Simple Ternary Grassmann Algebras

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Abstract. The paper introduces a new class of simple comtrans algebras obtained in the tangent space of a Grassmann manifold. It is shown that no simple algebra of this new Grassmann type appears as a simple algebra of any of the previously known types.

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1 Introduction

A *comtrans algebra* E over a unital commutative ring R is an R-module E equipped with two trilinear ternary operations, a *commutator* $[x, y, z]$ and a *translator* $\langle x, y, z \rangle$, such that the commutator satisfies the *left alternative identity*

$$[x, x, y] = 0, \qquad (1.1)$$

the translator satisfies the *Jacobi identity*

$$\langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle = 0, \qquad (1.2)$$

and together the commutator and translator satisfy the *comtrans identity*

$$[x, y, x] = \langle x, y, x \rangle. \qquad (1.3)$$

Comtrans algebras were originally introduced [12] in answer to a problem from differential geometry, asking for the algebraic structure in the tangent

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bundle corresponding to the coordinate n-ary loop of an $(n+1)$-web (cf. [2]). In this context, the role played by comtrans algebras is analogous to the role played by the Lie algebra of a Lie group. The standard Lie algebra multiplication is the binary commutator $[x,y] = xy - yx$ of a bilinear and associative operation $(x,y) \mapsto xy$. Similarly, the standard ternary comtrans algebra operations are the ternary commutator

$$[x,y,z] = xyz - yxz \tag{1.4}$$

and translator

$$\langle x,y,z \rangle = xyz - yzx \tag{1.5}$$

of a trilinear operation

$$(x,y,z) \mapsto xyz. \tag{1.6}$$

Indeed, over a ring R in which 6 is a unit, any comtrans algebra arises from the commutator (1.4) and translator (1.5) of a suitably defined trilinear operation (1.6) (see [12]).

The present paper forms part of a research programme devoted to the study of comtrans algebras from a purely algebraic point of view, concerned especially with the discovery and classification of simple comtrans algebras [1, 3, 6–11]. Currently, there are three known general types of simple comtrans algebra:

(1) rectangular;
(2) Lie;
(3) Hermitian.

An algebra $CT(A,B)$ of the first type is defined by two square matrices A and B (not necessarily of the same size) over a field R. It is obtained by (1.4) and (1.5) from the trilinear operation

$$(X,Y,Z) \mapsto XAY^T BZ, \tag{1.7}$$

where X,Y,Z are rectangular matrices sized so that the matrix product in (1.7) is defined. The exact criteria for the simplicity of $CT(A,B)$ are given in [9]. A simple comtrans algebra $CT(L)$ of the second (Lie) type is obtained by setting

$$[x,y,z] = \langle x,y,z \rangle = [[x,y],z] \tag{1.8}$$

in a Lie algebra L. Then $CT(L)$ is simple if and only if L is simple (Theorem 3.2 of [9]). Simple comtrans algebras of the third type are obtained from spaces of Hermitian and generalized Hermitian operators [1, 10, 11].

In this paper, a new type of real simple comtrans algebras is introduced. The algebras are said to be of *Grassmann* type. For each positive integer n, let $O(n)$ be the group of orthogonal $n \times n$ matrices. Let $\mathfrak{o}(n,\mathbb{R})$ be the corresponding Lie algebra, and let O^n be the corresponding comtrans algebra $CT(\mathfrak{o}(n,\mathbb{R}))$. The underlying set of O^n may be taken to be the set of skew-symmetric real $n \times n$ matrices. Let p and q be integers larger than 1. Then the *Grassmann comtrans algebra* $G^{p,q}$ is defined to be the

subalgebra of O^{p+q} consisting of those matrices X whose entries X_{ij} are zero if i and j are either both not greater than p, or else both greater than p. From a geometrical point of view, the Grassmann comtrans algebra $G^{p,q}$ appears in the tangent space to the symmetric space $O(p+q)/O(p) \times O(q)$, the Grassmann manifold of p-dimensional hyperplanes in $\mathbb{R}^{p+q}$ (cf. Example XI.10.3 of [5]).

The plan of the paper is as follows. Section 2 summarizes some of the prerequisite details of the general algebraic theory of comtrans algebras, particularly concerning the universal enveloping algebra of a comtrans algebra. In Section 3, it is shown that the Grassmann comtrans algebras $G^{p,q}$ are simple if p and q are not both equal to 2. The remainder of the paper is devoted to the problem of showing that Grassmann comtrans algebras are not isomorphic to other types of simple comtrans algebras. Now it is certainly clear that the Grassmann algebras are not of Hermitian type. Indeed, the Grassmann algebras are *monic* in the sense that their commutators and translators agree according to (1.8), while the algebras of Hermitian type are not monic. In Section 4, it is shown that no simple Grassmann comtrans algebra is of rectangular type. In Section 5, it is shown that no simple Grassmann comtrans algebra $G^{p,q}$ is of Lie type. The techniques used to prove the non-isomorphisms may be of independent interest as adumbrations of a Cartan-type structure theory for comtrans algebras.

For concepts and conventions of algebra that are not otherwise explained here, readers are referred to [13].

2 Action of the Enveloping Algebra

The class $\mathfrak{CT}_R$ of all comtrans algebras over a fixed ring R forms a variety in the sense of universal algebra. This variety becomes (the class of objects of) a bicomplete category whose morphisms are the homomorphisms between comtrans algebras (cf. Theorems IV.2.1.3 and 2.2.3 of [13]). For a member E of $\mathfrak{CT}_R$, let $E[X]$ denote the coproduct of E in $\mathfrak{CT}_R$ with the free $\mathfrak{CT}_R$-algebra on a singleton $\{X\}$. For x, y in E, there are R-module homomorphisms

$$K(x,y) : E[X] \to E[X]; \quad z \mapsto [z,x,y], \tag{2.1}$$

$$R(x,y) : E[X] \to E[X]; \quad z \mapsto \langle z,x,y \rangle, \tag{2.2}$$

and

$$L(x,y) : E[X] \to E[X]; \quad z \mapsto \langle y,x,z \rangle. \tag{2.3}$$

The *universal enveloping algebra* $U(E)$ of E is the R-subalgebra of the endomorphism ring of the R-module $E[X]$ generated by

$$\{K(x,y), R(x,y), L(x,y) \mid x, y \in E\}$$

(see [7]). Note that the maps

$$(x,y) \mapsto K(x,y), \quad (x,y) \mapsto R(x,y), \quad (x,y) \mapsto L(x,y) \tag{2.4}$$

from $E \times E$ to $U(E)$ are bilinear. For a comtrans algebra E, the restrictions to E of the maps (2.4) play a role analogous to that of the adjoint mappings

$$\text{Ad}(x) : L \to L; \quad z \mapsto [z, x] \tag{2.5}$$

determined by elements x of a Lie algebra L. In particular, (1.8) shows that

$$K(x, x) = \text{Ad}(x)^2 \tag{2.6}$$

in $CT(L)$.

An R-submodule of a comtrans algebra E is an *ideal* of E if and only if it is a $U(E)$-submodule. The ideals of the comtrans algebra E are the kernels of the comtrans algebra homomorphisms with domain E (Proposition 3.1 of [9]). Thus E is simple if and only if it is irreducible as a $U(E)$-module. A comtrans algebra E is said to be *abelian* if and only if E is a trivial $U(E)$-module, i.e. if and only if its commutators and translators are all zero.

The following definition is modelled on the concept of the characteristic polynomial of a Lie algebra (cf. §III.1 of [4]).

Definition 2.1. Let R be a field, and let E be a monic comtrans algebra of finite dimension n over R. Let $\{X_1, \dots, X_n\}$ be a set of n indeterminates over R, and let $\{e_1, \dots, e_n\}$ be a basis for E over R. Then the *characteristic polynomial* of the comtrans algebra E with respect to the basis $\{e_1, \dots, e_n\}$ is the characteristic polynomial of the endomorphism

$$K(X_1 e_1 + \cdots + X_n e_n, X_1 e_1 + \cdots + X_n e_n) \tag{2.7}$$

of the extension of the comtrans algebra E to the field of rational functions over the set $\{X_1, \dots, X_n\}$ of indeterminates.

For finite-dimensional monic comtrans algebras over a field, the dependence of the characteristic polynomial on the choice of basis is analogous to the dependence exhibited by characteristic polynomials of Lie algebras (cf. §III.1 of [4]). In particular, the multiplicities of the factors of the characteristic polynomial are independent of the choice of basis, and thus provide invariants of the algebra.

3 Simplicity

For the purposes of this and subsequent sections, it is convenient to use the notation of [3]. The main result of that paper (Theorem 7.1) decomposes each orthogonal comtrans algebra O^{n+1} as a cascading sum of subalgebras $E^n, E^{n-1}, \dots, E^1$. In accordance with the decomposition, define e_j^i to be the skew-symmetric difference $E^{(i+1),j} - E^{j,(i+1)}$ of elementary matrices. In this notation, the "Euclidean space" E^s is spanned by $\{e_j^s \mid 1 \le j \le s\}$. The algebra $G^{p,q}$ is then spanned by the basis

$$A = \{e_j^{p+t} \mid 1 \le j \le p, \ 0 \le t < q\}. \tag{3.1}$$

The following proposition is readily verified by direct computation with the matrices involved.

Proposition 3.1. *Inside $G^{p,q}$, the following actions take place:*
(1) $K(e_i^s, e_i^s)$ *negates* e_j^s *for* $j \neq i$ *and* e_i^t *for* $t \neq s$,
(2) $K(e_i^s, e_j^t)$ *for* $s \neq t, i \neq j$ *switches* e_i^t *and* e_j^s,
while the other basis elements of $G^{p,q}$ are annihilated by these maps.

Lemma 3.2. *Let p, q be a pair of integers bigger than 1. Then for each $1 \leq i \leq p$ and $0 \leq s < q$, the ideal J of $G^{p,q}$ generated by e_i^{p+s} is improper.*

Proof. For $1 \leq j \leq p$ with $j \neq i$, the left alternativity (1.1) and Proposition 3.1 yield

$$e_i^{p+s} K(e_j^{p+s}, e_i^{p+s}) = -e_j^{p+s} K(e_i^{p+s}, e_i^{p+s}) = e_j^{p+s}$$

so that J contains the whole p-dimensional Euclidean space spanned by $\{e_j^{p+s} \mid 1 \leq j \leq p\}$. Similarly, for $0 \leq t < q$ with $t \neq s$, one has

$$e_i^{p+s} K(e_i^{p+t}, e_i^{p+s}) = -e_i^{p+t} K(e_i^{p+s}, e_i^{p+s}) = e_i^{p+t}.$$

Applying the above argument with e_i^{p+s} replaced by the various e_j^{p+s} for $j \neq i$, it becomes clear that J contains the basis (3.1) of $G^{p,q}$.   □

Theorem 3.3. *For each pair of integers $p, q > 1$, not both equal to 2, the Grassmann comtrans algebra $G^{p,q}$ is simple.*

Proof. Let x be a non-zero element of a non-trivial ideal J of $G^{p,q}$. It will be shown that J contains a non-zero multiple of an element of the set A of (3.1). The result then follows by application of Lemma 3.2.

Suppose that $p \geq q$ (the case $q > p$ is similar). Consider the expression of x as a linear combination $\sum_{s=0}^{q-1} \sum_{i=1}^{p} \mu_i^{p+s} e_i^{p+s}$ of elements of the basis A of (3.1). Suppose that a particular coefficient μ_j^{p+t} is non-zero, that $\{t, u\}$ is a 2-element subset of $\{0, \ldots, q-1\}$, and that $\{j, k, l\}$ is a 3-element subset of $\{1, \ldots, p\}$. Using Proposition 3.1,

$$xK(e_j^{p+u}, e_k^{p+t}) = \mu_j^{p+t} e_k^{p+u} + \mu_k^{p+u} e_j^{p+t} = y,$$

where y is an element of J. Again by Proposition 3.1,

$$yK(e_l^{p+u}, e_k^{p+t}) = \mu_j^{p+t} e_l^{p+t} \in J,$$

as required.   □

Example 3.4. The case $p = q = 2$ has to be excluded from Theorem 3.3 since $e_1^2 - e_2^3$ and $e_2^2 + e_1^3$ span a proper, non-trivial ideal H of $G^{2,2}$. Similarly, $e_2^2 - e_1^3$ and $e_1^2 + e_2^3$ span a proper, non-trivial ideal J of $G^{2,2}$. In Shen's classification [6] of 2-dimensional algebras over an algebraically closed field of characteristic 0, the complexifications of both H and J are of type $E(I, I)$. The algebra $G^{2,2}$ decomposes as the direct sum $H \oplus J$.

4 Grassmann Algebras and Rectangular Matrices

In this section, it will be shown that no simple Grassmann comtrans algebra $G^{p,q}$ appears as a rectangular algebra.

Definition 4.1. A subalgebra H of a finite-dimensional monic comtrans algebra E over a field R of characteristic zero is said to be *semisimplicial* if for each pair of (not necessarily distinct) elements x and y in H, the endomorphism $K(x, y)$ of E is semisimple.

Proposition 4.2. *For each pair of integers $p, q > 1$, the subspace H of the simple Grassmann comtrans algebra $G^{p,q}$ spanned by $\{e_1^p, e_2^{p+1}\}$ is an abelian and semisimplicial subalgebra.*

Proof. Consider the basis

$$B = \{e_1^p, e_2^{p+1}, e_2^p - e_1^{p+1}, e_2^p + e_1^{p+1}, e_k^u \mid u > p + 1 \text{ or } k > 2\} \tag{4.1}$$

of $G^{p,q}$. By Proposition 3.1, H is certainly an abelian subalgebra of $G^{p,q}$. Moreover, the matrix of each $K(x, y)$ for (not necessarily distinct) elements x and y of H with respect to the basis B of $G^{p,q}$ is diagonal. $\square$

For a unital commutative ring R, let E be an R-module equipped with a bilinear form β. Then a comtrans algebra $CT(E, \beta)$ with underlying module E is defined by

$$[x, y, z] = y\beta(x, z) - x\beta(y, z) \tag{4.2}$$

and

$$\langle x, y, z \rangle = y\beta(z, x) - x\beta(y, z) \tag{4.3}$$

(see [8]). If β is symmetric, then $CT(E, \beta)$ is monic. For a vector space E of dimension at least 3, two forms β and γ on E are equivalent if and only if the comtrans algebras $CT(E, \beta)$ and $CT(E, \gamma)$ are isomorphic (Theorem 3.6 of [8]).

Proposition 4.3. *Let R be a field of characteristic different from 2. Let E be an R-space equipped with a symmetric bilinear form β. Let H be a subspace of E of dimension at least 2. Then H is an abelian subalgebra of $CT(E, \beta)$ if and only if it forms an isotropic subspace of E.*

Proof. If H is isotropic, then (4.2) shows that H is abelian. Conversely, suppose that H is an abelian subalgebra of $CT(E, \beta)$. By (4.2), one has

$$y\beta(x, z) - x\beta(y, z) = 0 \tag{4.4}$$

for all x, y, z in H. Consider a pair of (not necessarily distinct) elements y, z of H. Let x be an element of H that is linearly independent of y. Then (4.4) shows that $\beta(y, z) = 0$. $\square$

Proposition 4.4. *Let E be a complex vector space of finite dimension greater than 3, equipped with a non-degenerate, symmetric bilinear form β. Then the comtrans algebra $CT(E, \beta)$ contains no two-dimensional abelian semisimplicial subalgebra.*

Proof. Suppose that y and z span a two-dimensional abelian semisimplicial subalgebra H of $CT(E, \beta)$. Since H is abelian as a comtrans subalgebra of $CT(E, \beta)$, Proposition 4.3 shows that it is isotropic as a subspace of (E, β). Let x be an element of E such that $\beta(x, z) \neq 0$. (The non-degeneracy of β guarantees that such an element exists.) Then the restriction of $K(y, z)$ to its invariant subspace spanned by $\{x, y, z\}$ is a non-zero endomorphism whose characteristic polynomial is λ^3, contradicting the semisimpliciality of H. □

Theorem 4.5. *For each pair of integers $p, q > 1$, not both equal to 2, there are no square matrices A and B such that the simple Grassmann comtrans algebra $G^{p,q}$ is isomorphic to the algebra $CT(A, B)$.*

Proof. Suppose that $G^{p,q}$ is isomorphic to some $CT(A, B)$. Then the comtrans algebra $CT(A, B)$ is monic and simple. By Proposition 5.1 of [9], it follows that $CT(A, B)$ is of the form $CT(\mathbb{R}^{pq}, \beta)$ for a symmetric bilinear form β. By Theorem 3.4 of [8], the form β is non-degenerate. By Proposition 4.2, the complexification of $G^{p,q}$ contains a two-dimensional abelian semisimplicial subalgebra. On the other hand, by Proposition 4.4, the complexification of $CT(A, B)$ cannot contain any such subalgebra. Thus $G^{p,q}$ cannot be isomorphic to $CT(A, B)$. □

5 Grassmann Algebras and Lie Algebras

In this section, it will be shown that no simple Grassmann comtrans algebra $G^{p,q}$ appears as the comtrans algebra $CT(L)$ of a Lie algebra L.

Proposition 5.1. *For each pair of integers $p, q > 1$, not both equal to 2, the characteristic polynomial of the simple Grassmann comtrans algebra $G^{p,q}$ has at least one linear factor of multiplicity one.*

Proof. Take the basis (4.1) of $G^{p,q}$, and consider the specialization

$$K(2e_1^p + e_2^{p+1}, 2e_1^p + e_2^{p+1}) \tag{5.1}$$

of (2.7). Then the basis (4.1) consists entirely of eigenvectors of (5.1). By Proposition 3.1, the first two basis eigenvectors are annihilated, and the third $(e_2^p - e_1^{p+1})$ belongs to the eigenvalue -9, while the rest are all negated, annihilated, or multiplied by -4. □

Proposition 5.2. *Let L be a simple real Lie algebra of dimension larger than 3. Then each linear factor of the characteristic polynomial of $CT(L)$ appears with multiplicity at least 2.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for L over $\mathbb{R}$. Let $\{X_1, \ldots, X_n\}$ be a set of n indeterminates over $\mathbb{R}$, and let F be the algebraic closure of the field of rational functions over $\{X_1, \ldots, X_n\}$. Let L' be the extension of L to F. Let x be the generic element $X_1 e_1 + \cdots + X_n e_n$ of L'. Then x is a regular element of L' (cf. Exercise IX.6 of [4]), and its centralizer H is a Cartan

subalgebra of L'. Consider the corresponding Cartan decomposition

$$L' = H \oplus \sum_{\phi \in \Phi} L'_\phi$$

of L'. Then in the comtrans algebra $CT(L')$, the extension to F of $CT(L)$, the eigenspaces of (2.6) are H and the sums $L'_\phi \oplus L'_{-\phi}$ for roots ϕ in Φ. None of these is one-dimensional. $\qquad\square$

Theorem 5.3. *For each pair of integers $p, q > 1$, not both equal to 2, there is no real Lie algebra L such that the simple Grassmann comtrans algebra $G^{p,q}$ is isomorphic to the algebra $CT(L)$.*

Proof. Suppose that there is a real Lie algebra L such that $G^{p,q}$ is isomorphic to $CT(L)$. Then $CT(L)$ is simple, and its dimension is at least 6. By Theorem 3.2 of [9], the real Lie algebra L is also simple. By Proposition 5.2, each linear factor of the characteristic polynomial of $CT(L)$ has multiplicity at least 2. On the other hand, by Proposition 5.1, the characteristic polynomial of $G^{p,q}$ has at least one linear factor of multiplicity one. Thus $G^{p,q}$ and $CT(L)$ cannot be isomorphic. $\qquad\square$

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