On relation graphs of 3-class nonsymmetric association schemes

by Sung-Yell Song (Apr. 28, 2014)

Abstract: A 3-class nonsymmetric association scheme of order v consists of three nontrivial $v \times v$ $\{0,1\}$-matrices, A, A^{t}, B where A^{t} is the transpose of A such that:

(1) $A + A^{\mathrm{t}} + B = J - I$, where I and J are the identity and all-ones matrices, respectively;

(2) there exist integers p_{ij}^{h} for every triple $h, i, j \in \{0,1,2,3\}$ such that with labeling $A_0 = I, A_1 = A, A_2 = A^{\mathrm{t}}, A_3 = B$,

$$A_iA_j = \sum_{h=0}^{3} p_{ij}^{h} A_h = A_jA_i.$$

That is, the set $\{I, A, A^{\mathrm{t}}, B\}$ forms a subalgebra of the full matrix algebra $M_v(\mathbb{C})$ over the complex.

In other words, the matrix A is the adjacency matrix of one of the two nonsymmetric relations, and A^{t} is the adjacency matrix of the directed graph associated with the other nonsymmetric relation. The undirected graph associated with the symmetric relation has adjacency matrix $B = J - I - A - A^{\mathrm{t}}$, and is a 'strongly regular graph'. Being the relation graphs of an association scheme, it is required that these graphs are regular, in particular, that $AJ = JA = kJ$ for some number k, and that

(a) $AA^{\mathrm{t}} = A^{\mathrm{t}}A$ is a linear combination of I, A, A^{t} and B, and

(b) A^2 is a linear combination of I, A, A^{t} and B.

There are many combinatorial structures that possess either of these characteristics. In this talk, we discuss the construction of 'normally regular digraphs', (a directed version of strongly regular graphs,) i.e., regular directed graphs satisfying condition (a). We show that many normally regular digraphs arise from Cayley graphs, group rings and difference sets. We investigate the conditions on the connection set $S \subset G$ under which the Cayley graph $\mathrm{cay}(G,S)$ becomes a normally regular digraph. We then reformulate the condition in terms of group rings. We also show that if the Cayley graph of (G,D) with $D \subset G \setminus \{1\}$ is isomorphic to a certain normally regular digraph, then D gives rise to a certain 'relative difference set'. If time permits, we will discuss some open problems in this subject. (This talk is based on joint works with G. Jones, L. Jørgensen, M. Klin and K. Nowak.)