JOURNAL OF ALGEBRA 120, 263–283 (1989)

On the Structure of Subalgebra Systems of Idempotent Entropic Algebras

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Communicated by Walter Feit

Received July 1, 1982

Contents.

1. INTRODUCTION

A general study of the subalgebra systems of idempotent entropic algebras was begun in [21], where these systems were considered as abstract algebras called idempotent-entropic operator- or IEO-semilattices. The current paper examines the structure of these algebras by means of various decompositions of them, together with corresponding construction methods for recovering the algebras from their decompositions.

One of the basic construction methods in semigroup theory is A. G. Clifford's "strong semilattice of semigroups," generalised to other kinds of algebra without nullary operations by J. Płonka. This method is now referred to as the "Płonka sum." It arises from a simultaneous consideration of semilattices both as abstract algebras of general type and as categories with finite products. The method is recalled in Section 2, which then goes on to introduce a generalisation of the Płonka sum required in Section 5. Some of the properties of the generalised Płonka sum are

0021-8693/89 $3.00

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investigated. Section 3 then focuses attention on idempotent entropic algebras. Theorem 3.2 shows when such algebras are expressible as Płonka sums over their greatest semilattice quotients. It is followed by a brief discussion of the scope and limitations of the method. The subsequent section carries the projection of an idempotent entropic Ω-algebra onto its greatest semilattice quotient over to a decomposition of the IEO-semilattice of finitely generated subalgebras of the algebra into a meet-distributive bisemilattice of sub-IEO-semilattices. This is given in Thereom 4.1. Next, Theorem 4.2 lifts the Płonka sum construction of Theorem 3.2 to a Płonka sum construction of the Ω-reduct of the IEO-semilattice over the meet reduct of the bisemilattice, and the Corollary 4.4 examines the Ω-reducts of the sub-IEO-semilattices of the decomposition.

As Theorem 4.1 shows, meet-distributive bisemilattices play an important role in the structure of IEO-semilattices. The next two sections are thus devoted to decompositions of meet-distributive bisemilattices, which then combine with the decomposition in Theorem 4.1 to give more comprehensive decompositions of the IEO-semilattices. There are two major classes of meet-distributive bisemilattices, namely semilattices and distributive lattices. Section 5 considers an earlier result [20, Corollary 3.5] decomposing meet-distributive bisemilattices over semilattices, and in Theorem 5.1 shows how the generalised Płonka sum of Section 2 may be used to retrieve the meet-distributive bisemilattice from this decomposition. Theorem 6.1 of Section 6 decomposes meet-distributive bisemilattices over distributive lattices. Theorem 6.8 then shows how the meet reduct of a meet-distributive bisemilattice may be obtained from the join reduct and this decomposition. It is an example of a general idea for constructing bisemilattices, based on the observation that a bisemilattice is specified by two partial orders. Given one partial order, one may then be able to obtain the other on reversing some of the order relationships and preserving the rest. In Theorem 6.8, the decomposition of Theorem 6.1 tells which relationships to reverse and which to preserve. Propositions 6.4 and 6.5, required in the proof of Theorem 6.8, are also of independent interest. They show that, for a given semilattice, the partial order $\leqslant_+$ on the set of finite subsemilattices is a locally finite lower semimodular lattice order, while the partial order $\leqslant_.$ is locally finite if and only if the partial order on the semilattice is locally finite.

Under the hypotheses of Theorem 3.2, Section 7 investigates construction methods corresponding to the decomposition of IEO-semilattices over distributive lattices obtained by combining the decomposition of Theorem 4.1 with that of Theorem 6.1. Theorem 7.1 shows that the fibres of this combined decomposition are Płonka sums over the semilattices appearing as fibres in the decomposition of bisemilattices given by Theorem 6.1. Then Theorem 7.2 completes the construction task begun in Theorem 4.2 by showing how to recover the semilattice reduct of the IEO-semilattice there. Finally, Section 8 gives an example illustrating all of the decompositions and construction methods discussed in the paper.

2. Semilattices, Płonka Sums, and a Generalisation

Semilattices are usually thought of algebraically, as commutative idempotent semigroups, or order-theoretically, as partially ordered sets ("meet-semilattices") in which each pair of elements has a greatest lower bound [7, I.3; 20, Sect. 1]. For the purposes of this paper it is convenient to generalise these two points of view. Given a semilattice as a semigroup $(S, \cdot)$, and an operator domain Ω having an operation of arity at least 2, but no nullary operations, then one may regard the semilattice as an Ω-algebra (S, Ω) by defining the action of an n-ary operation ω from Ω on S as

$$\omega\colon S^n \to S\colon (s_1, s_2, ..., s_n) \mapsto s_1 \cdot s_2 \cdot \quad \cdots \quad \cdot s_n.$$

Conversely, given an Ω-algebra (S, Ω), there are identities on the operations of Ω ensuring the existence of a binary operation $\cdot$ on S such that $(S, \cdot)$ is a semilattice with $s_1 \cdots s_n \omega = s_n \cdot \ \cdots \ \cdot s_n$ (see [6]). Thus one may generalise the algebraic way of looking at semilattices to think of them as algebras of arbitrary non-nullary type. The generalisation of the order-theoretic way of looking at semilattices regards a (meet-) semilattice $(S, \leqslant_{\cdot})$ as a (small) category in which there is a unique morphism $x \to y$ iff $x \leqslant_{\cdot} y$ (cf. [11, I.2]). The product $x_1 \cdot \ \cdots \ \cdot x_n$ of a finite set $\{x_1, ..., x_n\}$ of elements of the semilattice is then their product in the categorical sense. Throughout this paper these two points of view will be assumed implicitly.

The two aspects come together in the notion of "Płonka sum," introduced in [16] under the name "sum of a direct system" as a generalisation of A. H. Clifford's "strong semilattice of semigroups" [7, Chap. IV].

Definition 2.1. Let Ω be an operator domain without nullary operations, S a meet semilattice, and F a contravariant functor from S to the category of Ω-algebras and homomorphisms. For $s \leqslant_{\cdot} t$ in S, set $F(s) = A_s$ and $F(s \to t) = \phi_{t,s}\colon A_t \to A_s$. Then the *Płonka sum* of the algebras A_s over the semilattice S by the functor F is the disjoint union A of the underlying sets A_s, $s \in S$, equipped with the Ω-algebra structure given by

$$\omega\colon A_{s_1} \times \cdots \times A_{s_n} \to A_{s_1 \cdots s_n \omega};$$
$$(x_1, ..., x_n) \mapsto x_1 \phi_{s_1, s_1 \cdots s_n \omega} \cdots x_n \phi_{s_n, s_1 \cdots s_n \omega} \omega.$$

In Section 5 of this paper a generalisation of the notion of Płonka sum is required. This generalisation is obtained by weakening the requirement of functoriality in Definition 2.1. Its properties are summarised in the following theorem. Recall that a *regular* identity is one in which the sets of variables involved in each side are precisely the same, and that semilattices satisfy all regular identities.

THEOREM 2.2. *Let Ω be an operator domain without nullary operations, and $(S, \leqslant_{\bullet})$ a meet semilattice. For each $s \in S$, let an Ω-algebra A_s be specified, and for each pair (s, t) in the relation $\leqslant_{\bullet}$ on S, a mapping $\phi_{t,s}\colon A_t \to A_s$. For each Ω-operation ω, of arity n, and for $s_1, ..., s_n, s', s$ in S with $s \leqslant_{\bullet} s_1 \cdots s_n \omega = s'$, let the diagram*

$$\begin{array}{ccccc} A_s^n & \xleftarrow{\phi_{s_1,s} \times \cdots \times \phi_{s_n,s}} & A_{s_1} \times \cdots \times A_{s_n} & \xrightarrow{\phi_{s_1,s'} \times \cdots \times \phi_{s_n,s'}} & A_{s'}^n \\ \downarrow{\scriptstyle\omega} & & & & \downarrow{\scriptstyle\omega} \\ A_s & & \xleftarrow{\quad\phi_{s',s}\quad} & & A_{s'} \end{array} \tag{2.1}$$

commute. Define an Ω-algebra structure on the disjoint union A of the underlying sets A_s, $s \in S$, by

$$\omega\colon A_{s_1} \times \cdots A_{s_n} \to A_{s_1 \cdots s_n \omega};$$
$$(x_1, ..., x_n) \mapsto x_1 \phi_{s_1, s_1 \cdots s_n \omega} \cdots x_n \phi_{s_n, s_1 \cdots s_n \omega} \omega.$$

Then the algebra (A, Ω) satisfies all the regular identities satisfied by each of the (A_s, Ω), with the possible exception of idempotence.

Proof. First, note that the coproduct of the functions $A_s \to \{s\}$ over s in S yields a homomorphism of the algebra (A, Ω) onto the semilattices (S, Ω). Next, it will be shown that for each derived operator $\bar{w}$ (notation as [4, p. 145]), of arity n, and for elements $a_i \in A_{s_i}$, $i = 1, ..., n$,

$$a_1 \cdots a_n \bar{w} = a_1 \phi_{s_1, s_1 \cdots s_n \bar{w}} \cdots a_n \phi_{s_n, s_1 \cdots s_n \bar{w}} \bar{w}. \tag{2.2}$$

The proof goes by induction on the number of Ω-operations constituting $\bar{w}$, the result holding by the definition of Ω on A if this number is 1. Otherwise, suppose $a_1 \cdots a_n \bar{w} = a_1 \cdots a_i a_{i+1} \cdots a_{i+m} \omega a_{i+m+1} \cdots a_n w$ for an m-ary operation ω of Ω and an $(n-m+1)$-ary derived operator w. Let $s' = s_{i+1} \cdots s_{i+m} \omega$, $s = s_1 \cdots s_n \bar{w} = s_1 \cdots s' \cdots s_n w \leqslant_{\bullet} s'$. Then by the induction hypothesis

$$\begin{aligned} a_1 \cdots a_n \bar{w} &= a_1 \cdots a_i a_{i+1} \phi_{s_{i+1},s'} \cdots a_{i+m} \phi_{s_{i+m},s'} \omega a_{i+m+1} \cdots a_n w \\ &= a_1 \phi_{s_1,s} \cdots a_i \phi_{s_i,s} (a_{i+1} \phi_{s_{i+1},s'} \cdots a_{i+m} \phi_{s_{i+m},s'} \omega)\, \phi_{s',s} \\ &\quad \times a_{i+m+1} \phi_{s_{i+m+1},s} \cdots a_n \phi_{s_n,s} w \\ &= a_1 \phi_{s_1,s} \cdots a_i \phi_{s_i,s} a_{i+1} \phi_{s_{i+1},s} \cdots a_{i+m} \phi_{s_{i+m},s} \omega \\ &\quad \times a_{i+m+1} \phi_{s_{i+m+1},s} \cdots a_n \phi_{s_n,s} w \\ &= a_1 \phi_{s_1,s} \cdots a_n \phi_{s_n,s} \bar{w} \end{aligned}$$

as required, the penultimate equality holding by the commuting of (2.1).

Now let $x_1 \cdots x_m w = y_1 \cdots y_n w'$ be a regular identity satisfied by each of the (A_s, Ω) in which w and w' are derived operators (i.e., the identity is not idempotent). Set $x_1 := a_1 \in A_{s_1}, ..., x_m := a_m \in A_{s_m}$, $y_1 := b_1 \in A_{t_1}, ..., y_n := b_n \in A_{t_n}$. Since $\{x_1, ..., x_m\} = \{y_1, ..., y_n\}$, one has $\{s_1, ..., s_m\} = \{t_1, ..., t_n\}$, whence $s_1 \cdots s_m w = s_1 \cdot \cdots \cdot s_m = t_1 \cdot \cdots \cdot t_n = t_1 \cdots t_n w' = s$, say. Then by (2.2) $a_1 \cdots a_m w = a_1 \phi_{s_1,s} \cdots a_m \phi_{s_m,s} w$ and $b_1 \cdots b_n w' = b_1 \phi_{t_1,s} \cdots b_n \phi_{t_n,s} w'$. Since $a_i \phi_{s_i,s}, b_j \phi_{t_j,s} \in A_s$, where the identity $x_1 \cdots x_m w = y_1 \cdots y_n w'$ holds, $a_1 \phi_{s_1,s} \cdots a_m \phi_{s_m,s} w = b_1 \phi_{t_1,s} \cdots b_n \phi_{t_n,s} w'$, whence $a_1 \cdots a_m w = b_1 \cdots b_n w'$, as required. ∎

COROLLARY 2.3. *Let each of the (A_s, Ω) be idempotent. Then (A, Ω) is idempotent iff each of the mappings $\phi_{s,s}: A_s \to A_s$ is the identity mapping.*

Proof. Let ω be an n-ary operation in Ω, and a an element of A_s for some s. Then by the definition of the action of Ω on A, $a \cdots a\omega$ in A is $a\phi_{s,s} \cdots a\phi_{s,s}$ in A_s, i.e., $a\phi_{s,s}$ by the idempotence of (A_s, Ω). Thus $a \cdots a\omega = a$ in A iff $a\phi_{s,s} = a$. ∎

The following proposition and its corollary summarise results that are well known, but for which explicit formulations are hard to find in the literature (the closest to Corollary 2.5 probably being [9, Corollary 2.8]).

PROPOSITION 2.4. *A non-trivial semilattice cannot satisfy a non-regular identity.*

Proof. Suppose that a semilattice S satisfies the non-regular identity $x_1 x_2 \cdots x_m w_1 = y_1 \cdots y_n w_2$, where without loss of generality $x_1 \notin \{y_1, ..., y_n\}$. Then S satisfies $x_1 x_2 \cdots x_2 w_1 = x_2 \cdots x_2 w_2$, i.e., $x_1 \geqslant_{\bullet} x_2$. For elements a, b of S, this implies $a \geqslant_{\bullet} b$ and $b \geqslant_{\bullet} a$, i.e., $a = b$. Thus S is trivial. ∎

COROLLARY 2.5. *Let Ω be an operator domain having an operation of arity at least 2, but no nullary operations. Let* **V** *be a variety of Ω-algebras, and* **S** *the variety of semilattices regarded as Ω-algebras. Then*

(i) $\mathbf{S}$ *is specified by the set of all regular identities;*

(ii) $\mathbf{V} \supseteq \mathbf{S}$ *iff* $\mathbf{V}$ *is specified purely by regular identities.*

Consideration of Proposition 2.4 allows a more precise specification of the identities satisfied by the algebra (A, Ω) of Theorem 2.2.

PROPOSITION 2.6. *If the semilattice* $(S, \leqslant_{\bullet})$ *of Theorem* 2.2 *is non-trivial, and each* $\varphi_{s,s} = I_{A_s}$, *then the identities satisfied by the algebra* (A, Ω) *there are precisely the regular identities satisfied by each of the* (A_s, Ω).

Proof. By Theorem 2.2, (A, Ω) satifies all the regular identities satisfied by each of the (A_s, Ω), with the possible exception of idempotence. Since each (A_s, Ω) is a subalgebra of (A, Ω), these are the only regular identities that (A, Ω) can satisfy. Were (A, Ω) to satisfy a non-regular identity, then so would its quotient (S, Ω), contradicting the non-triviality of $(S, \leqslant_{\bullet})$ by Proposition 2.4. ∎

Note that Płonka sums are a special case of the construction of Theorem 2.2, since the functoriality in Definition 2.1 guarantees the commuting of diagram (2.1). Conversely, under certain conditions algebras constructed according to Theorem 2.2 may also be obtained as Płonka sums. Let $\mathbf{K}$ be a class of Ω-algebras for an operator domain Ω without nullary operations. Let $\mathbf{V}$ be the variety of Ω-algebras satisfying the regular identities satisfied by the algebras in $\mathbf{K}$. Then by Theorem 2.2 (and Corollary 2.3 if appropriate), algebras (A, Ω) constructed by the generalisation (with $\phi_{s,s} = 1_{A_s}$ if necessary) from algebras (A_s, Ω) in $\mathbf{K}$ belong to the variety $\mathbf{V}$. Now under particular hypotheses on the class $\mathbf{K}$, Theorem I of [18] shows that algebras in $\mathbf{V}$ can be represented as Płonka sums of algebras in $\mathbf{K}$. However this, or indeed any, representation of the algebra (A, Ω) as a Płonka sum may well decompose it into subalgebras other than the (A_s, Ω) used for the generalised sum. As an elementary example, take $\mathbf{K}$ and $\mathbf{V}$ to be the variety of semilattices. These certainly satisfy the hypotheses of Płonka's theorem. Let $(S, \leqslant_{\bullet})$ be the meet-semilattice given by natural numbers less than 3 under the usual ordering, so that $s \cdot t = \min(s, t)$. Then the set A of subsets of S containing 2 under the semilattice operation of union is obtained as the generalised sum of its subalgebras $A_2 = \{\{2\}\}$, $A_1 = \{\{1, 2\}\}$, $A_0 = \{\{0, 2\}, \{0, 1, 2\}\}$ on setting $\{1, 2\}\phi_{1,0} = \{0, 1, 2\}$, $\{2\}\phi_{2,0} = \{0, 2\}$, and $\phi_{s,s} = 1_{A_s}$ for each s in S. But A cannot be obtained as a Płonka sum of these subalgebras. If it could be, there would be just two possible choices for $\phi_{1,0}$. But taking $\{1, 2\}\phi_{1,0} = \{0, 1, 2\}$ would imply $\{0, 2\} = \{2\} \cup \{0, 2\} = \{2\}\phi_{2,0} \cup \{0, 2\}\phi_{0,0} = \{0, 1, 2\} \cup \{0, 2\} = \{0, 1, 2\}$, while setting $\{1, 2\}\phi_{1,0} = \{0, 2\}$ would imply $\{0, 1, 2\} = \{1, 2\} \cup \{0, 2\} = \{1, 2\}\phi_{1,0} \cup \{0, 2\}\phi_{0,0} = \{0, 2\} \cup \{0, 2\} = \{0, 2\}$. Płonka's theorem decomposes A as a sum of one copy of A.

3. Płonka Sums of Idempotent Entropic Algebras

Let Ω be an operator domain having an operation of arity at least 2, but no nullary operations. Let **V** be a variety of Ω-algebras, and **S** the variety of semilattices regarded as Ω-algebras. By [12, V.11.3, Theorem 5] the forgetful functor $\mathbf{V} \cap \mathbf{S} \to \mathbf{V}$ has a left adjoint, the unit of the adjunction being the projection θ of a **V**-algebra A onto its "largest" semilattice quotient. Since semilattices are idempotent, the fibres $\theta^{-1}(a\theta)$ for each element a of A are subalgebras of A. Let $\kappa\mathbf{V}$ denote the class consisting of all such fibres for each element of each **V**-algebra.

Definition 3.1. A variety **V** of algebras having an operation of arity at least 2, but no nullary operations, is said *to have non-regular fibres over semilattices* iff a non-regular identity holds in the (variety generated by the) class $\kappa\mathbf{V}$.

Theorem 3.2. *Let* **V** *be an idempotent entropic variety having non-regular fibres over semilattices. Then each* **V***-algebra is a Płonka sum over its greatest semilattice quotient.*

Proof. Let $x_1 x_2 \cdots x_m w_1 = y_1 \cdots y_n w_2$ be a non-regular identity holding in $\kappa\mathbf{V}$, say without loss of generality $x_1 \notin \{y_1, ..., y_n\}$. Then in particular $x_1 x_2 \cdots x_2 w_1 = x_2 \cdots x_2 w_2 = x_2$. Set $x_1 x_2 p := x_1 x_2 \cdots x_2 w_1$, so that

$$x_1 x_2 p = x_2 \tag{3.1}$$

holds in $\kappa\mathbf{V}$.

Now let A be a **V**-algebra, and $\theta: A \to S$ its projection onto its greatest semilattice quotient. Note that

$$\forall a, b, c \in A,\ a\theta \geqslant_{\bullet} b\theta = c\theta \Rightarrow bap = cap \tag{3.2}$$

since $bap = bcpbapp = bbpcapp = cap$, the first and third equalities following from (3.1) and the second from the entropic law. Define $F: \mathbf{S} \to \mathbf{V}$ by $F(s) = \theta^{-1}(s)$ and $F(s \leqslant_{\bullet} t) = \phi_{t,s}: A_t \to A_s;\ a \mapsto bap$ for some b with $b\theta = s$. By (3.2) the definition of $\phi_{t,s}$ is independent of the particular choice of b. By (3.1) each $\phi_{s,s}$ is the identity on $\theta^{-1}(s)$. The $\phi_{t,s}$ are **V**-homomorphisms, since for an n-ary operation ω and $a_1, ..., a_n, b$ in A with $a_i\theta = t$, $b\theta = s$, one has $a_1 \cdots a_n \omega\phi_{t,s} = ba_1 \cdots a_n \omega p = b \cdots b\omega a_1 \cdots a_n \omega p = ba_1 p \cdots ba_n p\omega = a_1\phi_{t,s} \cdots a_n\phi_{t,s}\omega$. Then F is a functor, since for $s \leqslant_{\bullet} t \leqslant_{\bullet} u$ and $a\theta = u$, $b\theta = t$, $c\theta = s$, one has $a\phi_{u,t}\phi_{t,s} = bap\phi_{t,s} = cbapp = ccpbapp = cbpcapp = cap = a\phi_{u,s}$. Finally, the algebra A is recovered as the Płonka sum of the algebras A_s over S since for each operation ω of arity n, and for $a_1, ..., a_n$ in A with $a_i\theta = s_i$, $s = s_1, ..., s_n\omega$, $a \in \theta^{-1}(s)$, the action of ω on $(a_1, ..., a_n)$ in the

Płonka sum is $a_1\phi_{s_1,s}\cdots a_n\phi_{s_n,s}\omega = aa_1p\cdots aa_np\omega = a\cdots a\omega a_1\cdots a_n\omega p = a_1\cdots a_n\omega$. ▌

Of the examples of idempotent entropic algebras given in [21], semilattices, normal bands, and quasigroups are covered by Theorem 3.2. The class $\kappa\mathbf{S}$ is trivial, so (3.1) becomes $x_1 = x_2$. If $\mathbf{V}$ is the variety of normal bands, then $\kappa\mathbf{V}$ is the class of rectangular bands, (3.1) taking say the form $x_2x_1x_2 = x_2$ (cf. [7, Proposition IV.5.14]; [10]). If $\mathbf{V}$ is the variety of idempotent entropic quasigroups, then $\mathbf{V}\cap\mathbf{S}$ is trivial, so that $\kappa\mathbf{V} = \mathbf{V}$. In this case (3.1) becomes, say, $x_1\backslash(x_1x_2) = x_2$ (cf. [3, p. 9]). Further examples are given by the varieties of groupoids introduced in [5, 13], and by the idempotent medial algebras dealt with in [14, 15, 17].

For an example of a variety of idempotent entropic algebras not having non-regular fibres over semilattices, one may take the variety $\mathbf{V}$ generated by the bounded convex sets in finite-dimensional Euclidean space as in [21, Example 2.4], but with just those operations $(a, b)\mapsto \lambda a + (1-\lambda)b$ coming from λ in the open interval $(0, 1)$. The free algebra on two generators is just the closed interval $[0, 1]$. Its greatest semilattice quotient is the free semilattice on two generators, the corresponding congruence classes being $\{0\}$ and $\{1\}$ (the free generators of the semilattice) and $(0, 1)$. The class just consisting of the free semilattice on two generators generates the variety $\mathbf{S}$, so by Corollary 2.5, $\mathbf{V}$ is specified purely by regular identities. However, $\mathbf{V}$ is the variety generated by $\kappa\mathbf{V}$, since the open interval $(0, 1)$ in $\mathbf{V}$ generates $\mathbf{V}$. Altogether, the problem of constructing an algebra out of its congruence classes becomes somewhat bizarre if the algebra itself appears as a subalgebra of one of these classes. A similar situation, namely that occurring here on just taking $\lambda = 1/2$, is discussed in [8].

4. Decompositions of IEO-Semilattices over Bisemilattices

From now on, let A denote an idempotent entropic Ω-algebra, and $\theta\colon A\to S$ the projection onto its greatest semilattice quotient. By [21, Theorem 6.1], there is an IEO-semilattice homomorphism $\Theta\colon(\mathrm{Fg}\,A, +, \Omega)\to(\mathrm{Fg}\,S, +, \Omega)$ such that the following diagram commutes:

$$\begin{array}{ccccc} (A,\Omega) & \xrightarrow{\ \theta\ } & (S,\Omega) & \xrightarrow{\ \iota\ } & (\mathrm{Fg}\,S,\Omega) \\ \Big\downarrow{\scriptstyle\iota} & & & & \Big\downarrow{\scriptstyle 1_{\mathrm{Fg}\,S}} \\ (\mathrm{Fg}\,A, +, \Omega) & & \xrightarrow{\qquad\Theta\qquad} & & (\mathrm{Fg}\,S, +, \Omega). \end{array} \tag{4.1}$$

If $G = \langle x \mid x\in X\rangle$ for a finite subset X of A, then $G\Theta = \langle x\theta \mid x\in X\rangle = \{a\theta \mid a\in G\} = G\theta$.

Just as semilattices may be taken to be Ω-algebras, so meet-distributive bisemilattices may be taken to be IEO-semilattices with operator domain $\{+\}\cup\Omega$. Conversely, if the Ω-reduct (B, Ω) of an IEO-semilattice $(B, +, \Omega)$ is a semilattice, then $(B, +, \Omega)$ is a meet-distributive bisemilattice in this sense. This is the case here for $(\mathrm{Fg}\, S, +, \Omega)$, which is just the free meet-distributive bisemilattice over the semilattice $(S, \cdot)$, as considered in [20].

THEOREM 4.1. *Let A be an idempotent entropic Ω-algebra, and $\theta\colon A \to S$ the projection onto its greatest semilattice quotient. Then the* IEO*-semilattice* $(\mathrm{Fg}\, A, +, \Omega)$ *has the projection* $\Theta\colon (\mathrm{Fg}\, A, +, \Omega) \to (\mathrm{Fg}\, S, +, \Omega)$; $G \mapsto G\theta$ *onto the meet-distributive bisemilattice* $(\mathrm{Fg}\, S, +, \cdot)$.

Proof. It merely remains to check that Θ surjects. Let K be a finitely generated subsemilattice of S. Then K is finite, since the variety of semilattices is locally finite. One may thus choose a finite subset X of A such that $X\theta = K$. Then $\langle X\rangle$ in $\mathrm{Fg}\, A$ satisfies $\langle X\rangle \Theta = \langle X\theta\rangle = K$.

The following theorem shows that one can recover $(\mathrm{Fg}\, A, \Omega)$ from the decomposition in Theorem 4.1 if the conditions of Theorem 3.2 obtain. Later, Theorem 7.2 will show how $(\mathrm{Fg}\, A, +)$ may be recovered.

THEOREM 4.2. *Let* $\mathbf{V}$ *be a variety of idempotent entropic algebras having non-regular fibres over semilattices. Let A be a* $\mathbf{V}$*-algebra, and $\theta\colon A \to S$ the projection onto its greatest semilattice quotient. Then there is a functor F from the semilattice* $(\mathrm{Fg}\, S, \cdot)$ *to the category of Ω-algebras and homomorphisms with*

(i) $F(K) = \Theta^{-1}(K)$ *and*

(ii) $F(L \leqslant_{\cdot} K) = \Phi_{K,L}\colon \Theta^{-1}(K) \to \Theta^{-1}(L)$; $G \mapsto \{g\phi_{g\theta, g\theta \cdot l} \mid g \in G, l \in L\}$.

(*for finite(-ly generated) subsemilattices K, L of S, the mapping ϕ being as in the proof of Theorem* 3.2), *by which the Ω-reduct* $(\mathrm{Fg}\, A, \Omega)$ *of the* IEO*-semilattice* $(\mathrm{Fg}\, A, +, \Omega)$ *is a Płonka sum.*

Proof. It first has to be checked that F is indeed a functor as claimed. To begin with, note that, for an n-ary operation ω and elements $g_1, \ldots, g_n$ of A, $l_1, \ldots, l_n$ of S, with $g = g_1 \cdots g_n\omega$, $l = l_1 \cdots l_n\omega$,

$$g_1\phi_{g_1\theta, g_1\theta \cdot l_1} \cdots g_n\phi_{g_n\theta, g_n\theta \cdot l_n}\omega = g\phi_{g\theta, g\theta \cdot l}. \tag{4.2}$$

This holds since

$$\begin{aligned}
&g_1\phi_{g_1\theta,\, g_1\theta\cdot l_1}\cdots g_n\phi_{g_n\theta,\, g_n\theta\cdot l_n}\omega\\
&\quad= g_1\phi_{g_1\theta,\, g_1\theta\cdot l_1}\phi_{g_1\theta\cdot l_1,\, g\theta\cdot l}\cdots g_n\phi_{g_n\theta,\, g_n\theta\cdot l_n}\phi_{g_n\theta\cdot l_n,\, g\theta\cdot l}\omega,\\
&\qquad\text{by Definition 2.1}\\
&\quad= g_1\phi_{g_1\theta,\, g\theta\cdot l}\cdots g_n\phi_{g_n\theta,\, g\theta\cdot l}\omega,\\
&\qquad\text{by the functoriality of } \phi\\
&\quad= g_1\phi_{g_1\theta,\, g\theta}\phi_{g\theta,\, g\theta\cdot l}\cdots g_n\phi_{g_n\theta,\, g\theta}\phi_{g\theta,\, g\theta\cdot l}\omega,\\
&\qquad\text{by the functoriality of } \phi\\
&\quad= g_1\phi_{g_1\theta,\, g\theta}\cdots g_n\phi_{g_n\theta,\, g\theta}\omega\phi_{g\theta,\, g\theta\cdot l},\\
&\qquad\text{since } \phi_{g\theta,\, g\theta\cdot l} \text{ is a homomorphism}\\
&\quad= g_1\cdots g_n\omega\phi_{g\theta,\, g\theta\cdot l}, \qquad \text{by Definition 2.1.}
\end{aligned}$$

From (4.2) it follows immediately that $G\Phi_{K,L}$ is a subalgebra of A.

To show that $G\Phi_{K,L}$ is a finitely generated subalgebra of A, let $G=\langle g_1, ..., g_m\rangle$ and $L=\langle l_1, ..., l_n\rangle$. Let $g\in G$, $l\in L$, say $g=g_1\cdots g_m w_1$ and $l=l_1\cdots l_n w_2$ as words in the generators. Let the mn-ary derived operation ω be defined by $x_{11}\cdots x_{m1}\cdots x_{1n}\cdots x_{mn}\omega = x_{11}\cdots x_{m1}w_1\cdots x_{1n}\cdots x_{mn}w_1w_2$. Then $g=g_1\cdots g_m w_1 = g_1\cdots g_1 w_2\cdots g_m\cdots g_m w_2 w_1 = g_1\cdots g_m w_1\cdots g_1\cdots g_m w_1 w_2 = g_1\cdots g_m\cdots g_1\cdots g_m\omega$ and $l=l_1\cdots l_n w_2 = l_1\cdots l_1 w_1\cdots l_n\cdots l_n w_1 w_2 = l_1\cdots l_1\cdots l_n\cdots l_n\omega$, so that the typical element $g\phi_{g\theta,l}$ of $G\Phi_{K,L}$ may be expressed as $g\phi_{g\theta,l}=g_1\cdots g_m\cdots g_1\cdots g_m\omega\phi_{g_1\cdots g_m\cdots g_1\cdots g_m\omega\theta,\, l_1\cdots l_1\cdots l_n\cdots l_n\omega} = g_1\phi_{g_1\theta,\, g_1\theta\cdot l_1}\cdots g_m\phi_{g_m\theta,\, g_m\theta\cdot l_1}\cdots g_1\phi_{g_1\theta,\, g_1\theta\cdot l_n}\cdots g_m\phi_{g_m\theta,\, g_m\theta\cdot l_n}\omega$, the latter equality following from (4.2). This shows that $G\Phi_{K,L}$ is generated by the finite set $\{g_i\phi_{g_i\theta,\, g_i\theta\cdot l_j} \mid 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\}$.

Now $L\leqslant_{.} K$ means $K\cdot L=L$. Let $l\in L$. Then $l=k\cdot l'$ for some $k\in K$, $l'\in L$. Since $G\Theta=G\theta=K$, there is an element g of G with $g\theta=k$. Then the element $g\phi_{g\theta,\, g\theta\cdot l'}$ of $G\Phi_{K,L}$ has $g\theta\cdot l'=k\cdot l'=l$ as its image under θ, i.e., $G\Phi_{K,L}\theta\supseteq L$. Conversely, the θ-image of the typical element $g\phi_{g\theta,\, g\theta\cdot l}$ of $G\Phi_{K,L}$ is $g\theta\cdot l\in K\cdot L=L$, so that $G\Phi_{K,L}\subseteq L$. Thus $\Phi_{K,L}$ does indeed map $\Theta^{-1}(K)$ into $\Theta^{-1}(L)$. To check that $\Phi_{K,L}$ is a homomorphism, let ω be an n-ary operation and $G_1, ..., G_n\in\Theta^{-1}(K)$. Then $G_1\cdots G_n\omega\Phi_{K,L} = \{g_1\cdots g_n\omega\phi_{g_1\cdots g_n\omega\theta,\, g_1\cdots g_n\omega\theta\cdot l_1\cdots l_n\omega} \mid g_i\in G_i,\ l_i\in L\} = \{g_1\phi_{g_1\theta,\, g_1\theta\cdot l_1}\cdots g_n\phi_{g_n\theta,\, g_n\theta\cdot l_n}\omega \mid g_i\in G_i,\ l_i\in L\} = G_1\Phi_{K,L}\cdots G_n\Phi_{K,L}\omega$ as required, the middle equality following from (4.2).

To verify the functoriality, suppose first that $G\Theta=L$. Then $G\Phi_{L,L}=\{g\phi_{g\theta,\, g\theta\cdot l}\mid g\in G,\ l\in L\}\supseteq\{g\phi_{g\theta,\, g\theta}\mid g\in G\}=G$. Conversely, for $l\in L=G\theta$, there is some g' in G with $g'\theta=l$. Then the typical element $g\phi_{g\theta,\, g\theta\cdot l}$ of

$G\Phi_{L,L}$ may be written as $g\phi_{g\theta, g\theta \cdot l} = g\phi_{g\theta, g\theta \cdot g'\theta} = g\phi_{g\theta, g\theta \cdots g\theta g'\theta\omega} = g \cdots gg'\omega gp \in G$ with some operation ω of arity at least 2, the last equality following from (3.2). Thus $\Phi_{L,L} = 1_{\Theta^{-1}(L)}$. Now suppose $K \geqslant_{.} L \geqslant_{.} M$ in $(\mathrm{Fg}\, S, \leqslant_{.})$; in particular $L \cdot M = M$. Then for $G \in \Theta^{-1}(K)$, $G\Phi_{K,L}\Phi_{L,M} = \{g\phi_{g\theta, g\theta \cdot l} \mid g \in G, l \in L\}\, \Phi_{L,M} = \{g\phi_{g\theta, g\theta \cdot l}\phi_{g\theta \cdot l, g\theta \cdot l \cdot m} \mid g \in G, l \in L, m \in M\} = \{g\phi_{g\theta, g\theta \cdot l \cdot m} \mid g \in G, l \in L, m \in M\} = \{g\phi_{g\theta, g\theta \cdot m'} \mid m' \in M\} = G\Phi_{K,M}$, so that $\Phi_{K,L}\Phi_{L,M} = \Phi_{K,M}$, and F is indeed a functor with the required properties.

Finally, it remains to show that $(\mathrm{Fg}\, A, \Omega)$ is the algebra constructed as a Płonka sum of the $\Theta^{-1}(K)$ over $\mathrm{Fg}\, S$ in this way. Let ω be an n-ary operation, and $G_i \in \Theta^{-1}(G_i)$, $i = 1, \ldots, n$. Then in the Płonka sum, $G_1 \cdots G_n\omega = G_1\Phi_{G_1\theta, G_1 \cdots G_n\omega\theta} \cdots G_n\Phi_{G_n\theta, G_1 \cdots G_n\omega\theta}\omega = \{g_1\phi_{g_1\theta, g_1\theta \cdot l_1} \cdots g_n\phi_{g_n\theta, g_n\theta \cdot l_n} \mid g_i \in G_i,\ l_i \in G_1 \cdots G_n\omega\theta\} = \{g_1 \cdots g_n\omega\phi_{g_1 \cdots g_n\omega\theta, g_1 \cdots g_n\omega\theta \cdot l_1 \cdots l_n\omega} \mid g_i \in G_i, l_i \in G_1 \cdots G_n\omega\theta\} = \{g_1 \cdots g_n\omega\phi_{g_1 \cdots g_n\omega\theta, g_1 \cdots g_n\omega\theta \cdot g\theta} \mid g_i \in G_i, g \in G_1 \cdots G_n\omega\} = \{g_1 \cdots g_n\omega g \cdots g\omega' g_1 \cdots g_n\omega p \mid g_i \in G_i, g \in G_1 \cdots G_n\omega\} = G_1 \cdots G_n\omega$ in $(\mathrm{Fg}\, A, \Omega)$ as required, ω' being some operation of arity at least 2, the third equality holding by (4.2). ▮

COROLLARY 4.3. *The mapping* $\Phi_{K,L}\colon \Theta^{-1}(K) \to \Theta^{-1}(L)$ *of Theorem* 4.2 *is also a* $+$*-homomorphism.*

Proof. For G, G' in $\Theta^{-1}(K)$,

$$\begin{aligned}(G+G')\,\Phi_{K,L} &= \{g_1 \cdots g_n\omega\phi_{g_1 \cdots g_n\omega\theta, g_1 \cdots g_n\omega\theta \cdot l} \mid g_i \in G \cup G', l \in L\} \\ &= \{g_1 \cdots g_n\omega\phi_{g_1 \cdots g_n\omega\theta, g_1 \cdots g_n\omega\theta \cdot l_1 \cdots l_n\omega} \mid g_i \in G \cup G', l_i \in L\} \\ &= \{g_1\phi_{g_1\theta, g_1\theta \cdot l_1} \cdots g_n\phi_{g_n\theta, g_n\theta \cdot l_n}\omega \mid g_i \in G \cup G', l_i \in L\} \\ &= G\Phi_{K,L} + G'\Phi_{K,L},\end{aligned}$$

the penultimate equality following from (4.2). ▮

An identity on an algebra is said to be *subregular* if the set of variables appearing on one side is a subset of those appearing on the other. (The identity is regular if this subset is improper.) An identity is said to be *linear* if variables appear at most once on each side. The following corollary of Theorem 4.2 may be compared with [21, Proposition 2.2], which remarked that $(\mathrm{Fg}\, A, \Omega)$ satisfies all the regular linear identities satisfied by (A, Ω).

COROLLARY 4.4. *In the context of Theorem* 4.2, *the* Θ*-fibres* $(\Theta^{-1}(K), \Omega)$ *in* $(\mathrm{Fg}\, A, \Omega)$ *satisfy all the subregular linear identities satisfied by the* θ*-fibres* $(\theta^{-1}(s), \Omega)$ *in* (A, Ω).

Proof. Let $x_1 \cdots x_n w_1 = x_{1j} \cdots x_{nj} w_2$, with injection $j\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$, be a subregular linear identity holding in the θ-fibres. Let K be a finite subsemilattice of S, and $X_1, \ldots, X_n$ elements of $\Theta^{-1}(K)$. Then

$$
\begin{aligned}
&X_1 \cdots X_n w_1 \\
&= \{x_1 \cdots x_n w_1 \mid x_i \in X_i\} \\
&= \{x_1 \phi_{x_1\theta, x_1 \cdots x_n w_1 \theta} \cdots x_n \phi_{x_n\theta, x_1 \cdots x_n w_1 \theta} w_1 \mid x_i \in X_i\} \\
&= \{x_{1j} \phi_{x_{1j}\theta, x_1 \cdots x_n w_1 \theta} \cdots x_{mj} \phi_{x_{mj}\theta, x_1 \cdots x_n w_1 \theta} w_2 \mid x_i \in X_i\} \\
&= \{x_{1j} \phi_{x_{1j}\theta, x_{1j} \cdots x_{mj} w_2 \theta} \phi_{x_{1j} \cdots x_{mj} w_2 \theta, x_1 \cdots x_n w_1 \theta} \\
&\quad \cdots x_{mj} \phi_{x_{mj}\theta, x_{1j} \cdots x_{mj} w_2 \theta} \phi_{x_{1j} \cdots x_{mj} w_2 \theta, x_1 \cdots x_n w_1 \theta} w_2 \mid x_i \in X_i\} \\
&= \{x_{1j} \phi_{x_{1j}\theta, x_{1j} \cdots x_{mj} w_2 \theta} \cdots x_{mj} \phi_{x_{mj}\theta, x_{1j} \cdots x_{mj} w_2 \theta} w_2 \phi_{x_{1j} \cdots x_{mj} w_2 \theta, x_1 \cdots x_n w_1 \theta} \mid x_i \in X_i\} \\
&= \{x_{1j} \cdots x_{mj} w_2 \phi_{x_{1j} \cdots x_{mj} w_2 \theta, x_1 \cdots x_n w_1 \theta} \mid x_i \in X_i\}.
\end{aligned}
$$

It must be shown that this latter set is $X_{1j} \cdots X_{mj} w_2$. Certainly the typical element $x_{1j} \cdots x_{mj} w_2$ of $X_{1j} \cdots X_{mj} w_2$, with $x_{ij} \in X_{ij}$, takes the form $x_{1j} \cdots x_{mj} w_2 \phi_{x_{1j} \cdots x_{mj} w_2 \theta, x_1 \cdots x_n w_1 \theta}$ on choosing x_k such that $x_k \theta = x_{1j}\theta$ for k not in the image of the injection j, this being possible since $X_k \theta = X_{1j}\theta$. Conversely, consider $x_{1j} \cdots x_{mj} w_2 \phi_{x_{1j} \cdots x_{mj} w_2 \theta, x_1 \cdots x_n w_1 \theta}$ for x_i in X_i. Since $x_1 \cdots x_n w_1 \theta \in K = K \cdots K w_2 = X_{1j}\Theta \cdots X_{mj}\Theta w_2 = X_{1j} \cdots X_{mj} w_2 \Theta$, there are elements Y_{kj} of X_{kj} (for $k \in \{1, ..., m\}$) such that $x_1 \cdots x_n w_1 \theta = y_{1j} \cdots y_{mj} w_2 \theta$. Then by (3.2), $x_{1j} \cdots x_{mj} w_2 \phi_{x_{1j} \cdots x_{mj} w_2 \theta, x_1 \cdots x_n w_1 \theta} = x_{1j} \cdots x_{mj} w_2 \phi_{x_{1j} \cdots x_{mj} w_2 \theta, y_{1j} \cdots y_{mj} w_2 \theta} = y_{1j} \cdots y_{mj} w_2 x_{1j} \cdots x_{mj} w_2 p = y_{1j} x_{1j} p \cdots y_{mj} x_{mj} x_{mj} p w_2 \in X_{1j} \cdots X_{mj} w_2$, as required. ▮

5. Decomposition of Bisemilattices over Semilattices

The previous section showed that, given the projection $\theta: A \to S$ of the idempotent entropic algebra A onto its greatest semilattice quotient, then the IEO-semilattice $(\mathrm{Fg}\, A, +, \Omega)$ could be projected onto the meet-distributive bisemilattice $(\mathrm{Fg}\, S, +, \cdot)$. To decompose Fg A further one may decompose Fg S. This section and the next investigate such decompositions and some corresponding construction methods.

One decomposition introduced in [20, Corollary 3.5] arises on noting that $(S, \cdot, \cdot)$ is a meet-distributive bisemilattice, and then applying [20, Theorem 3.1] or [21, Theorem 6.1] to the identity mapping on S. This produces the homomorphism

$$\pi: (\mathrm{Fg}\, S, +, \cdot) \to (S, \cdot, \cdot);\ K \mapsto \prod_{k \in K} k.$$

The following theorem shows how the generalised Płonka sum of Theorem 2.2 may be used to recover $(\mathrm{Fg}\, S, +, \cdot)$ from this decomposition.

THEOREM 5.1. *For $s \leqslant_. t$ in S, define $\phi_{t,s}\colon \pi^{-1}(t) \to \pi^{-1}(s)$; $K \mapsto K \cup \{s\}$. Then the mappings $\phi_{t,s}$ yield* (Fg S, $+, \cdot$) *as the generalised Plonka sum of the meet-distributive bisemilattice* $(\pi^{-1}(s), +, \cdot)$ *over S as in Theorem 2.2.*

Proof. With ω respectively as $+$ and $\cdot$, and with $K_i \in \pi^{-1}(s_i)$ for $i = 1, 2$, the corresponding diagrams (2.1) on the element level are

$$\begin{array}{ccc} (K_1 \cup \{s\}, K_2 \cup \{s\}) & \leftarrow\!\!\dashv (K_1, K_2) \mapsto & (K_1 \cup \{s_1 s_2\}, K_2 \cup \{s_1 s_2\}) \\ \downarrow & & \downarrow \\ K_1 \cup K_2 \cup K_1 \cdot K_2 \cup \{s\} & \longleftarrow\!\!\dashv & K_1 \cup K_2 \cup K_1 \cdot K_2 \end{array}$$

and

$$\begin{array}{ccc} (K_1 \cup \{s\}, K_2 \cup \{s\}) & \leftarrow\!\!\dashv (K_1, K_2) \mapsto & (K_1 \cup \{s_1 s_2\}, K_2 \cup \{s_1 s_2\}) \\ \downarrow & & \downarrow \\ K_1 \cdot K_2 \cup \{s\} & \longleftarrow\!\!\dashv & K_1 \cdot K_2. \end{array}$$

Thus the conditions of Theorem 2.2 are satisfied. Further, since the upper right hand corner of (2.1) gives the action of ω in the generalised Płonka sum, these diagrams show that (Fg S, $+, \cdot$) is the algebra constructed. ∎

6. DECOMPOSITION OF BISEMILATTICES OVER LATTICES

The decomposition of (Fg S, $+, \cdot$) (and hence of (Fg A, $+, \Omega$)) studied in the previous section was over the semilattice $(S, \cdot, \cdot)$, considered as a meet-distributive bisemilattice. This section investigates a decomposition over a meet-distributive bisemilattice that is a distributive lattice, together with a corresponding construction method.

A subsemilattice U of the semilattice S is said to be *initial* if $U = \downarrow U$, the set of elements of S lying below elements of U. The complex product $U_1 \cdot U_2$ of two initial subsemilattices U_1, U_2 is just their set-theoretic intersection $U_1 \cap U_2$, since for elements u_i of U_i, $u_1 u_2 \leqslant_. u_i$, whence $u_1 u_2 \in U_1 \cap U_2$, and conversely each element u of $U_1 \cap U_2$ is the element $u = u \cdot u$ of $U_1 \cdot U_2$. It follows that the subsemilattice $U_1 + U_2 = U_1 \cup U_2 \cup U_1 \cdot U_2$ of S generated by U_1 and U_2 is just their set-theoretic union $U_1 \cup U_2$. In particular, the initial subsemilattices form a sub-bisemilattice (In S, $+, \cdot$) of (Sub S, $+, \cdot$) that is also a sublattice of the distributive power-set lattice $(2^S, \cup, \cap)$ of the set S.

THEOREM 6.1. *For a semilattice* $(S, \cdot)$, *there is a meet-distributive bisemilattice homomorphism*

$$\downarrow : (\mathrm{Sub}\, S, +, \cdot) \to (\mathrm{In}\, S, +, \cdot);\ K \mapsto \downarrow K$$

onto the distributive lattice $(\mathrm{In}\, S, +, \cdot)$ *of initial subsemilattices of S. For each initial subsemilattice U, the fibre* $(\downarrow^{-1}(U), +, \cdot)$ *is a semilattice considered as a meet-distributive bisemilattice.*

Proof. Let K_1, K_2 be subsemilattices of S. Then $\downarrow(K_1 \cdot K_2) = \{s \in S \mid \exists k_i \in K_i . s \leqslant k_1 k_2 \leqslant k_i\} = \downarrow K_1 \cap \downarrow K_2 = \downarrow K_1 \cdot \downarrow K_2$, so that $\downarrow$ is a $\cdot$-homomorphism. Also, $\downarrow(K_1 + K_2) = \downarrow(K_1 \cup K_2 \cup K_1 \cdot K_2) = \{s \in S \mid \exists k_i \in K_i . s \leqslant k_1$ or $s \leqslant k_2$ or $s \leqslant k_1 \cdot k_2\} = \{s \in S \mid \exists k_i \in K_i . s \leqslant k_1$ or $s \leqslant k_2\} = \downarrow K_1 \cup \downarrow K_2 = \downarrow K_1 + \downarrow K_2$, so that $\downarrow$ is a $+$-homomorphism. Now suppose that $\downarrow K_1 = \downarrow K_2$. Then $k_1 \in K_1 \Rightarrow k_1 \in \downarrow K_1 = \downarrow K_2 \Rightarrow \exists k_2 \in K_2 . k_1 \leqslant k_2$, i.e., $k_1 = k_1 k_2$, so that $K_1 \leqslant K_1 \cdot K_2$, and similarly $K_2 \leqslant K_1 \cdot K_2$. Thus $K_1 + K_2 = K_1 \cup K_2 \cup K_1 \cdot K_2 \subseteq K_1 \cdot K_2 \subseteq K_1 \cup K_2 \cup K_1 \cdot K_2 = K_1 + K_2$. ▌

COROLLARY 6.2. *Two subsemilattices K, L of S lie in a subsemilattice of* $(\mathrm{Sub}\, S, +, \cdot)$ *iff they lie in a* $\downarrow$*-fibre.*

Proof. Suppose $K <_+ L$ and $K >_\cdot L$. Then $K \subseteq L$ implies $\downarrow K \subseteq \downarrow L$. Conversely, let $x \in \downarrow L$, say $x \leqslant l \in L$. Then $K \cdot L = L \Rightarrow \exists k \in K,\ l' \in L$. $k \cdot l' = 1$, i.e., $l \leqslant k$. Thus $x \leqslant k$, so that $x \in \downarrow K$. ▌

The decomposition of $(\mathrm{Sub}\, S, +, \cdot)$ given by Theorem 6.1 induces a decomposition of its subalgebra $(\mathrm{Fg}\, S, +, \cdot)$. The image of $\mathrm{Fg}\, S$ in $\mathrm{In}\, S$ under $\downarrow$ consists of the set $\mathrm{Fmi}\, S$ of those initial subsemilattices U of S having a finite set $M(U)$ of maximal elements, since $M(\downarrow K) \subseteq K$ for K in $\mathrm{Fg}\, S$. Note that $\mathrm{Fmi}\, S$ is a sublattice of $(\mathrm{In}\, S, +, \cdot)$. Proposition 6.4 below shows that each $\downarrow$-fibre in $\mathrm{Fg}\, S$ has a lower semimodular lattice structure. The following lemma is a necessary preliminary.

LEMMA 6.3. *If K covers L in* $(\mathrm{Fg}\, S, \leqslant_+)$, *then there is an element k of K such that* $K = L \mathbin{\dot\cup} \{k\}$.

Proof. Since $K >_+ L$, there is an element k of K not in L. Since K covers L, $K = L + \{k\} = L \cup \{k\} \cup kL$. Suppose $kL \nsubseteq L \cup \{k\}$, i.e., $\exists l \in L$. $kl \notin L \cup \{k\}$. Then $L <_+ L + \{kl\}$, and further $L + \{kl\} <_+ L + \{k\}$, since otherwise $k \in L + \{kl\} = L \cup \{kl\} \cup klL = L \cup klL \Rightarrow \exists l' \in L.\ k = kll' \Rightarrow k \leqslant kl \leqslant k \Rightarrow kl = k \in L \cup \{k\}$, contrary to the choice of l. But $L <_+ L + \{kl\} <_+ L + \{k\} = K$ contradicts the hypothesis that K cover L, so that $kL \subseteq L \cup \{k\}$ after all. ▌

PROPOSITION 6.4. *The set of finite (possibly empty) subsemilattices of S forms a lower semimodular lattice* $(\mathrm{Fg}\, S \cup \{\varnothing\}, +, \cap)$ *under + and set-theoretic intersection. Each fibre* $\downarrow^{-1}(U)$ *in* $(\mathrm{Fg}\, S, +, \cdot)$ *forms a sublattice* $(\downarrow^{-1}(U), +, \cap)$ *of this lattice, having*

$$h\colon \downarrow^{-1}(U) \to N;\ K \mapsto |K| - |\langle M(U)\rangle|$$

as a height function.

Proof. Suppose $K+L$ covers L in $(\mathrm{Fg}\, S, \leqslant_+)$. For lower semimodularity it must be shown that K covers $K \cap L$ [1, II.2.8; 2, Sect. 8]. By Lemma 6.3, $K+L = L \cup \{k\}$ for some element k of $K+L$. Then $K - \{k\} \subseteq (K+L) - \{k\} = L$, so that $K - \{k\} \subseteq K \cap L$. Conversely $K \cap L \subseteq L \not\ni k$, so that $K \cap L = K - \{k\}$. It then follows immediately that K covers $K \cap L$.

Now consider $\downarrow^{-1}(U)$ in $\mathrm{Fg}\, S$ for an initial subsemilattice U of S with a finite non-empty set $M(U)$ of maximal elements. Since $\downarrow K = U$ iff $M(K) = M(U)$, the fibre is closed under set-theoretic intersection, and thus forms a sublattice of $(\mathrm{Fg}\, S \cup \{\varnothing\}, +, \cap)$. The explicit form of the height function [2, Sect. I.3] is a consequence of Lemma 6.3 and the fact that $\langle M(U)\rangle$ is the least element of $(\downarrow^{-1}(U), +, \cap)$. ∎

Theorem 6.8 below gives a construction method for recovering $(\mathrm{Fg}\, S, \cdot)$ from the semilattice $(\mathrm{Fg}\, S, +)$ and the decomposition of $\mathrm{Fg}\, S$ obtained from Theorem 6.1. The method works under appropriate conditions as follows. The partial order $(\mathrm{Fg}\, S, \leqslant_+)$ is locally finite [1, p. 9], so one may consider its Hasse diagram [1, p. 10], the directed graph of its covering relation. The arrows of this graph connecting vertices lying in the same $\downarrow$-fibre are coloured red (for "revolutionary"). The other arrows are coloured blue (for "conservative"). One then reverses the direction of the red arrows, and preserves the direction of the blue arrows, to obtain a new directed graph. The reflexive transitive closure of the corresponding relation on $\mathrm{Fg}\, S$ turns out to be the relation $\leqslant_.$, and thus determines $(\mathrm{Fg}\, S, \cdot)$. A finiteness hypothesis is needed for $(\mathrm{Fg}\, S, \leqslant_.)$ to be obtainable in this way: for simplicity local finiteness of the partial order on the meet semilattice $(S, \cdot)$ is assumed.

PROPOSITION 6.5. *The partial order* $\leqslant$ *on the meet semilattice* $(S, \cdot)$ *is locally finite if and only if the partial order* $\leqslant_.$ *on* $(\mathrm{Fg}\, S, \cdot)$ *is.*

Proof. The partial order $(S, \leqslant)$ is embedded in $(\mathrm{Fg}\, S, \leqslant_.)$ [20, (1.4); 21, (2.2)], so $(S, \leqslant)$ is locally finite if $(\mathrm{Fg}\, S, \leqslant_.)$ is. Conversely, suppose $(S, \leqslant)$ is locally finite, and consider $K <_. L$ in $\mathrm{Fg}\, S$. Let k_0 denote the minimum element $K\pi$ of K, and $l_1, ..., l_n$ the set of maximal elements of L. For an element h of a finite subsemilattice H of S with $K \leqslant_. H$, one has that

$k_0 h \in KH = K$, so that $k_0 \leqslant k_0 h \leqslant h$. In particular $k_0 \leqslant l_i$ for each i. Now by the local finiteness of $(S, \leqslant)$, each interval $[k_0, l_i] = \{s \in S \mid k_0 \leqslant s \leqslant l_i\}$ is finite. Suppose further that $K \leqslant_. H \leqslant_. L$. Then for h in $H = HL$, there is some l in L with $h \in Hl$, i.e., $h \leqslant l$. Thus h lies below some maximal element l_i of L, so that $h \in [k_0, l_i]$. Summarising, each finite subsemilattice H of S in the interval $[K, L]$ of $(\mathrm{Fg}\, S, \leqslant_.)$ is a subset of the finite set $\bigcup_{i=1}^{n} [k_0, l_i]$. Since there are only finitely many such subsets, the interval $[K, L]$ is finite, as required. ∎

Note that if $(S, \cdot)$ is the semilattice $\mathbb{R}$ with $xy = \min(x, y)$, then the interval $[\{0\}, \{1\}]$ of $(\mathrm{Fg}\, S, \leqslant_.)$ contains $\{x\}$ for each x in $[0, 1]$. Here $(\mathrm{Fg}\, S, \leqslant_+)$ is locally finite, as usual, but $(\mathrm{Fg}\, S, \leqslant_.)$ is not.

LEMMA 6.6 [19, Corollary 2.5]. *If K covers L in one of the partial orders $\leqslant_+$, $\leqslant_.$ on* Fg S, *then K and L are comparable in the other.*

Proof. (i). Suppose K covers L in $\leqslant_+$. Then $L <_+ K$, i.e., $L + K = K$, implies $KL + K = K$ and $L + LK = LK$, whence $L \leqslant_+ KL \leqslant_+ K$. Since K covers L, either $L = KL$, i.e., $L <_. K$, or $K = KL$, i.e., $K <_. L$.

(ii) Suppose K covers L in $\leqslant_.$. Then $L <_. K$, i.e., $KL = L$, implies $K + L = K + KL = K(K + L)$ and $L(K + L) = LK + L = L$, whence $L \leqslant_. K + L \leqslant_. K$. Since K covers L, either $L = K + L$, i.e., $K <_+ L$, or $K = K + L$, i.e., $L <_+ K$.

LEMMA 6.7. *If K covers L in the partial order $\leqslant_.$ on* Fg S, *and $\downarrow K \neq \downarrow L$, then K also covers L in the partial order $\leqslant_+$.*

Proof. By Lemma 6.6, K and L are comparable in $\leqslant_+$. By Corollary 6.2, $L \leqslant_+ K$. Since $(\mathrm{Fg}\, S, \leqslant_+)$ is locally finite, there is a subsemilattice K' such that $L <_+ K' \leqslant_+ K$ and K' covers L in $\leqslant_+$. By Lemma 6.6 either $L <_. K'$ or $K' <_. L$. Now if $K' <_. L$, i.e., $LK' = K'$, then $L = LK = L(K + K') = L + K' = K'$, a contradiction. Thus $L <_. K'$. Further, since $L <_. K$, one has $L \leqslant_. KK' \leqslant_. K$, whence $KK' = K$ or $KK' = L$. But $KK' = L$ is impossible, since $KK' = (K + K')K' = KK' + K'$, i.e., $K' \leqslant_+ KK'$, and $L <_+ K'$. Thus $KK' = K$, i.e., $K \leqslant_+ K'$. By Corollary 6.2, $\downarrow K' = \downarrow K$. By Lemma 6.3 there is an element m of K' such that $K' = L \,\dot\cup\, \{m\}$. Since $\downarrow K' = \downarrow K \neq \downarrow L$, the element m is a maximal element of K. Let $H = K - \{m\}$, a subsemilattice of S. Then $HK = H(H \,\dot\cup\, \{m\}) = H \cup Hm = H$, so that $H <_. K$. Further, $L \subseteq HL \subseteq KL = L$, whence $L \leqslant_. H$. Since K covers L in $\leqslant_.$, it follows that $L = H = K - \{m\}$, i.e., $K' = K$. Since K' was chosen to cover L in $\leqslant_+$, one has that K covers L in $\leqslant_+$, as required. ∎

THEOREM 6.8. *Let the partial order $\leqslant$ on the meet semilattice $(S, \cdot)$ be locally finite. Let* ker$\downarrow$ *be the kernel congruence of the meet-distributive* *bisemilattice homomorphism* $\downarrow : (\operatorname{Fg} S, +, \cdot) \to (\operatorname{In} S, +, \cdot)$. *Define relations* $<_r$ *and* $<_b$ *on* $\operatorname{Fg} S$ *by setting* $<_r = <_r \cap \ker \downarrow$, *and* $L <_b K$ *iff* K *covers* L *in* $\leqslant_+$ *and* $\downarrow K \neq \downarrow L$. *Then the relation* $<_{\bullet}$ *on* $\operatorname{Fg} S$ *is the transitive closure* $\operatorname{Tr}(<_b \cup >_r)$ *of the union of* $<_b$ *and the converse* $>_r$ *of* $<_r$.

Proof. By Theorem 6.1 the $\downarrow$-fibres are semilattices, so $>_r \subseteq <_{\bullet}$. Suppose $L <_b K$. By Corollary 6.2, $L \not>_{\bullet} K$, so by Lemma 6.6, $L <_{\bullet} K$. Thus $<_b \cup >_r \subseteq <_{\bullet}$. Since $<_{\bullet}$ is transitive, it follows that $\operatorname{Tr}(<_b \cup >_r) \subseteq <_{\bullet}$.

Conversely, suppose that $K <_{\bullet} K'$. By Proposition 6.5 there is a positive integer n and a chain of finite subsemilattices K_i of S such that $K = K_0 <_{\bullet} K_1 <_{\bullet} \cdots <_{\bullet} K_{n-1} <_{\bullet} K_n = K'$, where each K_i covers K_{i-1} in $\leqslant_{\bullet}$ for $i = 1, \ldots, n$. Then by Lemma 6.6 either $K_{i-1} >_+ K_i$ or $K_{i-1} <_+ K_i$. If $K_{i-1} >_+ K_i$, then by Corollary 6.2, $K_{i-1} >_r K_i$. Otherwise, by Lemma 6.7, K_i covers K_{i-1} in $\leqslant_+$, so that $K_{i-1} <_b K_i$. Thus $<_{\bullet} \subseteq \operatorname{Tr}(<_b \cup >_r)$. ∎

7. Decompositions of IEO-Semilattices over Lattices

For an idempotent entropic algebra A with projection $\theta: A \to S$ onto its greatest semilattice quotient, Theorem 4.1 decomposed the IEO-semilattice $(\operatorname{Fg} A, +, \Omega)$ over the meet-distributive bisemilattice $(\operatorname{Fg} S, +, \cdot)$. Then Theorem 6.1 induced a decomposition of $(\operatorname{Fg} S, +, \cdot)$ over the distributive lattices $(\operatorname{Fmi} S, +, \cdot)$. Putting these two decompositions together gives a distributive lattice decomposition

$$\alpha: (\operatorname{Fg} A, +, \Omega) \to (\operatorname{Fmi} S, +, \Omega);\ G \mapsto \downarrow G\theta \tag{7.1}$$

of the IEO-semilattice $(\operatorname{Fg} A, +, \Omega)$. The current section examines two construction methods associated with this decomposition: a Płonka sum construction for the α-fibres, and the promised recovery of the semilattice $(\operatorname{Fg} A, +)$ from the decomposition in Theorem 4.1 paralleling the recovery of the Ω-reduct $(\operatorname{Fg} A, \Omega)$ given in Theorem 4.2. The Płonka sum construction for the α-fibres is given by the following theorem.

Theorem 7.1. *Let* **V** *be a variety of idempotent entropic algebras having non-regular fibres over semilattices. Let* A *be a* **V**-*algebra, and* $\theta: A \to S$ *the projection onto its greatest semilattice quotient. Let* U *be an initial subsemilattice of* S *with only finitely many maximal elements. Then the* IEO-*semilattice* $(\alpha^{-1}(U), +, \Omega)$ *is a Płonka sum over the subsemilattice* $(\downarrow^{-1}(U), \cdot)$ *of* $(\operatorname{Fg} S, \cdot)$.

Proof. Theorem 4.2 shows that there is a functor F from $(\downarrow^{-1}(U), \cdot)$ to the category of Ω-algebras by which $(\alpha^{-1}(U), \Omega)$ is a Płonka sum. It remains to show that F is a functor to the corresponding category of

IEO-semilattices by which $(\alpha^{-1}(U), +, \Omega)$ is a Płonka sum. Certainly each $(\Theta^{-1}(K), +)$ is a semilattice, and by Corollary 4.3 the $\Phi_{K,L}$ for $L \leqslant_{\bullet} K$ are $+$-homomorphisms. Now suppose G° and G' are finitely generated subalgebras of A with $G^\circ\alpha = G'\alpha$. Then $\downarrow(G^\circ\theta) = \downarrow(G'\theta)$, whence by Theorem 6.1 $G^\circ\theta \cdot G'\theta = G^\circ\theta + G'\theta$. It must be shown that $G^\circ + G'$ in $(\mathrm{Fg}\,A, +)$ is $G^\circ\Phi_{G^\circ\theta, G^\circ\theta\cdot G'\theta} + G'\Phi_{G'\theta, G^\circ\theta\cdot G'\theta}$ in $\Theta^{-1}(G^\circ\theta\cdot G'\theta)$. But this latter sum is just

$$\{g_1\phi_{g_1\theta,\, g_1\theta\cdot g_1^\circ\theta\cdot g_1'\theta} \cdots g_n\phi_{g_n\theta,\, g_n\theta\cdot g_n^\circ\theta\cdot g_n'\theta}\omega \mid \omega\in\Omega,\ \mathrm{ar}(\omega)=n;\ g_i\in G^\circ\cup G',\ g_i^j\in G^j\}$$

or, by (4.2),

$$\{g_1\cdots g_n\omega\phi_{g_1\cdots g_n\omega\theta,\, g_1\cdots g_n\omega\theta\cdot g_1^\circ\cdots g_n^\circ\omega\theta\cdot g_1'\cdots g_n'\omega\theta} \mid \omega\in\Omega,\ \mathrm{ar}(\omega)=n;\ g_i\in G^\circ\cup G',\ g_i^j\in G^j\}.$$

By (3.2) the typical element of this set is

$$g_1\cdots g_n\omega\phi_{g_1\cdots g_n\omega\theta,\, g_1\cdots g_n\omega\theta\cdot g_1^\circ\cdots g_n^\circ\omega\theta\cdot g_1'\cdots g_n'\omega\theta} = g_1\cdots g_n\omega g_1^\circ\cdots g_n^\circ\omega p g_1'\cdots g_n'\omega p g_1\cdots g_n\omega p \in G^\circ + G'.$$

Conversely, suppose $g_1\cdots g_n\omega\in G^\circ + G'$. Now $g_i\in G^\circ\cup G' \Rightarrow g_i\theta\in G^\circ\theta\cup G'\theta\subseteq G^\circ\theta\cdot G'\theta \Rightarrow \exists g_i^j\in G^j\cdot g_i\theta = g_i^\circ\theta\cdot g_i'\theta$. Then $g_1\cdots g_n\omega\theta = g_1\cdots g_n\omega\theta\cdot g_1^\circ\cdots g_n^\circ\omega\theta\cdot g_1'\cdots g_n'\omega\theta$, and $g_1\cdots g_n\omega = g_1\cdots g_n\omega\phi_{g_1\cdots g_n\omega\theta,\, g_1\cdots g_n\omega\theta} = g_1\cdots g_n\omega\phi_{g_1\cdots g_n\omega\theta,\, g_1\cdots g_n\omega\theta\cdot g_1^\circ\cdots g_1^\circ\omega\theta\cdot g_1'\cdots g_n'\omega\theta} \in G^\circ\Phi_{G^\circ\theta, G^\circ\theta\cdot G'\theta} + G'\Phi_{G'\theta, G^\circ\theta\cdot G'\theta}$, completing the proof of the required equality. ∎

The construction method for recovering $(\mathrm{Fg}\,A, +)$ from the decomposition in Theorem 4.1 is based on the "revolutionary/conservative" method of Theorem 6.8, and thus demands a finiteness assumption. It also uses the functor yielding the Płonka sum in Theorem 4.2, in particular regarding the (graphs of the) homomorphisms $\Phi_{K,L}$ arising there for $L \leqslant_{\bullet} K$ as subsets of $\Theta^{-1}(K)\times\Theta^{-1}(L)$. Theorem 7.1 supplies the information about the $+$-reducts of the α-fibres needed to apply Theorem 7.2.

THEOREM 7.2. *Let* **V** *be a variety of idempotent entropic algebras having non-regular fibres over semilattices. Let A be a* **V**-*algebra having $\theta\colon A\to S$ as projection onto its greatest semilattice quotient. Let $>_B$ denote the union of the graphs of the homomorphisms $\Phi_{K,L}$ for those ordered pairs (K, L) of finite subsemilattices of S for which K covers L in* $(\mathrm{Fg}\,S, \leqslant_+)$ *and $\downarrow K \neq \downarrow L$. Let* $\ker\alpha$ *be the kernel congruence of the homomorphism α of* (7.1), *and let $>_R = >_+ \cap \ker\alpha$. Then if the order relation $>_+$ on* $\mathrm{Fg}\,A$ *is locally finite, it is obtained as the transitive closure* $\mathrm{Tr}(>_B\cup >_R)$ *of the union of $>_B$ and $>_R$.*

Proof. Clearly $>_R \subseteq >_+$. Suppose $G >_B G'$ for finitely generated subalgebras G, G' of A. Let $K = G\theta$, $L = G'\theta$. Then K covers L in $(\mathrm{Fg}\, S, \leqslant_+)$ and $\downarrow K \neq \downarrow L$, so by Lemma 6.6 and Corollary 6.2, $L <_\cdot K$. Further, $G' = G\Phi_{K,L} = \{g\phi_{g\theta, g\theta \cdot l} \mid g \in G, \quad l \in L\}$. Consider the typical element $g\phi_{g\theta, g\theta \cdot l}$ of G'. Now $g\theta \cdot l \in K \cdot L = L \subset K = G\theta$ implies there is an element g' of G with $g'\theta = g\theta \cdot l$. Then by (3.2), $g\phi_{g\theta, g\theta \cdot l} = g\phi_{g\theta, g'\theta} = g'gp \in G$. Thus $G >_+ G'$. This shows that $>_B \cup >_R = >_+$, whence $\mathrm{Tr}(>_B \cup >_R) \subseteq >_+$, since $>_+$ is transitive.

Conversely, suppose G and G' are finitely generated subalgebras of A for which G covers G' in $(\mathrm{Fg}\, A, \leqslant_+)$. If $G\alpha = G'\alpha$, then $G >_R G'$. If not, then $G\alpha \neq G'\alpha$, so that $\downarrow K \neq \downarrow L$ for $K = G\theta$, $L = G'\theta$. Since $G >_+ G'$, it follows that $K >_+ L$. Further, K must cover L in $(\mathrm{Fg}\, S, \leqslant_+)$, for if not, say $K >_+ H >_+ L$, one may choose H to cover L, and then $H = L \dot{\cup} \{h\}$ by Lemma 6.3. Since $h \in K = G\theta$, there is element g of G with $g\theta = h$. Then $(G' + \{g\})\theta = L + \{h\} = H$, so $G >_+ G' + \{g\} >_+ G'$, a contradiction. Now since K covers L in $\leqslant_+$, one has that $G >_B G\Phi_{K,L}$. It will be shown that $G' = G\Phi_{K,L}$. Certainly, since $>_B \subseteq >_+$, one has that $G >_+ G\Phi_{K,L}$. Now let g be an element of G'; as such, it is also an element of G. Further, $g\theta \in G'\theta = L$. Then $g = g\phi_{g\theta, g\theta} = g\phi_{g\theta, g\theta \cdot g\theta} \in G\Phi_{K,L}$. Thus $G >_+ G\Phi_{K,L} \geqslant G'$. Since G covers G' in $(\mathrm{Fg}\, A, \leqslant_+)$, one has that $G\Phi_{K,L} = G'$, whence $G >_B G'$. Summarising, if G covers G' in $(\mathrm{Fg}\, A, \leqslant_+)$, then $G(>_R \cup >_B)G'$. By the local finitenes of $>_+$, it follows that $>_+ \subseteq \mathrm{Tr}(>_R \cup >_B)$, as required to complete the proof. ∎

8. An Illustration

Let $\mathbf{V}$ be the variety of right normal bands, i.e., of idempotent semigroups satisfying the identity $xyz = yxz$. By [7, Corollary IV.5.18], this variety has non-regular fibres over semilattices, the fibres being right zero semigroups satisfying the identity $xy = y$. Thus Theorem 3.2 applies, the operation p of (3.1) just being multiplication. The top of Fig. 1 displays the right normal band $A = \{a, b, c, d, e, f\}$ as a Płonka sum of right zero semigroups $\{a, b\}$, $\{c, d\}$, $\{e, f\}$ over the semilattice $S = \{s, t, u\}$. For example, $cf = c(f\phi_{u,t}) = cd = d$. The "boxes" are the θ-fibres. Below this, Fig. 1 shows the Hasse diagrams of the join and meet reducts of the meet-distributive bisemilattice $(\mathrm{Fg}\, S, +, \cdot)$. The subsemilattices, such as $\{s, u\}$, are written as concatenations of their elements, such as su. The projection π onto $(S, \cdot, \cdot)$ as in Section 5 is shown by projecting the Hasse diagrams towards the bottom right hand corner, while projection onto the three-element chain $(\mathrm{Fmi}\, S, +, \cdot)$ as in Section 6 is shown by projecting the Hasse diagrams towards the bottom left hand corner. Covering pairs

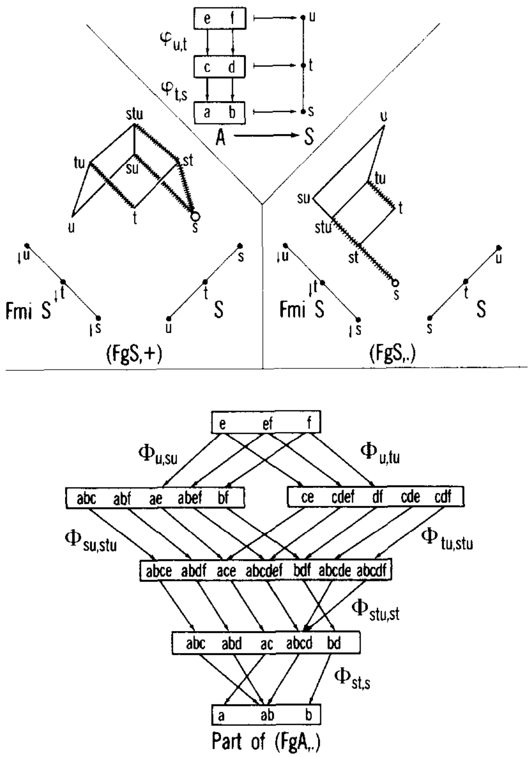

FIGURE 1

related by $<_r$ as in Theorem 6.8 are joined by plain lines, and pairs related by $<_b$ are joined by crossed lines. As an illustration of Theorem 6.8, one obtains $\{s\} <_{\cdot} \{s,u\}$ from $\{s\} <_b \{s, t\} <_b \{s, t, u\} >_r \{s, u\}$. The bottom of Fig. 1 displays part of (Fg $A, \cdot$) as a Płonka sum over (Fg $S, \cdot$), according to Theorem 4.2. For clarity $\Theta^{-1}(\{t\})$ has been omitted. By Corollary 4.5 the Θ-fibres, represented by the "boxes," are themselves right zero semigroups, since the identity $xy = y$ is subregular and linear. As an illustration of Theorem 7.1, one obtains $\{a, b, e\} + \{d, f\} = \{a, b, c, e\} + \{b, d, f\} = \{a, b, c, d, e, f\}$. As an illustration of Theorem 7.2, one obtains $\{a, b, e\} >_+ \{a\}$ from $\{a, b, e\}\ \Phi_{su,s} = \{a, b\} >_R \{a\}$.

ACKNOWLEDGMENT

This paper was written while the first author held a Fellowship from the Alexander von Humboldt Foundation at the Technische Hochschule Darmstadt.

REFERENCES

1. M. AIGNER, "Kombinatorik, I," Springer, Berlin/Heidelberg/New York, 1975.
2. G. BIRKHOFF, "Lattice Theory," 3rd ed., Amer. Math. Soc., Providence, RI, 1967.
3. R. H. BRUCK, "A Survey of Binary Systems," Springer, Berlin/Heidelberg/New York, 1958.
4. P. M. COHN, "Universal Algebra," Harper and Row, New York, 1965.
5. J. DUDEK, Varieties of idempotent commutative groupoids, manuscript.
6. G. GRÄTZER AND J. PŁONKA, A characterization of semilattices, *Colloq. Math.* **22** (1970), 21–24.
7. J. M. HOWIE, "An Introduction to Semigroup Theory," Academic Press, London, 1976.
8. J. JEŽEK AND T. KEPKA, Idealfree CIM-groupoids and open convex sets, manuscript.
9. B. JÓNSSON AND E. NELSON, Relatively free products in regular varieties, *Algebra Universalis* **4** (1974), 14–19.
10. N. KIMURA AND M. YAMADA, Note on idempotent semigroups, II, *Proc. Japan. Acad.* **34** (1958), 110–112.
11. S. MAC LANE, "Categories for the Working Mathematician," Springer, New York, 1971.
12. A. I. MAL'CEV, "Algebraic Systems," Springer, Berlin/Heidelberg/New York, 1973.
13. A. MITSCHKE, On a representation of groupoids as sums of directed systems, *Colloq. Math.* **28** (1973), 11–18.
14. J. PŁONKA, Diagonal algebras, *Fund. Math.* **58** (1966), 309–321.
15. J. PŁONKA, Remarks on diagonal and generalized diagonal algebras, *Colloq. Math.* **15** (1966), 19–23.
16. J. PŁONKA, On a method of construction of abstract algebras, *Fund. Math.* **61** (1967), 183–189.
17. J. PŁONKA, A representation theorem for idempotent medial algebras, *Fund. Math.* **61** (1967), 191–198.
18. J. PŁONKA, On equational classes of abstract algebras defined by regular equations, *Fund. Math.* **64** (1969), 241–247.
19. A. ROMANOWSKA, Subdirectly irreducible .-distributive bisemilattices, I, *Demonstratio Math.* **13** (1980), 767–785.
20. A. B. ROMANOWSKA AND J. D. H. SMITH, Bisemilattices of subsemilattices, *J. Algebra* **70** (1981), 78–88.
21. A. B. ROMANOWSKA AND J. D. H. SMITH, Subalgebra systems of idempotent entropic algebras, *J. Algebra* **120** (1989), 247–262.