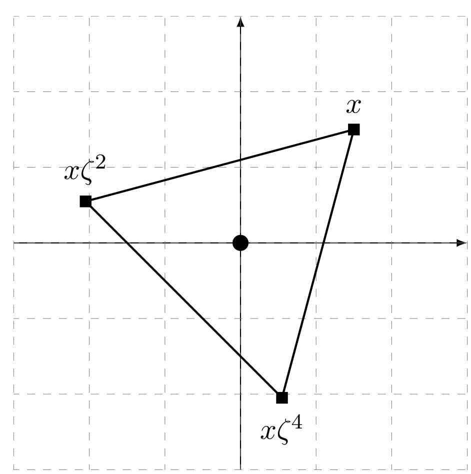


FIGURE 1. $x = \frac{3}{2} + \frac{3}{2}i$

Example 5.17. Let $\zeta = \exp(\pi i/3)$. As observed in Example 2.8(d), the idempotent operation $x * y = x\zeta + y\zeta^5$ endows $\mathbb{C}$ with the structure of an MTS. Note that $x\zeta + y\zeta^5$ is the third point of the equilateral triangle with counter-clockwise orientation that has the oriented segment $x \to y$ as one of its sides. The QMS $(\mathbb{C}, \zeta, \zeta^5, \zeta^4, \zeta^2)$ augments this MTS with a geometrically meaningful comultiplication displayed in Figure 1. Indeed, for each $x \in \mathbb{C}^*$, there is one equilateral triangle having x as one of its vertices and the origin as its barycenter; the other two vertices of this triangle are $x\zeta^4$ and $x\zeta^2$.

5.5. **Bisymmetric idempotence and quantum Steiner systems.**

Definition 5.18. A *quantum Steiner system* (QSS) is defined to be an idempotent and quantum totally symmetric quantum quasigroup within a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

Example 5.19. If $(Q, \cdot)$ is a totally symmetric, idempotent quasigroup, and thus a Steiner triple system in the sense of Definition 2.11, then it augments with the diagonal comultiplication to a QSS in the Cartesian category $(\mathbf{Set}, \times, \top)$.

Examples 4.27 and 4.32 showed that general bisymmetric linear quantum quasigroups need not be totally symmetric. The following result shows that this distinction collapses in the idempotent case.

Proposition 5.20. *Suppose that S is a commutative unital ring. Then a bisymmetric, idempotent linear quantum quasigroup within the Cartesian category $(\underline{\underline{S}}, \oplus, \{ 0 \})$ is totally symmetric.*

Proof. We show $\rho = \lambda$ in the bisymmetric structure $A(\rho, \lambda, -\lambda, -\rho)$. Now $1_A = L\rho + R\lambda = -\lambda\rho - \rho\lambda$. By the bimagma condition, $\rho\lambda = \lambda\rho$, so $1_A = -2\rho\lambda$. Thus $2_A \in \underline{\underline{S}}(A, A)^*$, and $2\lambda = -\rho$ since $\rho^2 = 1$. It follows that $4_A = (2\lambda)^2 = \rho^2 = 1_A$, so $\underline{\underline{S}}(A, A)$ has characteristic 3 and $\lambda = -\frac{1}{2}\rho = \rho$. □

Corollary 5.21. *A linear quantum quasigroup $A(\rho, \lambda, L, R)$ is a quantum Steiner system if and only if $\rho^2 = 1_A$, $\rho = \lambda = -L = -R$, and the ring $\underline{\underline{S}}(A, A)$ has characteristic* 3.

Corollary 5.22. *Suppose that K is a field of characteristic* 3. *Then each finite-dimensional, totally symmetric quantum quasigroup in $(\underline{\underline{K}}, \oplus, \{ 0 \})$ is a quantum Steiner system.*

Proof. If K has characteristic 3, Corollary 5.21 shows that $K(1, 1, -1, -1)$ and $K(-1, -1, 1, 1)$ are quantum Steiner systems. But by Theorem 4.33(a), each finite-dimensional, totally symmetric quantum quasigroup within the category $(\underline{\underline{K}}, \oplus, \{ 0 \})$ is a direct sum of such structures. □

Theorem 5.23. *Let K be a field.*

(a) *Quantum Steiner systems exist in $(\underline{\underline{K}}, \oplus, \{ 0 \})$ if and only if K has characteristic* 3.
(b) *Suppose that K has characteristic* 3. *Then each finite-dimensional quantum Steiner system in $(\underline{\underline{K}}, \oplus, \{ 0 \})$ is isomorphic to a direct sum of subquasigroups $K(1, 1, -1, -1)$ and $K(-1, -1, 1, 1)$.*
(c) *If K has characteristic* 3 *and $n \in \mathbb{N}$, then there are $n+1$ isomorphism classes of n-dimensional quantum Steiner systems in $(\underline{\underline{K}}, \oplus, \{ 0 \})$.*

Proof. (a) Restriction on the characteristic of the ground field follows from Corollary 5.21.

(b) is a special case of Theorem 4.33(a).

(c) follows from Theorem 4.33(b) and Corollary 5.22. □

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