SUPERPRODUCTS, HYPERIDENTITIES, AND ALGEBRAIC STRUCTURES OF LOGIC PROGRAMMING

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Abstract. Recent developments in logic programming are based on bilattices (algebras with two separate lattice structures). This paper provides characterizations and structural descriptions for bilattices using the algebraic concepts of superproduct and hyperidentity. The main structural description subsumes the many variants that have appeared in the literature.

1. Introduction

Bilattices, algebras with two separate lattice structures, are used for the algebraization of systems of inference in artificial intelligence and logic programming. The two lattice orderings of a bilattice are viewed as respectively representing the relative degree of truth and knowledge of possible events. Bilattices were first introduced by M. Ginsberg and M. Fitting in 1989-90 (see [2] – [4]) as a general framework for a variety of applications such as truth maintenance systems, default inference and logic programming. Their structure and applications were investigated further by these and other authors. (See e.g. [1], [7], [8], [23], [26].) Although the main common feature of these algebras is that they have two separate lattice structures defined on the same set, different authors consider different connections between the two lattice structures. Moreover, bilattices used in applications are often assumed to satisfy some additional finiteness conditions, to possess bounds for both lattices, or to incorporate some type of unary operation representing negation of truth values. Occasionally, algebras with more than two lattice structures are encountered, for example in [6] where a third lattice ordering represents a degree of precision.

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In this paper, we consider a bilattice as an algebra with just two separate lattice structures. Using the algebraic concepts of superproduct and hyperidentity, which are succinctly introduced in Sections 2 and 3, we present a general structure theorem for the bilattices of logic programming (Theorem 4.3) that subsumes the many variants that have appeared in the literature. The concluding Section 5 discusses some variants of the basic construction, considering the possibilities of adding constants, adding different types of unary "negation," or of extending the number of lattices defined on the underlying set.

2. Superproducts

A *type* is a function $\tau : \Omega \to T$ with $T \subseteq \mathbb{N}$. Elements ω of the domain Ω of τ are called *operators*. An *algebra* (A, τ) or A *of type* τ is a set A equipped with an *operation*

$$\omega : A^{\omega\tau} \to A; (a^1, \dots, a^{\omega\tau}) \mapsto a^1 \dots a^{\omega\tau}\omega$$

for each operator ω in Ω. If $n = \omega\tau$, the operator ω is described as being n*-ary.* These definitions are just a slightly more precise version of the usual definitions of universal algebra (compare [27, IV§1.1]).

An *arithmetic type* T is a subset of $\mathbb{N}$. A T*-algebra* or *algebra of arithmetic type* T is an algebra (A, τ) with $\tau(\Omega) \subseteq T$. Given two algebras $(A, \tau : \Omega \to T)$ and $(B, \sigma : \Psi \to T)$ of the same arithmetic type T, a *bihomomorphism* $f : A \to B$ consists of a pair

$$f = (f_0 : A \to B, f_1 : \Omega \to \Psi)$$

of functions with $f_1\sigma = \tau$, such that

$$\forall\, \omega \in \Omega\,, \forall\, a^1, \dots, a^{\omega\tau} \in A\,,\ (a^1 f_0) \dots (a^{\omega\tau} f_0)(\omega f_1) = (a^1 \dots a^{\omega\tau}\omega) f_0\,.$$

Alternatively, given $f_1 : \Omega \to \Psi$ with $f_1\sigma = \tau$, one may define the *pullback* of (B, σ) along f_1 as the algebra (B, τ) of type τ with

$$b^1 \dots b^{\omega\tau}\omega := b^1 \dots b^{\omega\tau}(\omega f_1)$$

for $b^i \in B$ and $\omega \in \Omega$. The pair (f_0, f_1) is then seen to be a bihomomorphism if and only if f_0 is a homomorphism from the algebra (A, τ) to the pullback (B, τ) of (B, σ) along f_1.

For a given arithmetic type T, let $\underline{\underline{T}}$ consist of two classes: the class of all T-algebras and the class of all bihomomorphisms between them. One readily checks that $\underline{\underline{T}}$ forms a large category, under the usual compositions of the components of bihomomorphisms, with these classes as the respective object class $\mathrm{Ob}(\underline{\underline{T}})$ and morphism class $\mathrm{Mor}(\underline{\underline{T}})$. For example, the identity at a T-algebra $(A, \tau : \Omega \to T)$ is the pair $1_{(A,\tau)} = (1_A, 1_\Omega)$.

Theorem 2.1. *For each arithmetical type T, the category $\underline{\underline{T}}$ possesses arbitrary products.*

Proof. Consider an indexing function $I \to \mathrm{Ob}(\underline{\underline{T}}); i \mapsto (A_i, \tau_i : \Omega_i \to T)$. Define $A = \prod_{i \in I} A_i$ as the usual product in the category of sets, with corresponding projections $\pi_0^i : A \to A_i$ for each i in I. Define a type $\tau : \Omega \to T$ by setting $\tau^{-1}\{n\} = \prod_{i \in I} \tau_i^{-1}\{n\}$ for each element n of T. For each element i of I, and for each element n of T, consider the projection $\pi_1^{i,n} : \tau^{-1}\{n\} \to \tau_i^{-1}\{n\}$. Let $\pi_1^i : \Omega \to \Omega_i$ be the coproduct $\sum_{n \in T} \pi_1^{i,n}$ in the category of sets of the functions $\pi_1^{i,n} : \tau^{-1}\{n\} \to \tau_i^{-1}\{n\}$. An algebra

$$(A, \tau) \tag{1}$$

of arithmetical type T is then specified uniquely by the requirement that for each n in T, each n-ary operator ω from Ω makes the projection

$$\pi_0^i : (A, \{\omega\} \to \{n\}) \to (A_i, \{\omega \pi_1^{i,n}\} \to \{n\})$$

a homomorphism of algebras with a single n-ary operation for each i in I. It follows that for each i in I, there is a bihomomorphism

$$\pi^i = (\pi_0^i, \pi_1^i) : (A, \tau) \to (A_i, \tau_i)\,. \tag{2}$$

Moreover, given a T-algebra $(B, \sigma : \Psi \to T)$ and bihomomorphisms $f^i = (f_0^i, f_1^i) : (B, \sigma) \to (A_i, \tau_i)$ for each i in I, a bihomomorphism

$$f = (f_0, f_1) : (B, \sigma) \to (A, \tau)$$

with $f\pi^i = f^i$ for each i in I is uniquely specified by the requirements $f_0 \pi_0^i = f_0^i$ and $f_1 \pi_1^i = f_1^i$ for each i in I. Thus the object (1), equipped with the projections (2), is the product of the (A_i, τ_i) in the category $\underline{\underline{T}}$. □

Corollary 2.2. *The T-algebra $(\{0\}, 1_T : T \to T)$ is a terminal object of $\underline{\underline{T}}$.*

Proof. Set $I = \varnothing$ in the proof of Theorem 2.1. Each $\tau^{-1}\{n\}$ for $n \in T$, as an empty product in the category of sets, may be implemented by the singleton $\{n\}$, so that τ becomes the identity function $1_T : T \to T$. □

Definition 2.3. For an arithmetical type T, the product of two objects A_1 and A_2 in the category $\underline{\underline{T}}$ is called the *superproduct* $A_1 \bowtie A_2$ of the T-algebras A_1 and A_2. More generally, products of objects A_i in $\underline{\underline{T}}$ are described as *superproducts* of the algebras A_i.

Example 2.4. Suppose that two algebras A_1 and A_2 both have binary operations ω_1 and ω_2. Then the superproduct $A_1 \bowtie A_2$ has four binary operations (ω_1, ω_1), (ω_1, ω_2), (ω_2, ω_1), (ω_2, ω_2). Writing the original operations with infix notation $xy\omega_i = x \times_i y$, one has

$$(x_1, x_2)(y_1, y_2)(\omega_i, \omega_j) = (x_1 \times_i y_1, x_2 \times_j y_2)$$

in the superproduct $A_1 \bowtie A_2$ for $i, j \in \{1, 2\}$.

Example 2.5. Suppose that A_i for $i \in I$ are algebras of the same type, each possessing just one n-ary operation for some n in T. Then the superproduct of the A_i reduces to their usual product.

We conclude this section with some observations that may be skipped by readers who are unfamiliar with the concepts of category theory (as detailed in [27], for example).

Theorem 2.6. *There is a forgetful functor G from the category $\underline{\underline{T}}$ to the product $\underline{\underline{S}}$ of the category $\underline{\underline{\mathrm{Set}}}$ of sets with the slice category $\underline{\underline{\mathrm{Set}}}/T$. This forgetful functor has a left adjoint F sending an $\underline{\underline{S}}$-object $(X, \tau : \Omega \to T)$ to the free τ-algebra over the set X.*

Proof. For a $\underline{\underline{T}}$-algebra (A, σ) and an $\underline{\underline{S}}$-morphism

$$(f_0, f_1) : (X, \tau) \to (A, \sigma)G\,,$$

pull the algebra (A, σ) back along f_1 to a τ-algebra (A, τ). Let $\overline{f_0}$ be the unique extension of $f_0 : X \to A$ to a τ-algebra homomorphism from the free τ-algebra over the set X to the τ-algebra (A, τ). Then the unique extension of (f_0, f_1) to a bihomomorphism is the pair $(\overline{f_0}, f_1)$. □

Remark 2.7. The proof of Theorem 2.1 just notes that the forgetful functor G of Theorem 2.6 creates products. Thus the underlying set of a superproduct of algebras is the product (in $\underline{\underline{\mathrm{Set}}}$) of their underlying sets, while the type of the superproduct is given by the product in the slice category $\underline{\underline{\mathrm{Set}}}/T$ of the respective types of the factor algebras.

3. Hyperidentities

Let T be an arithmetic type. Define

$$P = \{x^1, x^2, \dots\}$$

and

$$P_t = \{X_1^t, X_2^t \dots\}$$

for each t in T. Then set $P_T = \bigcup_{t \in T} P_t$, and define the projection

$$\pi : P_T \to T; X_i^t \mapsto t\,.$$

Inside the free monoid $(P \cup P_T)^*$ (the set of all words in the alphabet $P \cup P_T$ under the binary operation of concatenation, with the empty word as identity), define a subset PT inductively by the rules:

(1) $P \subseteq PT$;
(2) $\forall\, t \in T\,,\ \forall\, X_i^t \in P_t\,,\ p^1, \dots, p^t \in PT \Rightarrow p^1 \dots p^t X_i^t \in PT$.

An algebra (PT, π) of arithmetic type T is then obtained by defining the operation

$$X_i^t : PT^t \to PT; (p^1, \dots, p^t) \mapsto p^1 \dots p^t X_i^t$$

for each operator X_i^t in P_T. A *hyperidentity* of arithmetic type T is defined to be a pair (p,q) (more informally written $p=q$) of elements of PT. An algebra (A,τ) of arithmetic type T is said to *satisfy* the hyperidentity $p=q$ if $pf_0 = qf_0$ in A for each bihomomorphism (f_0, f_1) from (PT,π) to (A,τ) ([9], [10], [14], [15], [28]). Less formally, lower case letters are used for the elements of P, and upper-case letters for the operators from P_T. Binary operations are usually written with infix notation, while unary operations are written as superscripts. The key example for this paper is given by the following [12] [18].

Theorem 3.1. *Each lattice satisfies the hyperidentities:*

(i) $x\,X\,x = x$;
(ii) $x\,X\,y = y\,X\,x$;
(iii) $x\,X\,(y\,X\,z) = (x\,X\,y)\,X\,z$;
(iv) $((x\,X\,y)\,Y\,z)\,X\,(y\,Y\,z) = (x\,X\,y)\,Y\,z$.

Conversely, each hyperidentity satisfied by any lattice is a consequence of the hyperidentities (i) – (iv).

Remark 3.2. (For category theorists.) In the language of Theorem 2.6, the algebra (PT,π) is the image of the pair (P,π) under the left adjoint functor $F : \underline{\underline{S}} \to \underline{\underline{T}}$. In turn, one may interpret the pair (P,π) as the natural number object of the topos $\underline{\underline{S}}$.

4. Bilattices

An algebra $(B,\wedge,\vee,\cdot,+)$ with four binary (infix) operations is called a *bilattice* if both the reducts $B_1 = (B,\wedge,\vee)$ and $B_2 = (B,\cdot,+)$ are lattices. A bilattice is *bounded* if both its reducts B_1 and B_2 are bounded lattices. A bilattice is called *interlaced* (see [2]) if each of the four basic operations preserves both the ordering relations: $\leq_1$ in the lattice B_1 and $\leq_2$ in the lattice B_2. Each lattice $(L,\wedge,\vee)$ determines two interlaced bilattices:

$$L^+ = (L,\wedge,\vee,\wedge,\vee)$$

and

$$L^- = (L,\vee,\wedge,\wedge,\vee).$$

Note that the condition of being interlaced is equivalent to the satisfaction of the following identities for each pair of operations $\times_1, \times_2$ in the set $\{\wedge,\vee,\cdot,+\}$:

$$((x \times_1 y) \times_2 z) \times_1 (y \times_2 z) = (x \times_1 y) \times_2 z. \tag{3}$$

These identities are always satisfied in each lattice, and in each *stammered semilattice* (a set with two equal semilattice operations). They were first noted by Padmanabhan ([17]–[20]), who showed that they define the regularization of the variety of lattices, the variety of so-called *quasi-lattices* or *Padmanabhan bisemilattices*. (See [22], [24] or [25, Ch. 4] for the definition and basic properties of the regularization of a variety.) In [26], bilattices satisfying the above identities were called *Padmanabhan bilattices* or briefly *P-bilattices*. In particular, the class of interlaced bilattices is a variety. (In fact, as was shown in [26]. not all these identities are necessary to define the variety of interlaced bilattices.) It is easy to see that the identities (3) may be replaced by the single hyperidentity (iv) of Theorem 3.1, so the variety $\mathcal{IBL}$ of interlaced bilattices may be defined as the class of bilattices satisfying this hyperidentity. Indeed, we may summarize as follows.

Theorem 4.1. *A bilattice is interlaced iff it satisfies the hyperidentities satisfied in the variety of lattices.*

Theorem 4.1 extends to some subvarieties of the variety $\mathcal{IBL}$. First recall that a bilattice is *distributive* (and hence is interlaced) if it satisfies both distributive identities for each pair of basic operations. It is *modular* if it satisfies the hyperidentity

$$(x\,Y\,(y\,X\,z))\,X\,(y\,Y\,z) = (x\,X\,(y\,Y\,z))\,Y\,(y\,X\,z)\,.$$

Proposition 4.2. *Let B be an interlaced bilattice.*

(a) *B is distributive iff it satisfies the hyperidentities satisfied in the variety of distributive lattices.*
(b) *B is modular iff it satisfies the hyperidentities satisfied in the variety of modular lattices.*

Proof. This follows directly by results of [11] and [13], where it was shown that the hyperidentities satisfied in the variety of distributive lattices are consequences of the hyperidentities of Theorem 3.1 and

$$x\,X\,(y\,Y\,z) = (x\,X\,y)\,Y\,(x\,X\,z)\,,$$

while the hyperidentities satisfied in the variety of modular lattices are consequences of the hyperidentities of Theorem 3.1 and

$$(x\,Y\,(y\,X\,z))\,X\,(y\,Y\,z) = (x\,X\,(y\,Y\,z))\,Y\,(y\,X\,z)\,.$$

□

The paper [26] contains a representation theorem for bounded interlaced bilattices with the unary negation introduced by Ginsberg. This theorem was first discovered in the case of distributive bounded bilattices by Ginsberg and Fitting. For the case of bounded interlaced bilattices, it was then rediscovered in different forms in [1], [8], and [23]. We now provide a general version of the representation theorem considering arbitrary interlaced bilattices. The new theorem subsumes all the previous forms, and offers the additional advantage of providing an equational basis for the variety $\mathcal{IBL}$.

Let $(L_1, \wedge_1, \vee_1)$ and $(L_2, \wedge_2, \vee_2)$ be lattices. Let B be the superproduct of the lattices L_1 and L_2. Specifically, consider $B = (L_1 \times L_2, \wedge, \vee, \cdot, +)$ with basic operations defined by

$$\wedge = (\wedge_1, \vee_2),\ \vee = (\vee_1, \wedge_2),\ \cdot = (\wedge_1, \wedge_2),\ + = (\vee_1, \vee_2).$$

The reducts B_1 and B_2 of B are lattices. In fact $B_1 \cong L_1 \times L_2^d$, where L_2^d is the dual of L_2 and $B_2 \cong L_1 \times L_2$. Moreover, as in [26, Prop. 2.2], each reduct of B with one basic operation from B_1 and one basic operation from B_2 belongs to the regularization $\bar{\mathcal{L}}$ of the variety $\mathcal{L}$ of lattices. Thus it is a quasi-lattice, satisfying all the Padmanabhan identities. It follows that B is an interlaced bilattice.

We will show that each interlaced bilattice can be obtained in this way. The proof differs from the proofs of the theorems mentioned above, since we do not assume that the bilattice is bounded. However, the idea of the proof is very close to the idea of the proof in [26], so we will omit many of the details that are similar to that proof.

Theorem 4.3. *An algebra $(B, \wedge, \vee, \cdot, +)$ is an interlaced bilattice iff it is isomorphic to the superproduct $L_1 \bowtie L_2$ of two lattices L_1 and L_2.*

Proof. We already know that a superproduct of two lattices is an interlaced bilattice. We will sketch the proof of the converse implication. Assume that the bilattice B is interlaced. Then each of the quasi-lattice reducts $(B, \times, \circ)$ with $\times \in \{\wedge, \vee\}$ and $\circ \in \{\cdot, +\}$ belongs to $\bar{\mathcal{L}}$, and hence is a Płonka sum of lattices. In fact we have four decompositions of these reducts of B into Płonka sums of lattices. Denote the Płonka sum of lattices $(L_i, \times, \circ)$ over a stammered semilattice $(I, \times = \circ)$ by the symbol $\sum_{i \in I}(L_i, \times, \circ)$. Then we obtain the following Płonka sums:

$$\sum_{i \in I}(B_i, \wedge, +) \text{ over } (I, \wedge = +), \tag{4}$$

$$\sum_{i \in I'}(B'_i, \cdot, \vee) \text{ over } (I', \cdot = \vee), \tag{5}$$

$$\sum_{j \in J}(A_j, \wedge, \cdot) \text{ over } (J, \wedge = \cdot), \tag{6}$$

$$\sum_{j \in J'}(A'_j, +, \vee) \text{ over } (J', + = \vee). \tag{7}$$

Each of these decompositions determines a congruence relation of the corresponding quasi-lattice with the lattice summands as congruence classes. Call these congruences σ_1, σ_2, σ_3 and σ_4 respectively. From the proof of [20, Th. 2.1] it follows that the congruences σ_i are in fact the bilattice congruences, so that the lattice summands are subbilattices of B. In particular, each is a sublattice of B_1 and a sublattice of B_2. However, as observed in [5], if a *quadrisemilattice* (an algebra with four semilattice operations) satisfying the Padmanabhan identities has two lattice (basic) reducts, then the sets of basic operations of these two lattices must either be disjoint or coincide. It follows for example that in each lattice B_i we have $\wedge = \cdot$ and $\vee = +$. The same holds for each lattice B'_i. As B_i and B'_i are maximal subalgebras of B that are sublattices of B_1 and of B_2, it follows that the sets I and I' coincide, forming dual semilattices, and for each $i \in I$, we also have $B_i = B'_i$. Moreover, the lattices $(B_i, \wedge, +)$ and $(B'_i \cdot, \vee)$ are isomorphic to $(B_i, \wedge, \vee) = (B_i, \cdot, +)$. Similarly, one may show that $A_j = A'_j$, that $J = J'$, that in A_j, one has $\wedge = +$ and $\vee = \cdot$, and that the lattices $(A_j, \wedge, \cdot)$ and $(A'_j, +, \vee)$ are isomorphic to $(A_j, +, \cdot)$, the latter being isomorphic to $(A_j, \wedge, \vee)$.

The final step of the proof is to show that for each $i \in I$ and $j \in J$, the intersection $A_i \cap B_j$ has precisely one element. This is routine. It follows that the congruences $\sigma_1 = \sigma_2$ and $\sigma_3 = \sigma_4$ form a pair of factor congruences of the bilattice B, and hence the bilattice B is the superproduct of the lattices I and J. □

5. Extending the type

Consider bounded lattices as algebras with two binary and two nullary operations. If L_1 is a lattice bounded by 0_1 and 1_1, while L_2 is a lattice bounded by 0_2 and 1_2, then their superproduct $L_1 \bowtie L_2$ has four nullary operations

$$\bot_1 = (0_1, 1_2)\,, \top_1 = (1_1, 0_2)\,, \bot_2 = (0_1, 0_2)\,, \top_2 = (1_1, 1_2)\,.$$

Note however that the identities defining the bounds of a lattice are no longer hyperidentities. This raises the following questions.

Problem 5.1. Is the class of bounded bilattices defined by hyperidentities? Is the class of bilattices defined by all the hyperidentities satisfied in the class of bounded bilattices?

The proof of Theorem 4.3 may easily be adjusted to show that it also holds in the case of bounded lattices. However all the previous versions of this theorem gave more direct proofs (with or without use of the structure theorem for quasi-lattices), making essential use of the existence of the nullary operations. In fact, in the bounded case, the components of the superproduct obtained in the decomposition of a bounded bilattice may be defined more directly as being isomorphic to the intervals $[0_2, 1_1]$ of L_1 (isomorphic to $[0_2, 1_1]$ of L_2) and $[0_2, 0_1]$ of L_2 (dually isomorphic to $[0_1, 0_2]$ of L_1).

The Ginsberg negation may easily be defined in the superpower $L \bowtie L$, where L is a bounded lattice, by

$$(x, y)' = (y, x).$$

However, the bounded bilattice with negation obtained in this fashion cannot be described as a superproduct of two bounded lattices with unary operations. Nevertheless, Theorem 4.3 also holds in this case, and in fact the first version proved for interlaced bilattices in [26] dealt with exactly this situation.

One may define other types of negation in a bilattice that better fit to the approach considered in this paper. However, their interpretation in various applications is not yet clear. Let us consider one example. Define a bilattice B with one unary operation $'$ to be *Boolean* if it is non-empty, distributive, and satisfies the hyperidentities

$$x^{CC} = x$$

and

$$x\, X\, (y\, X\, y^C)^C = x\,.$$

Instead of the latter hyperidentity we can consider four corresponding identities:

$$x \wedge_1 (y \wedge_1 y')' = x\,,$$
$$x \vee_1 (y \vee_1 y')' = x\,,$$
$$x \wedge_2 (y \wedge_2 y')' = x\,,$$
$$x \vee_2 (y \vee_2 y')' = x\,.$$

These identities show that B is in fact bounded, with bounds defined by

$$\top_1 = (y \wedge_1 y')',\ \top_2 = (y \wedge_2 y')',\ \bot_1 = (y \vee_1 y')',\ \bot_2 = (y \vee_2 y')'.$$

This, together with the first (hyper)identity, shows that

$$\top_1 = y \vee_1 y',\ \ \top_2 = y \vee_2 y',\ \ \bot_1 = y \wedge_1 y',\ \ \bot_2 = y \wedge_2 y'.$$

Hence the operation $'$ is a complement in both (bounded distributive) lattices B_1 and B_2, which implies that both these lattices are in fact Boolean algebras. It is an open question which other types of unary operations considered in distributive lattices would also give bilattices with unary operations defined by hyperidentities. Boolean bilattices can be characterized in similar fashion to bilattices. In particular, we have the following (compare [16]).

Proposition 5.2. *A non-empty bilattice is Boolean iff it satisfies the hyperidentities satisfied in the variety of Boolean algebras.*

Theorem 5.3. *An algebra* $(B, \wedge, \vee, \cdot, +, ')$ *is a Boolean bilattice if and only if it is isomorphic to the superproduct* $L_1 \bowtie L_2$ *of two Boolean algebras* L_1 *and* L_2*.*

We omit the proofs, since they are similar to the proofs of Proposition 4.2 and Theorem 4.3. We only note that the proof of Theorem 5.3 requires a definition of the operation $'$ different from that implicit in the definition of a Płonka sum (see [13]).

The concept of an interlaced bilattice may easily be extended to the case of (bounded or unbounded) n-lattices, having n different lattice structures, and satisfying all possible Padmanabhan identities. Note that such algebras form a subvariety of the variety of $2n$-semilattices as defined in [5] ($2n$-quasilattices in the terminology of [20] and [21].) An analysis of the proof of Theorem 4.3 shows that the number of lattices involved is not essential for the proof, giving the possibility of proving a similar representation theorem for n-lattices. Note also that *trilattices*, algebras with three separate lattice structures, were already used to model systems with three lattice orderings in [6].

We conclude with the following.

Problem 5.4. Which varieties of algebras admit a representation similar to the representation for bilattices given by Theorem 4.3?

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