QUANTIZATION OF WEB GEOMETRY: SEMISYMMETRIZATION OF LINEAR QUANTUM QUASIGROUPS

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Abstract. Classical quasigroups coordinatize structures called 3-nets in combinatorics, and 3-webs in geometry. The coordinatization is up to isotopy, a relation coarser than isomorphism. The semisymmetrization of a classical quasigroup is built on the cube of the underlying set of the quasigroup. Isotopic quasigroups have isomorphic semisymmetrizations.

Quantum quasigroups provide a self-dual unification (with both a multiplication and a comultiplication) of quasigroups and Hopf algebras, in the general setting of symmetric monoidal categories. Linear quantum quasigroups are quantum quasigroups in categories of vector spaces or modules over a commutative ring, with the direct sum as the Cartesian monoidal product.

With a view to addressing the quantization of web geometry, the paper determines linear quantum quasigroup structures that provide comultiplications to extend the semisymmetrization multiplication of a linear quasigroup. In particular, if the linear quasigroup structure comes from a real or complex affine plane, a complete classification of the quantum semisymmetric comultiplications is provided, based on the solution of a system of cubic equations.

Contents

2020 *Mathematics Subject Classification.* 20N05; 14C21, 16T99, 51D20, 81S30.

Key words and phrases. pregeometry; quantum quasigroup; quasigroup; web geometry; semisymmetric.

1. Introduction

1.1. **Motivation.** Classical web or net geometry is the study of spaces which decompose (globally or locally) in multiple ways as products of smaller spaces. Its structures manifest themselves in various settings. Within the combinatorial context of the category of sets, the "net" terminology is more established [1, 2],[5, pp.19,],[12, §15.1],[35, Def'n. I.4.2]. Also, Hall's "Latin square designs" are dual to 3-nets [12, Ch. 3]. The "web" terminology is generally used in algebraic or differential geometric settings [7, 11, 23]. Quasigroups, which retain the cancellativity of groups while relaxing the associativity, are the algebraic structures which emerge naturally in the coordinatization of web geometry (§2.4.4).

A prime motivation for the work of this paper is the ongoing search for a "pregeometry" [22, §§44.4–5] as a spacetime geometry that is quantized at the Planck length of 10^{-38} miles. Classical web geometry is well placed to provide a background for this search. It already underlies both quantum mechanics (compare, say, Wooters' use of nets [36, Fig. 2] in his approach to finite state Wigner functions) and general relativity (compare, say, the expression [11, p.269] of curvature within web geometry). Thus, the problem of quantizing web geometry emerges as a potentially important part of the general problem of appropriately quantizing spacetime.

Classical web geometry uses ordered pairs and projections to factors (§2.4.3). However, in the general quantum setting of a symmetric, monoidal structure, this approach is no longer available. An alternative is needed. In [29], the points of a classical web appear as categorical points (morphisms out of the terminal object) in a suitable category of semisymmetrizations of classical quasigroups (compare §2.4.5 below). Semisymmetrization of quantum quasigroups thus presents itself as one possible approach to the quantization of web geometry. The current paper initiates this program by examining the most amenable setting, Cartesian symmetric monoidal categories of modules over a commutative, unital ring.

1.2. **Quantum quasigroups.** Quasigroups and Hopf algebras represent two distinct extensions of the concept of a group. Like groups, quasigroups are set-theoretical objects with a cancellative multiplication. Unlike group multiplications, however, quasigroup multiplications are not required to be associative. On the other hand, Hopf algebras extend the group concept to a linear setting, say to a vector space A, with a linear *multiplication* $\nabla\colon A \otimes A \to A; x \otimes y \mapsto x \cdot y$ required to be associative. The concept of a Hopf algebra is self-dual, so along with the multiplication, there is a *comultiplication* $\Delta\colon A \to A \otimes A$ that is coassociative and compatible (mutually homomorphic) with the multiplication. In the context of this paper, where (co-)associativity is relaxed, the compatibility is expressed as a *bimagma condition* (2.3), (2.4).

Initial extensions of the concept of a Hopf algebra to comprise a non-associative multiplication, such as [3, 6, 20, 21, 24] for example, took what are now regarded as *semi-classical* approaches. Restricting themselves to the linearization of certain equationally defined classes of quasigroups, such as inverse-property loops [5, §III.4],[12, §2.4],[35, §I.4.1], these approaches impose a linearized version of the defining equations on the non-associative multiplication. For example, the inverse property is linearized by the *Hopf quasigroups* of Klim and Majid [20, Prop. 4.2(1)], while the *Moufang-Hopf algebras* of Benkhart *et al.* linearize Moufang properties [3, Def'n 1.2]. These semi-classical approaches lack the self-duality that is characteristic of Hopf algebras.

Quantum quasigroups were introduced [31] as a self-dual framework for the unification of quasigroups and Hopf algebras, within the general setting of any symmetric monoidal category. Viewed from the quasigroup side, they linearize an elegant characterization of quasigroups given by the topologist

I.M. James [18]. Viewed from the Hopf algebra side, they abstract the property (well-known to experts, but usually obscured under cohomological conditions) that each Hopf algebra A is an A-A-bi-Galois object [4, Ex. 1.2]. Much as groups are characterized as (non-empty) associative quasigroups, finite-dimensional Hopf algebras may be characterized as (co-)associative, (co-)unital quantum quasigroups [31, Th. 4.5].

Beyond the original work of [31], the theory of quantum quasigroups was analyzed further from the semi-classical standpoint in [16]. Combe, Manin and Marcolli have discussed operadic aspects in [8], while symmetry aspects were treated in [17]. Among quantum quasigroups, members of the recently identified class of quantum T-quasigroups [34] have a full and exact triality symmetry, as shown in Figure 2. This symmetry nicely matches the triality symmetry of classical equational quasigroups that is treated in §2.1.3. For clarification, it should be noted that "triality" here refers to the S_3-symmetry of the language of quasigroups, which is richer than the S_2-symmetry of the language of groups. This triality is related to, but distinct from, "triality" associated with the Coxeter-Dynkin diagram D_4, as discussed in [12], for example.

1.3. **Plan of the paper.** Starting with a summary of classical quasigroups in §2.1 and quantum quasigroups in §2.2, background for understanding the paper is presented in Chapter 2. To familiarize readers with the more recent "equational" approach to quantum quasigroups (§§2.2.1–2.2.4), Section 2.3 interprets classical quasigroups as equational quantum quasigroups. In the context of a discussion that involves equational quantum quasigroups, it is convenient to speak of "combinatorial" quantum quasigroups when referring to quantum quasigroups as originally introduced in [31]. The distinction is that, while Definition 2.8 of a combinatorial quantum quasigroup merely requires invertibility of the left and right composites (2.5),(2.6), equational quantum quasigroups incorporate additional structure actually witnessing the invertibility.

Web geometry, in conjunction with the fundamental classical concept of semisymmetrization, is introduced in §2.4. The remainder of Chapter 2 is devoted to *linear quantum quasigroups*: equational quantum quasigroups in the monoidal category $\big(\underline{\underline{S}}, \oplus, \{0\}\big)$ of S-modules under the biproduct $\oplus$ over a unital, commutative ring S. Quantum quasigroups in $\big(\underline{\underline{S}}, \oplus, \{0\}\big)$ are very well behaved. In particular, each combinatorial quantum quasigroup in $\big(\underline{\underline{S}}, \oplus, \{0\}\big)$ extends to a well-determined set of quantum T-quasigroups that are parametrized by an automorphism Ω of the underlying S-module (Corollary 2.36).

Chapter 3 introduces the main theme of the paper: semisymmetrization of linear quantum quasigroups. The basis is established in §3.1, taking the classical semisymmetrization of a linear quantum T-quasigroup Q, without regard for its comultiplication structure. The multiplicative part of the classical semisymmetrization is encoded succinctly in the *Rho-matrix* P (a capital Greek rho) of (3.3). In the sense of Definition 3.8(a), the Rho-matrix is *monomial*: a monomial matrix over a cyclic permutation matrix of order dividing three, with S-module automorphisms of Q as its non-zero entries.

In the linear setting of the category $\left(\underline{S}, \oplus, \{0\}\right)$, two main problems arise:

- Problem 3.4 seeks **general** combinatorial quantum quasigroups to extend the classical semisymmetrization;
- Problem 3.19 asks which **quantum semisymmetric** combinatorial quantum quasigroups extend the classical semisymmetrization.

It is convenient to refer to these problems respectively as the *easy* and the *hard* problem. Quantum semisymmetry, as demanded in the hard problem, is summarized in §3.2. It is a natural analogue of its classical counterpart presented in §2.1.4, to which it reduces in the Cartesian monoidal category $(\mathbf{Set}, \times, \{0\})$ of sets under the direct product.

A formal solution to Problem 3.4 is presented in Theorem 3.13, in terms of the dual L of the Rho-matrix P that encoded the multiplication. Thus, the matrix L encodes the comultiplication in dual fashion. As a consequence of the bimagma condition imposed on a combinatorial quantum quasigroup, the matrix L must lie in the commutant of the Rho-matrix P. The solution relies on Lemma 3.5 for a specification of these commutant matrices, in terms of the S-automorphisms of Q that appear as the nonzero entries of the Rho-matrix. The remaining condition, recognized in the theorem, is that L be invertible. Monomial matrices L that satisfy the requirements imposed by Theorem 3.13 are provided in Corollary 3.6. They include the identity matrix that yields the classical diagonal comultiplication.

The formal solution to Problem 3.19 that is presented in Theorem 3.20, as a specialization of Theorem 3.13, requires the comultiplication matrix L to satisfy the additional cubic matrix equation $L^3 = \Omega^{-1}$ (3.20). Here, the right hand side of the cubic equation involves the parameter Ω from the Five-Parameter Representation of the linear quantum T-quasigroup Q (Theorem 2.35). Technically, the right hand side of (3.20) denotes the "scalar matrix" $\Omega^{-1} \oplus \Omega^{-1} \oplus \Omega^{-1}$.

Following the statements and formal solutions of the two main problems in Chapter 3, the remainder of the paper is devoted to examining explicit solutions to the hard problem. Chapter 4 focuses on monomial solutions, as presented in Theorem 4.1. In particular, monomial solutions within the Cartesian monoidal category of abelian groups exhibited in Corollary 4.2 are available over any commutative, unital ring S. Theorem 4.4 shows that monomial solutions are rigid, resistant to any local deformation towards a non-monomial solution.

Section 4.3 begins the more geometrical considerations of the paper by examining the linear quantum quasigroup structure, within the category $(\underline{\underline{S}}, \oplus, \{0\})$, of the affine plane over the commutative ring S. Wooters' phase space example [36, Fig. 2] takes $S = \mathbb{Z}/_5$. As observed in Section 4.4, the entries of the Rho-matrix P, for the pencil of parallel lines with given (invertible) slope m, constitute the respective additive and multiplicative identity elements 0 and 1 of the ring S, together with the respective additive and multiplicative inverse elements $-m$ and m^{-1} of m. Over any ring, Theorem 4.1 identifies the monomial comultiplications given by $L = 1, \mathrm{P}^2$ and $-\mathrm{P}$. Section 4.5 records that these monomial comultiplications are the only comultiplications in the real case, and indeed for any subring of $\mathbb{R}$. On the other hand, Table 1 in Section 4.6 exhibits nine monomial solutions in the complex case, all of which are actually defined over the cyclotomic extension field $\mathbb{Q}(\zeta)$ for a primitive twelfth root ζ of unity.

Chapter 5 classifies solutions to the affine case of the hard problem over the complex numbers, and, as it transpires, over the cyclotomic field $\mathbb{Q}(\zeta)$. Essentially, the classification problem reduces to solution of the cubic matrix equation $L^3 = 1$ for commutants L of the Rho-matrix (4.9) with $m = 1$. Section 5.1 translates this classification problem to the location of common roots of the three cubics (5.2)–(5.4). Definition 5.3 identifies a group of isometries of the Hilbert space $\mathbb{C}^3$, the so-called *little Galois group* Γ, which permutes the set $\mathcal{R}_\mathbb{C}$ of common roots. Under the translation, Γ acts on the set of comultiplication matrices, with the set of nine monomial matrices from Table 1 as one orbit.

Table 2 in Section 5.2 presents 18 members of the solution set $\mathcal{R}_\mathbb{C}$ where there are no zero components. By Lemma 5.5, these solutions determine 18 distinct non-monomial comultiplication matrices over $\mathbb{C}$. Starting from the particular solution (5.8), Theorem 5.7 produces these 18 solutions as a single Γ-orbit. The full set of 27 solutions is represented graphically in Figure 4 of Section 5.3. Theorem 5.11 in Section 5.4 establishes that the cardinality of $\mathcal{R}_\mathbb{C}$ is exactly 27, thereby completing the classification.

Section 5.6 provides a comprehensive overview of the set $\mathcal{M}_\mathbb{C}$ of all 27 complex comultiplication matrices. As recorded by Theorem 5.15, they form an elementary abelian subgroup of the unitary group $\mathsf{U}(3)$, displayed in diagonalized form in Table 4. Using the classical theory of semisymmetric and totally symmetric quasigroups from §2.1.4, $\mathcal{M}_\mathbb{C}$ acquires the structure of a 3-dimensional affine space over $\mathbb{Z}/_3$. The little Galois group Γ is recognized as a subgroup $C_3^3 \rtimes C_2^2$ of the corresponding affine group (which itself may be considered as the full Galois group).

1.4. **Notational and other conventions.** As the default options, this paper follows the notational conventions of [35]. For example, if (X, S) denotes structure S on an object X, then a *reduct* of (X, S) is (X, R) for a subset R of S, or more generally for a subset of the full set of structure on X derived from S. In order to minimize the occurrence of parentheses in our non-associative contexts, we adopt the "algebraic" or "diagrammatic" convention which composes functions in the natural reading order from left to right. Thus functions may be placed to the right of their arguments, either on the line or as a superfix (as in $n!$ or x^2, for example). The notation $\mathbb{Z}/_d$ is used for the set of residues modulo a positive divisor d. In §2.3 and elsewhere, it is often convenient to write "tuples" with tensor product notation. Thus (x, y) becomes $x \otimes y$, for example.

2. Background

2.1. **Classical quasigroups, isotopy and triality.** Semigroups extend groups by retaining the associative law and relaxing the cancellation. On the other hand, quasigroups retain the cancellation, but do not have to be associative. In a context that involves quantum quasigroups, it is often convenient to refer to quasigroups as being "classical."

2.1.1. *Classical quasigroups.*

Definition 2.1. A (*classical*) *quasigroup* $(Q, \cdot, /, \backslash)$ .is a set Q equipped with three binary operations: respectively a *multiplication*, a *right division* and a *left division*, such that the identities

$$(x \cdot y)/y \overset{\text{(IR)}}{=} x \overset{\text{(IL)}}{=} y\backslash(y \cdot x) \tag{2.1}$$

and

$$(x/y) \cdot y \overset{\text{(SR)}}{=} x \overset{\text{(SL)}}{=} y \cdot (y\backslash x) \tag{2.2}$$

are satisfied.

Remark 2.2. (a) If Q is a group, $x/y = xy^{-1}$ and $x\backslash y = x^{-1}y$.

(b) The identities (2.1) and (2.2) serve to express the cancellativity of the (potentially) non-associative multiplication of a quasigroup. Indeed, for an element y of a quasigroup Q, consider the *right multiplication*

$$R(y)\colon Q \to Q; x \mapsto x \cdot y\,.$$

Then the identity (IR) ensures that $R(y)$ is injective, while (SR) ensures its surjectivity. Dually, the bijectivity of the *left multiplication* $L(y)\colon x \mapsto y \cdot x$ is given by (IL) and (SL).

(c) Since the multiplication determines the divisions, the reduct $(Q, \cdot)$ will serve to completely specify a quasigroup $(Q, \cdot, /, \backslash)$.

2.1.2. *Isotopy.*

Definition 2.3. Consider quasigroups $(Q, \cdot, /, \backslash)$ and $(Q', \cdot, /, \backslash)$.

(a) A function $f\colon Q \to Q'$ is a *homomorphism* if
$$x^f \cdot y^f = (x \cdot y)^f$$
for all x, y in Q.

(b) A triple $(f, g, h)\colon Q \to Q'$ of functions from Q to Q' is a *homotopy* if
$$x^f \cdot y^g = (x \cdot y)^h$$
for all x, y in Q.

(c) A bijective homomorphism is an *isomorphism*.

(d) A homotopy (f, g, h) is an *isotopy* when each of its constituents is bijective.

(e) The quasigroups Q and Q' are *isomorphic*, $Q \cong Q'$, if they are related by an isomorphism.

(f) The quasigroups Q and Q' are *isotopic*, $Q \simeq Q'$, if they are related by an isotopy.

Remark 2.4. (a) Isotopic groups are isomorphic.

(b) If $Q_0 \times Q_1 \cong Q_0 \times Q_2$ for finite, nonempty quasigroups Q_0, Q_1, Q_2, then $Q_1 \simeq Q_2$, but Q_1 and Q_2 do not have to be isomorphic [28, §3.4].

2.1.3. *Triality symmetry of classical quasigroups.* Along with the operations of multiplication, right division and left division presented in Definition 2.1, the quasigroup Q also carries their respective *opposites*
$$x \circ y := y \cdot x\,, \quad x/\!/y := y/x\,, \quad \text{and} \quad x\backslash\!\backslash y := y\backslash x\,.$$
While the language of groups only has a duality symmetry $(Q, \cdot) \leftrightarrow (Q, \circ)$, an S_2-action, the language of quasigroups has a richer *triality* symmetry, an S_3-action, as displayed in Figure 1. At the macroscopic scale, disregarding

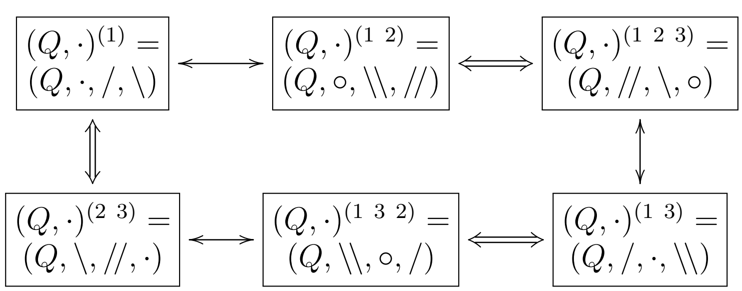


FIGURE 1. Triality symmetry of a classical quasigroup $(Q, \cdot)$.

the content of the boxes, the figure displays the Cayley diagram of the group S_3 of permutations of the set $\{ 1, 2, 3 \}$ with respect to the generators $(1\ 2)$ (single-shafted double-headed arrows $\leftrightarrow$) and $(2\ 3)$ (double-shafted double-headed arrows $\Leftrightarrow$). At the microscopic scale, the box representing a permutation π displays the quasigroup $(Q, \cdot)^\pi = (Q, *)$ with $x_1 \cdot x_2 = x_3$ if and only if $x_{1\pi} * x_{2\pi} = x_{3\pi}$ — compare Remark 2.2(c).

Remark 2.5. For many years, quasigroup structure has been presented by use of the multiplication along with the right and left divisions, as in Definition 2.1. However, since the subset $\{ (1), (1\ 3), (2\ 3) \}$ of S_3 has no particular group-theoretical significance, triality strongly suggests that a much more rational choice would be to take the opposite multiplication together with the right and left divisions, corresponding to the transposition coset $\{ (2\ 1), (1\ 3), (3\ 2) \}$ of the cyclic subgroup C_3 of S_3. Certainly, the traditional choice would make treatment of the topics of the current paper very awkward, obscuring the underlying symmetry, so the more rational choice will predominate here, as in §2.3, for example.

2.1.4. *Semisymmetric quasigroups.* Triality symmetry characterizes certain important classes of quasigroups. For example, commutative quasigroups $(Q, \cdot)$ are those for which $(Q, \cdot) = (Q, \circ)$ or $(Q, \cdot)^{(1)} = (Q, \cdot)^{(1\ 2)}$. In this paper, the following two classes play an important role.

Definition 2.6. Consider a quasigroup $(Q, \cdot)$.

(a) The quasigroup is *semisymmetric* if $(Q, \circ) = (Q, /) = (Q, \backslash)$ or $(Q, \cdot)^{(2\ 1)} = (Q, \cdot)^{(1\ 3)} = (Q, \cdot)^{(3\ 2)}$.

(b) The quasigroup is said to be *totally symmetric* if $(Q, \cdot)^\pi = (Q, \cdot)$ for each permutation π in S_3.

Example 2.7. (a) On (the underlying set A of) an abelian group $(A, +)$, the "multiplication" operation $(x, y) \mapsto -x - y$ yields a semisymmetric quasigroup.

(b) If an abelian group $(A, +)$ has exponent 3, the operation $(x, y) \mapsto -x-y$ yields an idempotent, totally symmetric quasigroup.

Recall that a binary operation $*$ is *idempotent* if it satisfies $x * x = x$ (while an *idempotent* of any binary operation $*$ means an element e with $e * e = e$). Idempotent, totally symmetric quasigroups $(Q, *)$ are equivalent to the combinatorial structures known as *Steiner triple systems*, with blocks ("triples") of the form $\{ x, y, x * y \}$ for $x \neq y \in Q$ [28, §1.6]. In the case of Example 2.7(b), these blocks are the lines of an affine geometry over $\mathsf{GF}(3)$.

2.2. **Quantum quasigroups.** Quantum quasigroups provide a self-dual unification of quasigroups and Hopf algebras [31].

Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ with swap morphism $\tau\colon A \otimes A \to A \otimes A$. A *weak bimagma* (A, ∇, Δ) is a $\mathbf{V}$-object A, equipped with a *multiplication* $\nabla\colon A \otimes A \to A$ and *comultiplication* $\Delta\colon A \to A \otimes A$. A *bimagma* is a weak bimagma (A, ∇, Δ) in which the multiplication and comultiplication are mutually homomorphic. The mutual homomorphism of the multiplication and comultiplication is expressed either by the *bimagma diagram*

$$\begin{array}{ccccc} a \otimes b & \xmapsto{\nabla} & a \cdot b & \xmapsto{\Delta} & (a \cdot b)^L \otimes (a \cdot b)^R \\ {\scriptstyle \Delta\otimes\Delta}\downarrow & & & & \uparrow{\scriptstyle \nabla\otimes\nabla} \\ a^L \otimes a^R \otimes b^L \otimes b^R & & \xmapsto[1_A\otimes\tau\otimes 1_A]{} & & a^L \otimes b^L \otimes a^R \otimes b^R \end{array} \tag{2.3}$$

[16, (2.1)] [31, (2.4)], or equationally as

$$x^L \cdot y^L = (x \cdot y)^L \quad \text{and} \quad x^R \cdot y^R = (x \cdot y)^R \tag{2.4}$$

in elementary form with $\nabla\colon a \otimes b \mapsto a \cdot b$ and the "non-coassociative Sweedler notation" $\Delta\colon a \mapsto a^L \otimes a^R$ (cf. [16, Rem. 2.2(b)], [19], [31]).

Definition 2.8. Let (A, ∇, Δ) be a bimagma in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$.

(a) The *left composite* is

$$\mathsf{G}\colon A \otimes A \xrightarrow{\Delta\otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A\otimes\nabla} A \otimes A\,; \tag{2.5}$$

$$x \otimes y \longmapsto x^L \otimes x^R \otimes y \longmapsto x^L \otimes x^R y$$

("G" for "Gauche").

(b) The *right composite* is

$$\rotatebox[origin=c]{180}{G}\colon A \otimes A \xrightarrow{1_A\otimes\Delta} A \otimes A \otimes A \xrightarrow{\nabla\otimes 1_A} A \otimes A\,; \tag{2.6}$$

$$x \otimes y \longmapsto x \otimes y^L \otimes y^R \longmapsto xy^L \otimes y^R$$

("$\rotatebox[origin=c]{180}{G}$" for "Droite"), the dual of the left composite.

(c) The bimagma (A, ∇, Δ) called is a *quantum quasigroup* if the left composite and right composite are invertible.

(d) The bimagma (A, ∇, Δ) is called a *left quantum quasigroup* if the left composite is invertible.

(e) The bimagma (A, ∇, Δ) is called a *right quantum quasigroup* if the right composite is invertible.

Remark 2.9. Given the alternative formulations of quantum quasigroups provided by the *equational quantum quasigroups* of §2.2.3, it is convenient to disambiguate by using the term *combinatorial quantum quasigroup* for quantum quasigroups as presented in Definition 2.8(c).

2.2.1. *Quantum triquasigroups.*

Definition 2.10. [34, Def'n. 3.1] A *quantum triquasigroup* $(Q, \nabla_i, \Delta_i)_{i\in\mathbb{Z}/_3}$ or

$$(Q, \nabla_i, \Delta_i)_{i=0}^{i=2} = (Q, \nabla_0, \nabla_1, \nabla_2, \Delta_0, \Delta_1, \Delta_2) \tag{2.7}$$

consists of

(0) a quantum quasigroup (Q, ∇_0, Δ_0);
(1) a left quantum quasigroup (Q, ∇_1, Δ_1); and
(2) a right quantum quasigroup (Q, ∇_2, Δ_2)

on an object Q of $\mathbf{V}$.

Use of residues modulo 3 for labelling constituent quantum quasigroups of a quantum triquasigroup implies equations such as $2+1 = 0$ and $2 = -1$. There is an implication that the constituent quantum quasigroup labelled by 0 is more fundamental than the other two constituents, which in general are only one-sided quantum quasigroups. This implicit understanding manifests itself, for example, in Definition 2.13 below.

2.2.2. *Composite diagrams.* Each bimagma (Q, ∇_i, Δ_i) within a quantum triquasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$, has its respective left and right composites G_i and $\reflectbox{$\mathsf{G}$}_i$. For $i \in \mathbb{Z}/_3$, the individual *composite diagrams*

$$\begin{array}{ccc} Q\otimes Q & \xrightarrow{\reflectbox{G}_i} & Q\otimes Q \\ \tau \updownarrow & & \updownarrow \tau \\ Q\otimes Q & \xleftarrow[\mathsf{G}_{i+1}]{} & Q\otimes Q \end{array} \tag{2.8}$$

in $\mathbf{V}$ [34, (3.3)] play an important role in the theory that initially arises from the following fundamental lemma.

Lemma 2.11. *Commutativity of the diagram* (2.8) *means that:*

(a) *the inverse of* $\reflectbox{$\mathsf{G}$}_i$ *is* $\tau\mathsf{G}_{i+1}\tau$*, and*
(b) *the inverse of* G_{i+1} *is* $\tau\reflectbox{$\mathsf{G}$}_i\tau$

for $i \in \mathbb{Z}/_3$.

Definition 2.12. [34, Def'n. 3.4] Suppose that $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ is a quantum triquasigroup.

(a) The diagram (2.8) is called the $(i + 1)i$*-diagram* of the quantum triquasigroup.

(b) The 02-diagram is called the *deuce* diagram.
(c) The 10-diagram is called the *tenner* diagram.
(d) The 21-diagram is called the *blackjack* diagram.

2.2.3. *Equational quantum quasigroups.* The following terminology is taken from [34].

Definition 2.13. A quantum triquasigroup in which the deuce and tenner diagrams commute is defined to be an *equational quantum quasigroup* or *quantum* S-*quasigroup*.

Definition 2.14. A *triality quantum quasigroup* or *quantum* T-*quasigroup* is a quantum triquasigroup in which the diagrams (2.8) commute for all $i \in \mathbb{Z}/_3$.

Remark 2.15. (a) The term "triality" appearing in Definition 2.14 refers to the symmetry displayed in Figure 2 below.

(b) Since quantum T-quasigroups possess quantum S-quasigroup structure, they are also considered to be equational quantum quasigroups.

Lemma 2.16. *In a triality quantum quasigroup* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$*, all three bimagmas* (Q, ∇_i, Δ_i) *form two-sided combinatorial quantum quasigroups.*

Definition 2.17. In the context of Lemma 2.16, the quantum quasigroup (Q, ∇_0, Δ_0) is described as the *leading* (combinatorial) quantum quasigroup of the triality quantum quasigroup. The same adjective is applied to its multiplication and comultiplication.

2.2.4. *Minimal definitions.* The respective Definitions 2.13 and 2.14 for quantum S- and T-quasigroups are presented on the basis of Definition 2.10 for quantum triquasigroups, to make the equational quantum quasigroup definitions more immediately comprehensible. Nevertheless, that approach introduces redundancies which may be removed as follows.

Proposition 2.18. [34, Prop. 3.12] *A triple* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ *of bimagmas forms a quantum S-quasigroup if and only if the deuce and tenner diagrams commute.*

Proposition 2.19. *A triple* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ *of bimagmas forms a quantum T-quasigroup if and only if all three composite diagrams commute.*

2.3. **Classical quasigroups as quantum quasigroups.** Suppose that $(Q, \circ, /, \backslash)$ is a classical quasigroup. Take the symmetric monoidal category $(\mathbf{Set}, \times, \top)$ of sets under the direct product, with a singleton set $\top$ (the terminal object) as monoidal unit. Here, we use tensor notation, writing

$X \times Y$ as $X \otimes Y$, and ordered tuples like (x, y) as $x \otimes y$. In this tensor notation, take diagonal comultiplications

$$\Delta_i \colon Q \to Q \otimes Q; x \mapsto x \otimes x \tag{2.9}$$

for $i \in \mathbb{Z}/_3$, and multiplications

$$\nabla_0 \colon Q \otimes Q \to Q; x \otimes y \mapsto x \circ y\,, \tag{2.10}$$

$$\nabla_1 \colon Q \otimes Q \to Q; x \otimes y \mapsto x \backslash y\,, \tag{2.11}$$

$$\nabla_2 \colon Q \otimes Q \to Q; x \otimes y \mapsto x / y\,. \tag{2.12}$$

It follows from [31, Prop. 3.11(a)] that each individual (Q, ∇_i, Δ_i) forms a quantum quasigroup. By (2.5) and (2.6), the composites are

$$\begin{aligned} &x \otimes y \overset{\mathsf{G}_0}{\longmapsto} x \otimes x \circ y\,, && x \otimes y \overset{\partial_0}{\longmapsto} x \circ y \otimes y\,, \\ &x \otimes y \overset{\mathsf{G}_1}{\longmapsto} x \otimes x \backslash y\,, && x \otimes y \overset{\partial_1}{\longmapsto} x \backslash y \otimes y\,, \\ &x \otimes y \overset{\mathsf{G}_2}{\longmapsto} x \otimes x / y\,, && x \otimes y \overset{\partial_2}{\longmapsto} x / y \otimes y\,, \end{aligned} \tag{2.13}$$

where the quasigroup operations bind tighter than the tensor product. Cocommutative and coassociative quantum T-quasigroups in $(\mathbf{Set}, \times, \top)$, where the multiplications and comultiplications are chosen as specified in (2.9)–(2.12), are equivalent to classical equational quasigroups.

Proposition 2.20. [34, Prop. 3.17] *Making assignments* (2.9)–(2.12) *for the multiplications and comultiplications, the commuting of the diagrams* (2.8) *are equivalent to the classical quasigroup identities.*

As a corollary, we obtain the following.

Theorem 2.21. [34, Th. 3.18] *Under the assignments* (2.9)–(2.12) *for the multiplications and comultiplications on an object Q of* $(\mathbf{Set}, \times, \top)$*, the structures*

(a) *A classical quasigroup* $(Q, \circ, /, \backslash)$;
(b) *A quantum* S*-quasigroup* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$;
(c) *A quantum* T*-quasigroup* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$

on Q are equivalent.

2.4. **Web geometry, triality, and semisymmetrization.**

2.4.1. *Triality symmetry of quantum* T*-quasigroups.* As already noted in Remark 2.15, quantum T-quasigroups possess a triality symmetry, which is displayed in Figure 2. As noted in Theorem 2.21, classical quasigroups are quantum T-quasigroups, and the quantum trialty of Figure 2 subsumes the classical triality of Figure 1.

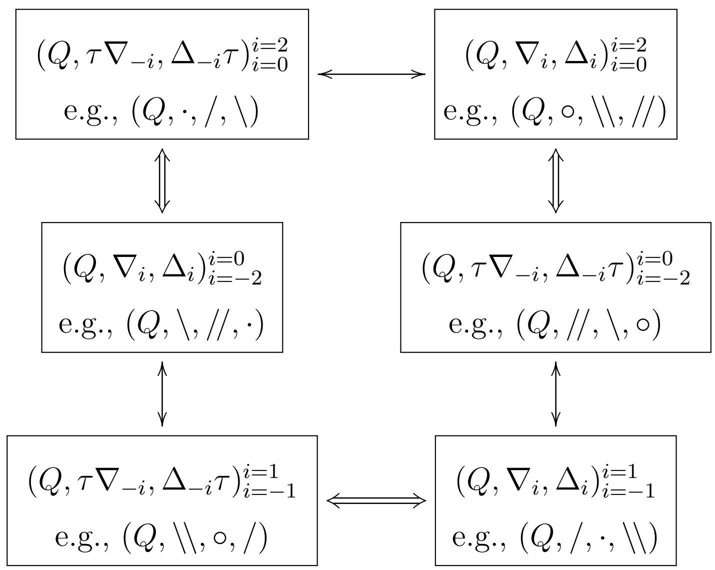


FIGURE 2. Triality symmetry of a quantum T-quasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$, including the case of a classical quasigroup $(Q, \circ, /, \backslash)$.

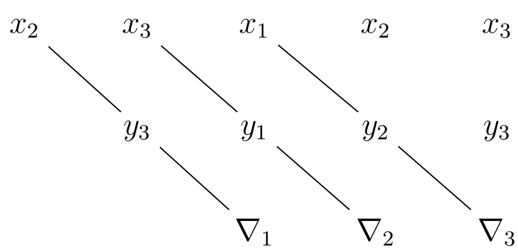
2.4.2. *Classical semisymmetrization.* The semisymmetrization of a classical quasigroup $(Q, \cdot, /, \backslash)$ was initially presented in terms of the multiplication and the opposites of the right and left divisions [27], [28, §1.4], [29]. In Figure 1, these operations form the C_3-orbit of the multiplication. In the present setting, following Remark 2.5, it is more convenient to take the operations from the C_3-orbit of the opposite multiplication, and to follow the quantum quasigroup notations of §2.3. In particular, the use of residues modulo 3 for labelling means that the leading multiplication ∇_0 (compare Definition 2.17) may be written as ∇_3.

Definition 2.22. Let Q be a classical quasigroup, interpreted as a quantum T-quasigroup according to Theorem 2.21 and the conventions of §2.3. Then the (*classical*) *semisymmetrization* is the structure on the cube $Q \otimes Q \otimes Q$ of the underlying set Q where

$$\begin{aligned} &[(x_1 \otimes x_2 \otimes x_3) \otimes \\ &(y_1 \otimes y_2 \otimes y_3)] \\ &(\nabla_1 \otimes \nabla_2 \otimes \nabla_3) = \\ (x_2 \otimes y_3)\nabla_1 \otimes (x_3 \otimes y_1)\nabla_2 &\otimes (x_1 \otimes y_2)\nabla_3 \end{aligned} \tag{2.14}$$

provides a multiplication.

Remark 2.23. The definition (2.14) follows the same "Sarrus' rule" pattern



as the original semisymmetrization definitions used in [27], [28, §1.4], [29].

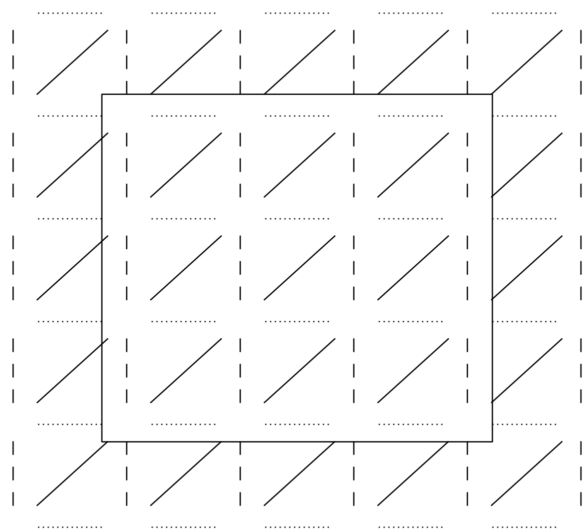
The terminology of Definition 2.22 is justified as follows [27, §4].

Lemma 2.24. *The semisymmetrization of a quasigroup is semisymmetric.*

2.4.3. *Web geometry.* A structure W known as a 3-*web* (in geometry) or a 3-*net* (in combinatorics) is a set of *points* which decomposes in three ways

$$W \cong H \times V \cong V \times D \cong D \times H \tag{2.15}$$

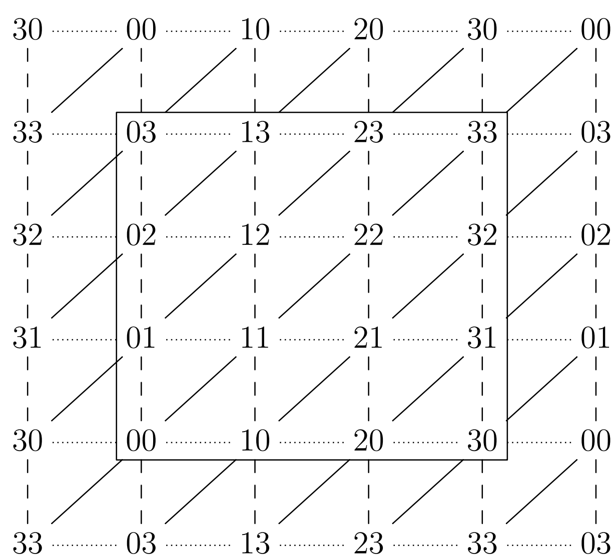
as a product of pencils H, V, D of *lines* respectively described as *horizontal*, *vertical*, and *diagonal* [35, p.88]. For example, the vertices of the (flat) torus triangulation



form a 3-web in which the horizontal lines appear dotted, the vertical lines appear dashed, and the diagonal lines appear solid.

2.4.4. *Web coordinatization.* In a 3-web structure (2.15), $|H| = |V| = |D|$ and $|W| = |H|^2 = |V|^2 = |D|^2$ [35, Prop. I.4.3]. Thus, there exists a set Q, in bijection with each pencil, such that $W \cong H \times V \cong Q^2$. The respective bijections with the three pencils H, V, D are said to *label* the horizontal, vertical, and diagonal lines, while the bijection $W \cong Q^2$ *labels* the points.

Example 2.25. The torus triangulation from §2.4.3 may be labeled with residues modulo 4 as follows:



Here, for compactness, the ordered pair (x, y) or $x \otimes y$ of residues labelling a point is exhibited simply as xy.

In the general situation, each horizontal line joins points $x \otimes y$ with a common second component y, while the vertical lines join points $x \otimes y$ with a common first component x. Then, a magma structure

$$\nabla\colon Q \otimes Q \to Q; x \otimes y \mapsto z \tag{2.16}$$

is defined by associating each point $x \otimes y$ with the label z of the unique diagonal line on which it lies. In Example 2.25, the magma (2.16) represents subtraction $x \otimes y \mapsto x - y$ of residues modulo 4.

The multiplication (2.16) endows Q with a quasigroup structure, which is described as a *coordinatization* of the web W. Conversely, each quasigroup Q yields a 3-web structure on Q^2, in which each diagonal line joins the points $x \otimes y$ with common product $x \cdot y$.

Web geometry clarifies the roles of triality and isotopy in the theory of classical quasigroups:

- Triality permutes the pencils (compare [1, p.432]); while
- The respective bijections f, g, h of an isotopy (f, g, h) permute the labels of the vertical, horizontal, and diagonal lines.

In the setting of quantum T-quasigroups, Figure 2 provides a sound notion of triality. On the other hand, the problem of finding a suitable isotopy concept for quantum quasigroups remains open — compare [32, §4].

2.4.5. *Web geometry and semisymmetrization.* In the classical setting, the process of semisymmetrization serves to provide a functor $\mathbf{Qtp} \to \mathbf{P}$ from the category of quasigroup homotopies to the category of homomorphisms between semisymmetric quasigroups; isotopic quasigroups have isomorphic semisymmetrizations [27, Th. 4.1] [28, (1.12)] [29, (6)]. The forgetful functor $\mathbf{P} \to \mathbf{Qtp}$ sending a homomorphism f to the homotopy (f, f, f) appears as a right adjoint to the semisymmetrization functor [27, Th. 5.2]. The points of the web coordinatized by a quasigroup Q correspond exactly to categorical points of the semisymmetrization Q^3 of Q in the category $\mathbf{P}$ — morphisms $v \otimes h \otimes d \mapsto x \otimes y \otimes z$ out of the terminal object $\top \cong \top \otimes \top \otimes \top$ [29, Th. 1(a)]. In other words, the 3-web determined by a quasigroup Q is the morphism set $\mathbf{Qtp}(\top, Q) \cong \mathbf{P}(\top, Q^3)$.

2.5. **Linear quantum quasigroups.** In this section, we will work with a commutative, unital ring S, taking the Cartesian symmetric monoidal category $\big(\underline{\underline{S}}, \oplus, \{\,0\,\}\big)$ of S-modules, with the direct sum as the monoidal product, the trivial module as the monoidal unit, and the swap matrix

$$\mathtt{swap}\colon A \oplus B \to B \oplus A; \begin{bmatrix} x & y \end{bmatrix} \mapsto \begin{bmatrix} y & x \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.17}$$

as the symmetry τ on the direct sum of modules A and B.

Definition 2.26. A quantum T-quasigroup $(Q,\nabla_i,\Delta_i)_{i=0}^{i=2}$ in $\big(\underline{\underline{S}},\oplus,\{\,0\,\}\big)$ is described as being *linear*. The multiplications

$$\nabla_i\colon Q\oplus Q\to Q;\ \begin{bmatrix}x & y\end{bmatrix}\mapsto\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}\rho_i\\ \lambda_i\end{bmatrix} \tag{2.18}$$

and comultiplications

$$\Delta_i\colon Q\to Q\oplus Q;\ \begin{bmatrix}x\end{bmatrix}\mapsto\begin{bmatrix}x\end{bmatrix}\begin{bmatrix}L_i & R_i\end{bmatrix} \tag{2.19}$$

are given by endomorphisms λ_i,ρ_i,L_i,R_i of Q for $i\in\mathbb{Z}/_3$.

For each $i\in\mathbb{Z}/_3$, the bimagma condition (2.4) amounts to the mutual commutativity of the two subalgebras $S(\lambda_i,\rho_i)$ and $S(L_i,R_i)$ within the endomorphism ring $\underline{\underline{S}}(Q,Q)$ of Q [16, Prop. 3.7][31, Prop. 3.39]. Using the standard bimagma notation for the Cartesian symmetric monoidal category $\big(\underline{\underline{S}},\oplus,\{\,0\,\}\big)$ of S-modules [16, Def'n. 3.5], such a bimagma is written as

$$Q(\rho_i,\lambda_i,L_i,R_i)\,. \tag{2.20}$$

For an alternative viewpoint, take the opposite $S(L_i,R_i)^{\mathsf{op}}$ of the subalgebra $S(L_i,R_i)$ of the endomorphism ring $\underline{\underline{S}}(Q,Q)$ of the S-module Q. Then ${}_{S(L_i,R_i)^{\mathsf{op}}}Q_{S(\rho_i,\lambda_i)}$, or

$$S(L_i,R_i)^{\mathsf{op}}\xrightarrow{\ Q\ }S(\rho_i,\lambda_i) \tag{2.21}$$

in the graphical notation which avoids multiple levels of subscripts, is a bimodule for each $i\in\mathbb{Z}/_3$ [16, Cor. 3.8].

In the quantum T-quasigroup of Definition 2.26, we have

$$\mathsf{G_i}=\big(\begin{bmatrix}L_i & R_i\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}\big)\left(\begin{bmatrix}1\end{bmatrix}\oplus\begin{bmatrix}\rho_i\\ \lambda_i\end{bmatrix}\right)=\begin{bmatrix}L_i & R_i\rho_i\\ 0 & \lambda_i\end{bmatrix} \tag{2.22}$$

and

$$\eth_i=\big(\begin{bmatrix}1\end{bmatrix}\oplus\begin{bmatrix}L_i & R_i\end{bmatrix}\big)\left(\begin{bmatrix}\rho_i\\ \lambda_i\end{bmatrix}\oplus\begin{bmatrix}1\end{bmatrix}\right)=\begin{bmatrix}\rho_i & 0\\ L_i\lambda_i & R_i\end{bmatrix} \tag{2.23}$$

for each $i\in\mathbb{Z}/_3$ [16, Lemma 3.9]. Consideration of the commutativity of the diagrams (2.8) is fundamental to the quantum T-quasigroup setting. We obtain the following [34, Lemma 4.2].

Lemma 2.27. *For linear quantum quasigroups (Q,∇_i,Δ_i) as presented in Definition 2.26, the commutativity of the diagrams* (2.8) *is equivalent to the* *equations*

$$R_i^{-1} = L_{i+1} \tag{2.24}$$

$$\rho_i^{-1} = \lambda_{i+1} \tag{2.25}$$

$$R_i R_{i+1} \rho_{i+1} = -L_i \lambda_i \lambda_{i+1} \tag{2.26}$$

$$R_{i+1} \rho_{i+1} \rho_i = -L_{i+1} L_i \lambda_i \tag{2.27}$$

for each $i \in \mathbb{Z}/_3$*. In particular, the endomorphisms* L_i, R_i, λ_i *and* ρ_i *are automorphisms of the module* Q*.*

The dualities in the respective equation pairs (2.24)–(2.25) and (2.26)–(2.27) of Lemma 2.27 should be noted. The first dual pair constitute the respective *dual* and *primal chiral shift* equations, where the mnemonic "low on the rights, high on the lefts" is useful for keeping track of the indexing. Using the chiral shift equations, we may rewrite the automorphisms L_i and λ_i in terms of R_i and ρ_i. The equations in the second dual pair may then be summarized as

$$R_{i-1} R_i R_{i+1} \rho_{i+1} \rho_i \rho_{i-1} = -1 \tag{2.28}$$

for each $i \in \mathbb{Z}/_3$. The composites (2.22)and (2.23) become

$$\mathsf{G_i} = \left(\begin{bmatrix} R_{i-1}^{-1} & R_i \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}\right) \left(\begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} \rho_i \\ \rho_{i-1}^{-1} \end{bmatrix}\right) = \begin{bmatrix} R_{i-1}^{-1} & R_i \rho_i \\ 0 & \rho_{i-1}^{-1} \end{bmatrix} \tag{2.29}$$

and

$$\partial_i = \left(\begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} R_{i-1}^{-1} & R_i \end{bmatrix}\right) \left(\begin{bmatrix} \rho_i \\ \rho_{i-1}^{-1} \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix}\right) = \begin{bmatrix} \rho_i & 0 \\ R_{i-1} \rho_{i-1}^{-1} & R_i \end{bmatrix} \tag{2.30}$$

respectively. The general linear bimagma notations (2.20) for $i \in \mathbb{Z}/_3$ are adequately compressed by the following.

Definition 2.28. Write $Q(R_i, \rho_i)_{i=0}^{i=2}$ or $Q(R_i, \rho_i)$ or $Q(R_0, R_1, R_2, \rho_0, \rho_1, \rho_2)$ as *triality notations* for the linear quantum T-quasigroup of Definition 2.26.

Lemma 2.29. [34, Lemma 4.4] *Given the linear quantum T-quasigroup of Definition 2.28, the subgroups*

$$\langle R_0, R_1, R_2 \rangle \quad \textit{and} \quad \langle \rho_0, \rho_1, \rho_2 \rangle \tag{2.31}$$

of the automorphism group $\underline{\underline{S}}(Q, Q)^*$ *of the* S*-module* Q *commute.*

Definition 2.30. (a) The subgroup B or

$$B(R_i, \rho_i)_{i=0}^{i=2} = \langle R_0, R_1, R_2, \rho_0, \rho_1, \rho_2 \rangle$$

of the automorphism group $\underline{\underline{S}}(Q, Q)^*$ of the underlying S-module Q of the linear quantum T-quasigroup $Q(R_i, \rho_i)_{i=0}^{i=2}$ is defined as its *bimultiplication group*. The subgroup $\langle R_0, R_1, R_2 \rangle$ is the *comultiplication group* or *Latin* *group*, while the subgroup $\langle \rho_0, \rho_1, \rho_2 \rangle$ is the *multiplication group* or *Greek group*.

(b) The group algebra SB of the bimultiplication group B is described as the *bimultiplication algebra* of the linear quantum T-quasigroup $Q(R_i, \rho_i)_{i=0}^{i=2}$. Its subalgebras $S\langle R_0, R_1, R_2 \rangle$ and $S\langle \rho_0, \rho_1, \rho_2 \rangle$ are the respective *multiplication algebra* and *comultiplication algebra*.

2.6. **Specification of linear quantum T-quasigroups.** Definition 2.26 of a linear quantum T-quasigroup was based on the three combinatorial quantum quasigroups which constitute the underlying structure. Later, the triality notation of Definition 2.28 specified a linear quantum T-quasigroup $Q(R_i, \rho_i)_{i=0}^{i=2}$ by the automorphisms $R_0, R_1, R_2, \rho_0, \rho_1, \rho_2$ of the S-module Q. The following lemma leads to a less redundant description (§2.6.1).

Lemma 2.31. [34, Lemma 4.6] *Consider an S-module Q. Suppose that the automorphism group $\underline{\underline{S}}(Q, Q)^*$ of the S-module Q has commuting subgroups* (2.31).

(a) *The nine equations*

$$R_{i-1} R_i R_{i+1} \rho_{j+1} \rho_j \rho_{j-1} = -1 \tag{2.32}$$

for $i, j \in \mathbb{Z}/_3$ are equivalent.

(b) *If (any one of) the equivalent equations of* (a) *hold, then there is a central element Ω in the group*

$$B = \langle R_0, R_1, R_2, \rho_0, \rho_1, \rho_2 \rangle \tag{2.33}$$

of automorphisms of the S-module Q such that the equations

$$\Omega = R_{i-1} R_i R_{i+1} = (-\rho_{j-1}^{-1})(-\rho_j^{-1})(-\rho_{j+1}^{-1}) \tag{2.34}$$

hold for each $i, j \in \mathbb{Z}/_3$.

(c) *If the chiral shift equations* (2.24)–(2.25) *hold, then*

$$\Omega = L_{i-1}^{-1} L_i^{-1} L_{i+1}^{-1} = (-\lambda_{j-1})(-\lambda_j)(-\lambda_{j+1})$$

for each $i, j \in \mathbb{Z}/_3$, and the equations (2.26)–(2.27) *are satisfied.*

Theorem 2.32. [34, Th. 4.7] *Let Q be a module over a commutative, unital ring S. Consider the following structures on Q as an object of the Cartesian symmetric monoidal category $\left(\underline{\underline{S}}, \oplus, \{0\}\right)$ of S-modules:*

(A) *A linear quantum T-quasigroup structure on Q given as $Q(R_i, \rho_i)_{i=0}^{i=2}$ in the triality notation of Definition 2.28;*

(B) *A subset $\mathcal{S} = \{R_0, R_1, R_2, \rho_0, \rho_1, \rho_2\}$ of the automorphism group $\underline{\underline{S}}(Q, Q)^*$ of the S-module Q such that the following relations hold:*

  (a) *The elements of $\{R_0, R_1, R_2\}$ commute with the elements of $\{\rho_0, \rho_1, \rho_2\}$;*

(b) *Any one of the equivalent relations* (2.32) *holds in the group algebra* $S\langle R_0, R_1, R_2, \rho_0, \rho_1, \rho_2\rangle$.

Then the two structures (A) *and* (B) *on* Q *are equivalent.*

Remark 2.33. (a) Theorem 2.32(B)(a) may be rewritten as saying that there is a bimodule

$$S\langle R_0, R_1, R_2\rangle^{\mathsf{op}} \xrightarrow{Q} S\langle \rho_0, \rho_1, \rho_2\rangle \tag{2.35}$$

over the opposite comultiplication algebra and the multiplication algebra — compare (2.21).

(b) Theorem 2.32(B)(b) eliminates one of the 6 apparent degrees of freedom in filling the set $\mathcal{S} = \{ R_0, R_1, R_2, \rho_0, \rho_1, \rho_2 \}$.

Corollary 2.34 (The Six-Parameter or Triality Representation)**.** *The three bimagmas that constitute the linear quantum T-quasigroup* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ *are*

$$(Q, \nabla_0, \Delta_0) = Q(\rho_0, \rho_2^{-1}, R_2^{-1}, R_0)\,, \tag{2.36}$$

$$(Q, \nabla_1, \Delta_1) = Q(\rho_1, \rho_0^{-1}, R_0^{-1}, R_1)\,, \tag{2.37}$$

$$(Q, \nabla_2, \Delta_2) = Q(\rho_2, \rho_1^{-1}, R_1^{-1}, R_2) \tag{2.38}$$

in the notation of (2.20).

2.6.1. *Five-parameter representations.* Corollary 2.34 gives linear quantum T-quasigroups using the six generators of the bimultiplication group. The number of parameters may be reduced to five, at the cost of breaking the triality symmetry of the six-parameter representations.

Consider a quantum T-quasigroup $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ in $\big(\underline{\underline{S}}, \oplus, \{ 0 \}\big)$, as in the original Definition 2.26. The basis for its five-parameter representation is to consider

$$(\rho, \lambda, L, R) = (\rho_0, \lambda_0, L_0, R_0) \tag{2.39}$$

from (2.20), together with Ω from Lemma 2.31. We then have the following.

Theorem 2.35 (The Five-Parameter Representation)**.** [34, Th. 4.10] *The three bimagmas constituting the linear quantum T-quasigroup* $(Q, \nabla_i, \Delta_i)_{i=0}^{i=2}$ *are*

$$(Q, \nabla_0, \Delta_0) = Q(\rho_0, \lambda_0, L_0, R_0) = Q(\rho, \lambda, L, R)\,, \tag{2.40}$$

$$(Q, \nabla_1, \Delta_1) = Q(\rho_1, \lambda_1, L_1, R_1) = Q(-\rho^{-1}\lambda\Omega^{-1}, \rho^{-1}, R^{-1}, R^{-1}L\Omega)\,, \quad \textit{and} \tag{2.41}$$

$$(Q, \nabla_2, \Delta_2) = Q(\rho_2, \lambda_2, L_2, R_2) = Q(\lambda^{-1}, -\lambda^{-1}\rho\Omega, L^{-1}R\Omega^{-1}, L^{-1}) \tag{2.42}$$

in the notation of (2.20).

Corollary 2.36. [34, Th. 4.14] *Within the category* $(\underline{\underline{S}}, \oplus, \{0\})$*, the full set of extensions to a quantum* T*-quasigroup of a combinatorial quantum quasigroup* $Q(\rho, \lambda, L, R)$ *is parametrized by* S*-module automorphisms* Ω *of* Q *that commute with each of the* S*-module automorphisms* ρ, λ, L, R*. In particular,* $\Omega = 1_Q$ *witnesses the nonemptiness of the extension set.*

3. Semisymmetrization of linear quantum quasigroups

3.1. **Classical semisymmetrization of linear quantum quasigroups.** In the Cartesian symmetric, monoidal category $(\underline{\underline{S}}, \oplus, \{0\})$ of modules over a commutative, unital ring S, consider a linear quantum T-quasigroup Q with the multiplications

$$\nabla_i \colon x \oplus y \mapsto x^{\rho_i} + y^{\lambda_i} = x^{\rho_i} + y^{\rho_{i-1}} \tag{3.1}$$

of (2.18) for $i \in \mathbb{Z}/_3$. The classical semisymmetrization of (2.14) yields the multiplication

$$\begin{aligned} &[(x_1 \oplus x_2 \oplus x_3) \oplus (y_1 \oplus y_2 \oplus y_3)](\nabla_1 \oplus \nabla_2 \oplus \nabla_3) = \\ &(x_2 \oplus y_3)\nabla_1 \oplus (x_3 \oplus y_1)\nabla_2 \oplus (x_1 \oplus y_2)\nabla_3 \\ &= (x_2^{\rho_1} + y_3^{\lambda_1}) \oplus (x_3^{\rho_2} + y_1^{\lambda_2}) \oplus (x_1^{\rho_3} + y_2^{\lambda_3}) \end{aligned} \tag{3.2}$$

on the direct cube $A = Q^3$ of the module Q. Writing elements of Q^3 as row matrices $a = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$, this multiplication may be expressed in the form $(a \oplus b)\nabla = a\mathrm{P} + b\Lambda$ with matrices

$$\mathrm{P} = \begin{bmatrix} 0 & 0 & \rho_3 \\ \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \\ \lambda_1 & 0 & 0 \end{bmatrix}. \tag{3.3}$$

The first matrix, sometimes described as the *Rho-matrix* in this context, is read as (capital) "rho." The mutual inverse relationship

$$\Lambda\mathrm{P} = 1_A \tag{3.4}$$

expresses the primal chiral shift equations (2.25). Thus, it is appropriate to describe (3.4) as the (*primal*) *shift equation*, noting the singular form.

3.2. **Quantum semisymmetry.** The equation (3.4) appears as part of [17, (4.6)] in the context of the quantum semisymmetry of combinatorial quantum quasigroups.

Definition 3.1. [17, Def'n. 4.11] A bimagma (A, ∇, Δ) in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is said to be (*quantum*) *semisymmetric* if the self-dual diagram

(3.5)
$$\begin{array}{ccc} A \otimes A & \xrightarrow{\;\partial\;} & A \otimes A \\ {\scriptstyle\tau}\updownarrow & & \updownarrow{\scriptstyle\tau} \\ A \otimes A & \xleftarrow[\mathsf{G}]{} & A \otimes A \end{array}$$

commutes.

The similarity of the diagram (3.5) to the diagrams (2.8) is apparent.

Theorem 3.2. [17, Th. 4.15] *Let $\mathcal{A} = A(\rho, \lambda, L, R)$ be a bimagma in the category $(\underline{\underline{S}}, \oplus, \{ 0 \})$ of modules over a commutative, unital ring S.*

(a) *If the bimagma $\mathcal{A}$ is quantum semisymmetric, then it actually forms a combinatorial quantum quasigroup.*

(b) *The bimagma $\mathcal{A}$ is quantum semisymmetric if and only if it has the form $\mathcal{A} = A(\rho, \rho^{-1}, L, L^{-1})$ with $L^3 = -\rho^3$.*

Remark 3.3. In Theorem 3.2, the bimagma condition on $A(\rho, \rho^{-1}, L, L^{-1})$ expresses mutual commutativity of the module automorphisms ρ and L [16, Prop. 3.7][31, Prop. 3.39] — compare (2.4).

3.3. **Extended semisymmetrization of linear quantum quasigroups.** In the Cartesian symmetric, monoidal category $(\underline{\underline{S}}, \oplus, \{ 0 \})$ of modules over a commutative, unital ring S, consider a linear quantum T-quasigroup Q with the multiplications

$$\nabla_i \colon x \oplus y \mapsto x^{\rho_i} + y^{\lambda_i}$$

of (3.1) and comultiplications

$$\nabla_i \colon x \mapsto x^{L_i} \oplus x^{R_i}$$

of (2.19) for $i \in \mathbb{Z}/_3$, as in Definition 2.26. Define the matrices of (3.3). This is the situation of §3.1, with $A = Q^3$. Note that $\lambda = \rho^{-1}$ within Theorem 3.2(b) gives new significance to the primal chiral shift equation (3.4). Motivated by the form of the combinatorial quantum quasigroup $A(\rho, \rho^{-1}, L, L^{-1})$ in Theorem 3.2(b), this section addresses the following:

Problem 3.4. Which combinatorial quantum quasigroups $A(\mathrm{P}, \mathrm{P}^{-1}, L, L^{-1})$ extend the classical semisymmetrization multiplication of §3.1 encoded in the Rho-matrix P of (3.3)?

Note that at this stage, it is only required that $A(\mathrm{P},\mathrm{P}^{-1},L,L^{-1})$ be a combinatorial quantum quasigroup in $\big(\underline{\underline{S}},\oplus,\{\,0\,\}\big)$. We are not expecting $A(\mathrm{P},\mathrm{P}^{-1},L,L^{-1})$ to be quantum semisymmetric. Furthermore, we are not taking account of any existing comultiplication structure (2.19) on Q. Thus, it is clear that the extension process involved in Problem 3.4 is not expected to be self-dual. The following important lemma, involving (3×3)-matrices over the endomorphism ring of the S-module Q, is motivated by Remark 3.3.

Lemma 3.5. *Suppose that the endomorphism*

$$C=\begin{bmatrix}\alpha & C_{12} & \gamma\\ C_{21} & C_{22} & C_{23}\\ \beta & C_{32} & C_{33}\end{bmatrix}$$

of the S-module A commutes with the Rho-matrix P. *Then*

$$C=C(\alpha,\beta,\gamma):=\begin{bmatrix}\alpha & \rho_3\beta\rho_1^{-1} & \gamma\\ \rho_1\gamma\rho_3^{-1} & \rho_1\alpha\rho_1^{-1} & \rho_2^{-1}\beta\rho_3\\ \beta & \rho_3^{-1}\gamma\rho_2 & \rho_3^{-1}\alpha\rho_3\end{bmatrix}. \tag{3.6}$$

Furthermore, the equations

$$\rho_3\rho_2\rho_1\alpha=\alpha\rho_3\rho_2\rho_1\,, \tag{3.7}$$

$$\rho_2\rho_1\rho_3\beta=\beta\rho_3\rho_2\rho_1\,, \tag{3.8}$$

$$\rho_3\rho_2\rho_1\gamma=\gamma\rho_2\rho_1\rho_3 \tag{3.9}$$

also hold in the endomorphism ring of the S-module Q.

Proof. We have

$$C\mathrm{P}=\begin{bmatrix}\alpha & C_{12} & \gamma\\ C_{21} & C_{22} & C_{23}\\ \beta & C_{32} & C_{33}\end{bmatrix}\begin{bmatrix}0 & 0 & \rho_3\\ \rho_1 & 0 & 0\\ 0 & \rho_2 & 0\end{bmatrix}=\begin{bmatrix}C_{12}\rho_1 & \gamma\rho_2 & \alpha\rho_3\\ C_{22}\rho_1 & C_{23}\rho_2 & C_{21}\rho_3\\ C_{32}\rho_1 & C_{33}\rho_2 & \beta\rho_3\end{bmatrix} \tag{3.10}$$

and

$$\mathrm{P}C=\begin{bmatrix}0 & 0 & \rho_3\\ \rho_1 & 0 & 0\\ 0 & \rho_2 & 0\end{bmatrix}\begin{bmatrix}\alpha & C_{12} & \gamma\\ C_{21} & C_{22} & C_{23}\\ \beta & C_{32} & C_{33}\end{bmatrix}=\begin{bmatrix}\rho_3\beta & \rho_3C_{32} & \rho_3C_{33}\\ \rho_1\alpha & \rho_1C_{12} & \rho_1\gamma\\ \rho_2C_{21} & \rho_2C_{22} & \rho_2C_{23}\end{bmatrix}. \tag{3.11}$$

(a) Equating the 23-entries of (3.10) and (3.11) yields $C_{21}=\rho_1\gamma\rho_3^{-1}$.

(b) Equating the 12-entries yields $C_{32}=\rho_3^{-1}\gamma\rho_2$.

(c) Equating the 31-entries yields $C_{32}=\rho_2C_{21}\rho_1^{-1}\overset{\text{(a)}}{=}\rho_2\rho_1\gamma\rho_3^{-1}\rho_1^{-1}$.

(d) Equating the expressions of C_{32} from (b) and (c) yields (3.9).

(e) Equating the 11-entries of (3.10) and (3.11) yields $C_{12}=\rho_3\beta\rho_1^{-1}$.

(f) Equating the 33-entries yields $C_{23}=\rho_2^{-1}\beta\rho_3$.

(g) Equating the 22-entries yields $C_{23} = \rho_1 C_{12} \rho_2^{-1} \overset{\text{(e)}}{=} \rho_1 \rho_3 \beta \rho_1^{-1} \rho_2^{-1}$.

(h) Equating the expressions of C_{23} from (f) and (g) yields (3.8).

(i) Equating the 13-entries of (3.10) and (3.11) yields $C_{33} = \rho_3^{-1} \alpha \rho_3$.

(j) Equating the 21-entries yields $C_{22} = \rho_1 \alpha \rho_1^{-1}$.

(k) Equating the 32-entries yields $C_{22} = \rho_2^{-1} C_{33} \rho_2 \overset{\text{(i)}}{=} \rho_2^{-1} \rho_3^{-1} \alpha \rho_3 \rho_2$.

(l) Equating the expressions of C_{22} from (j) and (k) yields (3.7).

This completes the identification of $C = C(\alpha, \beta, \gamma)$, and establishes the equations (3.7)–(3.9). □

Corollary 3.6. *The equations*

$$C(\alpha, 0, 0) C(\alpha^{-1}, 0, 0) = 1_A\,,$$
$$C(0, \beta, 0) C(0, 0, \beta^{-1}) = 1_A\,,$$
$$C(0, 0, \gamma) C(0, \gamma^{-1}, 0) = 1_A$$

all hold for automorphisms α, β, γ of the S-module Q, independently of the conditions (3.7)*–*(3.9)*.*

Definition 3.7. A matrix C as in Lemma 3.5 that commutes with the Rho-matrix P is described as a *commutant* matrix.

Definition 3.8. (a) For an endomorphism θ of the S-module Q, the module endomorphisms

$$C(\theta, 0, 0)\,, \quad C(0, \theta, 0)\,, \quad C(0, 0, \theta) \tag{3.12}$$

of A are described as *monomial*.

(b) The respective entries

$$C(\theta, 0, 0)_{11}\,, \quad C(0, \theta, 0)_{31}\,, \quad C(0, 0, \theta)_{13}$$

of the monomials (3.12) are described as *dominant*.

Remark 3.9. The terminology of Definition 3.8(a) is inspired by [14, §IV.1]. In (3.12), the non-zero entries of A-endomorphism matrices (3.6) appear at the non-zero entries of the permutation matrices of the cyclic action C_3. Huppert's context in [14, §IV.1] actually envisages full symmetric group actions S_n.

Example 3.10. The Rho-matrix $\mathrm{P} = C(0, 0, \rho_3)$ of (3.3) is monomial.

Lemma 3.11. *For endomorphisms α, β, γ of the S-module Q, the equation*

$$C(\alpha, \beta, \gamma) = C(\alpha, 0, 0) + C(0, \beta, 0) + C(0, 0, \gamma)$$

expresses general commutant endomorphisms (3.6) *as sums of monomial endomorphisms.*

Regarding monomial endomorphisms as non-commutative analogues of circulants, it is convenient to introduce the following nomenclature.

Definition 3.12. The notation $C(\alpha,\beta,\gamma)$ of (3.6) is described as *circulant notation.*

Now, a general answer to Problem 3.4 is provided as follows. Corollary 3.6 gives immediate assurance that the answer is not vacuous.

Theorem 3.13. *Consider a linear quantum* T*-quasigroup equipped with the multiplications of* (3.1) *for* $i \in \mathbb{Z}/_3$*. Take the Rho-matrix* P *of* (3.3)*, and an invertible matrix* $L = C(\alpha,\beta,\gamma)$ *as in Lemma 3.5. Then* $A(\mathrm{P},\mathrm{P}^{-1},L,L^{-1})$ *forms a combinatorial quantum quasigroup.*

A complete answer to Problem 3.4 awaits a detailed determination of the invertible commutant matrices $C(\alpha,\beta,\gamma)$.

3.4. **Monomial multiplication.** We now continue the investigation of the commutant matrices C from Lemma 3.5. Proposition 3.16 below assembles a multiplication table on the set of monomial endomorphisms that appear in Definition 3.8. The product of two commutant matrices is commutant. Also, the product of monomial matrices is monomial. It thus suffices to determine the dominant entry of each monomial product, thereby reducing the complexity of the matrix multiplications involved.

3.4.1. *The general case.* In this paragraph, we consider endomorphisms $\{\,\alpha_i,\beta_i,\gamma_i \mid i \in \mathbb{Z}/_2\,\}$ of the S-module Q that are subject to the commutant conditions (3.7)–(3.9). The first lemma may be regarded as a generalization of Corollary 3.6.

Lemma 3.14. *The equations*

$$C(\alpha_1,0,0)C(\alpha_2,0,0) = C(\alpha_1\alpha_2,0,0)\,, \tag{3.13}$$

$$C(0,\beta_1,0)C(0,0,\gamma_2) = C(\rho_3\beta_1\gamma_2\rho_3^{-1},0,0)\,, \tag{3.14}$$

$$C(0,0,\gamma_1)C(0,\beta_2,0) = C(\gamma_1\beta_2,0,0) \tag{3.15}$$

all hold.

Lemma 3.15. *The equations*

$$C(\alpha_1,0,0)C(0,\beta_2,0) = C(0,\rho_3^{-1}\alpha_1\rho_3\cdot\beta_2,0)\,;$$
$$C(\alpha_1,0,0)C(0,0,\gamma_2) = C(0,0,\alpha_1\gamma_2)$$
$$C(0,\beta_1,0)C(\alpha_2,0,0) = C(0,\beta_1\alpha_2,0)$$
$$C(0,0,\gamma_1)C(\alpha_2,0,0) = C(0,0,\gamma_1\cdot\rho_3^{-1}\alpha_2\rho_3)$$

all hold.

Proposition 3.16. *Together with the equations of Lemmas 3.14–3.15, the equations*

$$C(0,\beta_1,0)C(0,\beta_2,0)=C(0,0,\rho_3\beta_1\rho_1^{-1}\cdot\rho_2^{-1}\beta_2\rho_3) \quad \textit{and} \tag{3.16}$$

$$C(0,0,\gamma_1)C(0,0,\gamma_2)=C(0,\rho_3^{-1}\gamma_1\rho_2\cdot\rho_1\gamma_2\rho_3^{-1},0) \tag{3.17}$$

complete the multiplication table for the monomial endomorphisms of the S-module A.

3.4.2. *Cubing.* In this paragraph, S-module endomorphisms α,β,γ of Q are assumed to commute with the S-module automorphisms ρ_1,ρ_2,ρ_3 of Q. Motivated by the context of §2.6, we additionally assume that there is an S-automorphism Ω of Q such that

$$\Omega=(-\rho_{i-1}^{-1})(-\rho_i^{-1})(-\rho_{i+1}^{-1})=-(\rho_{i+1}\rho_i\rho_{i-1})^{-1} \tag{3.18}$$

for each $i\in\mathbb{Z}/_3$, as in (2.34). In this situation, the commutant conditions (3.7)–(3.9) reduce to the commuting of α,β,γ with Ω^{-1}, conditions which are then consequences of the commutations with ρ_1,ρ_2,ρ_3.

Lemma 3.17. *The equations*

$$\begin{aligned}C(\alpha,0,0)^3&=C(\alpha^3,0,0)\,,\\ C(0,\beta,0)^3&=C\big(-\Omega(\rho_3\beta)^3,0,0\big)\,,\\ C(0,0,\gamma)^3&=C\big(-\Omega^{-1}(\gamma\rho_3^{-1})^3,0,0\big)\end{aligned}$$

all hold.

Proof. The first equation is immediate from (3.13) in Lemma 3.14. For the second, we have

$$\begin{aligned}C(0,\beta,0)^3&\overset{(3.16)}{=}C(0,0,\rho_3\beta\rho_1^{-1}\cdot\rho_2^{-1}\beta\rho_3)C(0,\beta,0)\\&\overset{(3.15)}{=}C(\rho_3\beta\rho_1^{-1}\rho_2^{-1}\beta\rho_3\beta,0,0)\\&\overset{(3.18)}{=}C\big(-\Omega(\rho_3\beta)^3,0,0\big)\,,\end{aligned}$$

while the comparable (dual) computation

$$\begin{aligned}C(0,0,\gamma)^3&\overset{(3.17)}{=}C(0,0,\gamma)C(0,\rho_3^{-1}\gamma\rho_2\cdot\rho_1\gamma\rho_3^{-1},0)\\&\overset{(3.15)}{=}C(\gamma\rho_3^{-1}\gamma\rho_2\cdot\rho_1\gamma\rho_3^{-1},0,0)\\&\overset{(3.18)}{=}C\big(-\Omega^{-1}(\gamma\rho_3^{-1})^3,0,0\big)\end{aligned}$$

verifies the third equation. ∎

Example 3.18. It was pointed out in Example 3.10 that the Rho-matrix $\mathrm{P} = C(0, 0, \rho_3)$ of (3.3) is monomial. The equation

$$\mathrm{P}^3 = C(0, 0, \rho_3)^3 = C\big(-\Omega^{-1}, 0, 0\big) \overset{(3.18)}{=} -\Omega^{-1} 1_A \tag{3.19}$$

is now confirmed by Lemma 3.17.

3.5. **Quantum semisymmetrization.** Problem 3.4 may now be refined as follows.

Problem 3.19. Determine quantum semisymmetric quantum quasigroups $A(\mathrm{P}, \mathrm{P}^{-1}, L, L^{-1})$ extending the classical semisymmetrization multiplication of §3.1 encoded in the matrix P of (3.3).

By Theorem 3.2(b) and (3.19), the conditions on L are that

$$L^3 = -\mathrm{P}^3 = \Omega^{-1} \tag{3.20}$$

and $L\mathrm{P} = \mathrm{P}L$. Lemma 3.5 then provides the basis for a formal solution to Problem 3.19.

Theorem 3.20. *Suppose that Q is a linear quantum* T*-quasigroup with the multiplications of* (3.1) *for $i \in \mathbb{Z}/_3$. The quantum semisymmetric quantum quasigroups $A(\mathrm{P}, \mathrm{P}^{-1}, L, L^{-1})$ extending the classical semisymmetrization multiplication of §3.1 encoded in the matrix* P *of* (3.3) *have*

$$L = L(X_1, X_2, X_3) := \begin{bmatrix} X_1 & \rho_3 X_2 \rho_1^{-1} & X_3 \\ \rho_1 X_3 \rho_3^{-1} & \rho_1 X_1 \rho_1^{-1} & \rho_2^{-1} X_2 \rho_3 \\ X_2 & \rho_3^{-1} X_3 \rho_2 & \rho_3^{-1} X_1 \rho_3 \end{bmatrix} \tag{3.21}$$

for endomorphisms X_1, X_2, X_3 of the S-module Q that commute with Ω, subject to the condition $L^3 = \Omega^{-1}$.

Proof. It remains to be noted that $L^3 = \Omega^{-1}$ implies $L^{-1} = L^2\Omega$, so that L is indeed invertible. Furthermore, Theorem 2.32 ensures that (3.18) holds, so the commutant conditions (3.7)–(3.9) reduce to the commuting of X_1, X_2, X_3 with Ω. □

4. Monomial quantum semisymmetrization

4.1. **Monomial quantum-semisymmetric semisymmetrizations.** A quantum semisymmetric semisymmetrization as given by Theorem 3.20 is at best purely formal, in the absence of a fuller analysis of the solutions L to the cubic equation $L^3 = \Omega^{-1}$ in the commutant of the Rho-matrix P. More concrete specifications are available if the S-automorphism (3.21) of A is taken to be monomial. Then, the quantum semisymmetric quantum quasigroup $A(\mathrm{P}, \mathrm{P}^{-1}, L, L^{-1})$ is called a *monomial* extension of the classical semisymmetrization multiplication of §3.1 encoded in the matrix P of (3.3). In fact, in the situation of §4.5 below, it transpires that the only extensions are monomial, while non-monomial extensions do arise in the algebraically closed setting of §5.2.

General monomial solutions to the extension Problem 3.19 are treated by the following theorem.

Theorem 4.1. *Monomial quantum semisymmetric extensions are given in the context of Theorem 3.20 by*

$$L = C(\Omega^{-1/3}, 0, 0) = \Omega^{-1/3} 1_A \,, \tag{4.1}$$

$$L = C(0, -\rho_3^{-1}\Omega^{2/3}, 0) \,, \tag{4.2}$$

$$L = C(0, 0, -\zeta\rho_3) \,. \tag{4.3}$$

In (4.3)*, an S-automorphism ζ of Q commutes with ρ_1, ρ_2, and ρ_3, while satisfying $\zeta^3 = 1_Q$.*

Proof. Note that $C(\Omega^{-1}, 0, 0) = \Omega^{-1} 1_A$ by (3.18). For (4.1), Lemma 3.17 gives $C(\Omega^{-1/3}, 0, 0)^3 = C(\Omega^{-1}, 0, 0)$, as required to satisfy the condition $L^3 = \Omega^{-1}$ of Theorem 3.20. Again, for (4.2), Lemma 3.17 with $\beta = -\rho_3^{-1}\Omega^{-2/3}$, and thus $-\Omega(\rho_3\beta)^3 = \Omega(\Omega^{-2/3})^3 = \Omega^{-1}$, gives

$$C(0, -\rho_3^{-1}\Omega^{2/3}, 0)^3 = C(\Omega^{-1}, 0, 0) \,.$$

Finally, for (4.3), Lemma 3.17 with $\gamma = -\zeta\rho_3$ gives

$$C(0, 0, -\zeta\rho_3)^3 = C(\Omega^{-1}, 0, 0) \,,$$

since $-\Omega^{-1}(\gamma\rho_3^{-1})^3 = \Omega^{-1}$ in this case. □

A corollary considers monomial solutions working over any commutative, unital ring S.

Corollary 4.2. (a) *For linear quantum quasigroups that may be taken in the Cartesian monoidal category of abelian groups, monomial quantum semisymmetric extensions are given in the context of Theorem 3.20 by*

$$L = C(-\epsilon_1\epsilon_2\rho_3, 0, 0) = -\epsilon_1\epsilon_2\rho_3 1_A \,, \tag{4.4}$$

$$L = C(0, -\rho_3^{-3}, 0) \,, \tag{4.5}$$

$$L = C(0, 0, -\rho_3) = -\mathrm{P} \,. \tag{4.6}$$

Here, the first two cases apply when $\epsilon_1\rho_1 = \epsilon_2\rho_2 = \rho_3$ with $\epsilon_1, \epsilon_2 \in \{\pm 1\}$.

Proof. If $\epsilon_1\rho_1 = \epsilon_2\rho_2 = \rho_3$, then (3.18) gives $-\Omega^{-1} = \epsilon_1\epsilon_2\rho_3^3$, so $\Omega^{-1/3} = -\epsilon_1\epsilon_2\rho_3$ for (4.4). Also $\Omega^{2/3} = \rho_3^{-2}$, so $-\rho_3^{-1}\Omega^{2/3} = -\rho_3^{-3}$ for (4.5). □

4.2. **Local rigidity.** The aim of this section is to present some evidence concerning the place of the monomial extensions provided by Theorem 4.1, relative to the general set of extensions discussed in Theorem 3.20. It transpires that a monomial extension may not be deformable into a non-monomial extension.

For a given commutative, unital ring S, take $S[\varepsilon]$ to be the ring of dual numbers over S, the extension of S by an element ε with $\varepsilon^2 = 0$. For each residue $i \in \mathbb{Z}/_3$, and for general S-endomorphisms θ_i of the S-module Q, define infinitesimal endomorphisms $\varepsilon_i = \varepsilon\theta_i$ of the $S[\varepsilon]$-module $S[\varepsilon] \otimes_S Q$.

Definition 4.3. In the context of Theorem 3.20, the respective monomial quantum semisymmetric extensions given by the matrices L of (4.1)–(4.3) are said to be *locally rigid* if $[L + C(\varepsilon_1, \varepsilon_2, \varepsilon_3)]^3 = \Omega^{-1}$ implies that each infinitesimal endomorphism ε_i vanishes.

Theorem 4.4. *The extensions determined by each choice of matrix L from (4.1)–(4.3) are locally rigid.*

Proof. It will suffice to treat the case $L = C(0, 0, -\rho_3)$ from (4.3) — the two remaining cases are either similar, for (4.2), or much more direct, for (4.1). Furthermore, the S-automorphism ζ of Q in (4.3) would play no essential role in what follows.

Consider the deformed extension

$$L_\varepsilon = C(\varepsilon_1, \varepsilon_2, \varepsilon_3 - \rho_3) = \begin{bmatrix} \varepsilon_1 & \rho_3\varepsilon_2\rho_1^{-1} & \varepsilon_3 - \rho_3 \\ \rho_1\varepsilon_3\rho_3^{-1} - \rho_1 & \rho_1\varepsilon_1\rho_1^{-1} & \rho_2^{-1}\varepsilon_2\rho_3 \\ \varepsilon_2 & \rho_3^{-1}\varepsilon_3\rho_2 - \rho_2 & \rho_3^{-1}\varepsilon_1\rho_3 \end{bmatrix}$$

of $C(0, 0, -\rho_3)$, so that $L_\varepsilon^2 =$

$$C(-2\rho_3\varepsilon_2, \rho_2\rho_1 - \rho_3^{-1}\varepsilon_3\rho_2\rho_1 - \rho_2\rho_1\varepsilon_3\rho_3^{-1}, -2\varepsilon_1\rho_3) =$$

$$\begin{bmatrix} -2\rho_3\varepsilon_2 & \rho_3\rho_2 - 2\varepsilon_3\rho_2 & -2\varepsilon_1\rho_3 \\ -2\rho_1\varepsilon_1 & -\rho_1\rho_3\varepsilon_2\rho_1^{-1} - \rho_2^{-1}\varepsilon_2\rho_3\rho_2 & \rho_1\rho_3 - 2\rho_1\varepsilon_3 \\ \rho_2\rho_1 - \rho_3^{-1}\varepsilon_3\rho_2\rho_1 - \rho_2\rho_1\varepsilon_3\rho_3^{-1} & -\rho_2\rho_1\varepsilon_1\rho_1^{-1} - \rho_3^{-1}\varepsilon_1\rho_3\rho_2 & -2\varepsilon_2\rho_3 \end{bmatrix}.$$

Theorem 3.20 then demands

$$L_\varepsilon^3 = \begin{bmatrix} * & * & 3\rho_3\varepsilon_2\rho_3 \\ * & \Omega^{-1} - \rho_1\varepsilon_3\rho_2 & 3\rho_1\varepsilon_1\rho_3 \\ * & * & * \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ * & \Omega^{-1} & 0 \\ * & * & * \end{bmatrix},$$

whence $\varepsilon_i = 0$ for each $i \in \mathbb{Z}/_3$. □

4.3. **Affine planes.**

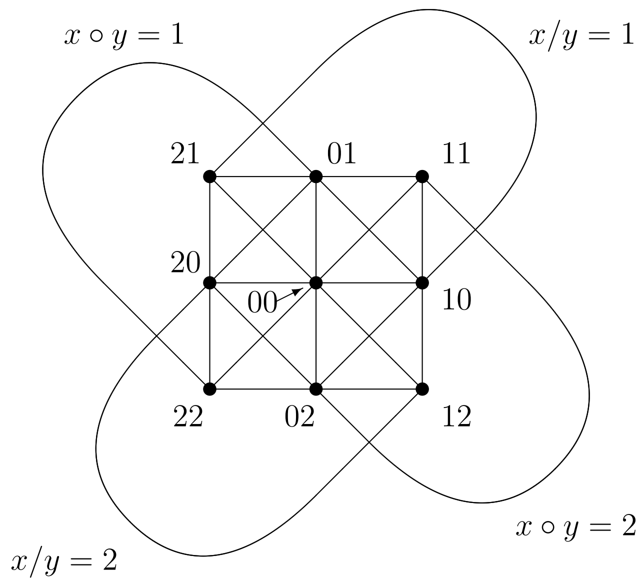


FIGURE 3. The affine plane over $\mathbb{Z}/_3 = \{ (-1 =)2, 0, 1 \}$, with lines determined by the operations of (4.7) for $m = 1$. Here, the unmarked non-horizontal and non-vertical lines are the main diagonal $\{ 22, 00, 11 \}$ specified by $x/y = 0$, and the reverse diagonal $\{ 21, 00, 12 \}$ specified by $x \circ y = 0$. Note that $x \backslash y = y/x$ in this case.

Let S be a commutative, unital ring, with group S^* of invertible elements. Linear quasigroup multiplications (3.1) are defined on S by

$$\begin{cases} xy\nabla_0 = & x \circ y = x + ym \in \Omega_S^{-+}\,, \\ xy\nabla_1 = & x\backslash y = -xm + y \in \Omega_S^{+-}\,, \\ xy\nabla_2 = & x/y = xm^{-1} - ym^{-1} \in \Omega_S^{++} \end{cases} \tag{4.7}$$

for each scalar $m \in S^*$. The memberships exhibited at the end of each equation in (4.7) refer to the *orthant structure* discussed in [30, (12.1)]. For each product ∇_i, and for each invertible element m of S, the set

$$\nabla_i^{-1}(c) = \{ xy\nabla_i = c \mid c \in S \} \tag{4.8}$$

forms a line in the affine plane over S. Setting the scalar m to be zero in the first two equations of (4.7) yields the additional *horizontal lines* $\nabla_0^{-1}(c)$ and *vertical lines* $\nabla_1^{-1}(c)$ for $c \in F$. Together with the lines (4.8), they form the *affine plane* over the ring S. Figure 3 illustrates the affine plane over $\mathbb{Z}/_3$ obtained in this way.

4.4. **Monomial semisymmetrization in affine planes.** Following the procedure of §3.1 as applied to the multiplications (4.7), we obtain the corresponding Rho-matrix

$$\mathrm{P} = C(0,0,\rho_3) = \begin{bmatrix} 0 & 0 & \rho_3 \\ \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -m & 0 & 0 \\ 0 & m^{-1} & 0 \end{bmatrix} \tag{4.9}$$

of (3.3), with $\mathrm{P}^3 = -1 = -\Omega^{-1}$ according to the relations (2.34) and (3.19). It is interesting to observe how the entries of the Rho-matrix include the respective additive and multiplicative identity elements 0 and 1 of the ring S, together with the respective additive and multiplicative inverse elements $-m$ and m^{-1} of the generic invertible element m of S.

Theorem 4.1 identifies three possible general[1] choices of the matrix L to create a quantum semisymmetric quantum quasigroup $A(\mathrm{P}, \mathrm{P}^{-1}, L, L^{-1})$ constituting a monomial extension of the classical semisymmetrization which is encoded in the Rho-matrix P of (4.9):

$$L = C(\Omega^{-1/3}, 0, 0) = \Omega^{-1/3} 1_A = 1_A \,, \tag{4.10}$$

$$L = C(0, -\rho_3^{-1}\Omega^{2/3}, 0) \overset{\text{Cor. 3.6}}{=} -\mathrm{P}^{-1} = \mathrm{P}^2 \quad \text{and} \tag{4.11}$$

$$L = C(0,0,-\rho_3) = -\mathrm{P} \,. \tag{4.12}$$

Here, (4.10) with diagonal comultiplication produces $A(\mathrm{P}, \mathrm{P}^{-1}, L, L^{-1})$ as the quantum quasigroup given directly according to the procedure of §2.3 from the classical semisymmetrization (§3.1). This option is described as *classical*.

The comultiplication encoded by the matrix $L = \mathrm{P}^2$ in option (4.11) is

$$\begin{aligned} \Delta\colon \begin{bmatrix} x & y & z \end{bmatrix} &\mapsto \begin{bmatrix} x & y & z \end{bmatrix} \mathrm{P}^2 \oplus \begin{bmatrix} x & y & z \end{bmatrix} (-\mathrm{P}) \\ &= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & m^{-1} & 0 \\ 0 & 0 & -m \\ -1 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ m & 0 & 0 \\ 0 & -m^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} -z & xm^{-1} & -ym \end{bmatrix} \oplus \begin{bmatrix} ym & -zm^{-1} & -x \end{bmatrix} . \end{aligned} \tag{4.13}$$

The comultiplication encoded by the matrix $L = -P$ in option (4.12), namely

$$\begin{aligned} \Delta\colon \begin{bmatrix} x & y & z \end{bmatrix} &\mapsto \begin{bmatrix} x & y & z \end{bmatrix} (-\mathrm{P}) \oplus \begin{bmatrix} x & y & z \end{bmatrix} \mathrm{P}^2 \\ &= \begin{bmatrix} ym & -zm^{-1} & -x \end{bmatrix} \oplus \begin{bmatrix} -z & xm^{-1} & -ym \end{bmatrix} , \end{aligned} \tag{4.14}$$

[1]Thus, available over any commutative, unital ring.

is just (4.13) followed by the swap (2.17).

4.5. **Real rigidity in the affine case.** This section briefly summarizes a complete solution to Problem 3.19 over the field of real numbers (and as it happens, over any subring), for the affine-geometrical Rho-matrix P of (4.9), corresponding to $\rho_3 = \Omega = 1$. Thus for real unknowns x, y, z, consider a matrix $L = C(-x, y, z)$ taken from the commutant of P.

Solutions (x, y, z) of the matrix equation

$$L^3 = C(-x, y, z)^3 = C(1, 0, 0) \tag{4.15}$$

(a system of 3 cubic scalar equations) are desired. Setting

$$X = x^2 + 2yz\,, \quad Y = y^2 + 2xz\,, \quad Z = z^2 + 2xy\,, \tag{4.16}$$

we have $L^3 = C(-xX - yY - zZ, xZ + yX + zY, xY + yZ + zX)$, noting that there is no dependence on m.

Theorem 5.11 below confirms that just three solutions for $(x, y, z) \in \mathbb{R}^3$ appear as

$$(-1, 0, 0)\,, \quad (0, -1, 0)\,, \quad (0, 0, -1)\,, \tag{4.17}$$

yielding respective L-matrices

$$C(1, 0, 0) = C(\Omega^{-1/3}, 0, 0) = 1_A\,, \tag{4.18}$$

$$C(0, -1, 0) = C(0, -\rho_3^{-1}\Omega^{2/3}, 0) = P^2\,, \text{ and} \tag{4.19}$$

$$C(0, 0, -1) = C(0, 0, -\rho_3) = -P \tag{4.20}$$

in agreement with (4.10)–(4.12).

4.6. **Complex monomial solutions.** Over the field of complex numbers, the complex triples

$$(-\omega, 0, 0)\,, \quad (0, -\omega, 0)\,, \quad (0, 0, -\omega) \tag{4.21}$$

and their three respective complex conjugates appear as monomial solutions of (4.15), as displayed in Table 1.[2] In (4.21) and henceforth, ω denotes a primitive cube root of unity. Note that

$$C(\omega, 0, 0) = C(\Omega^{-1/3}, 0, 0) = \omega 1_A\,, \tag{4.22}$$

$$C(0, -\omega, 0) = C(0, -\rho_3^{-1}\Omega^{2/3}, 0)\,, \text{ and} \tag{4.23}$$

$$C(0, 0, -\omega) = C(0, 0, -\omega\rho_3)\,, \tag{4.24}$$

[2] Lemma 5.9 below will in fact imply that the subfield $\mathbb{Q}(\zeta)$ of $\mathbb{C}$, for a primitive twelfth root ζ of unity, actually suffices to provide a home for all the monomial and non-monomial solutions under consideration.

Name	x	y	z
t_{11}	-1	0	0
t_{12}	0	-1	0
t_{13}	0	0	-1
t_{21}	$-\omega$	0	0
t_{22}	0	$-\omega$	0
t_{23}	0	0	$-\omega$
t_{31}	$-\omega^2$	0	0
t_{32}	0	$-\omega^2$	0
t_{33}	0	0	$-\omega^2$

Table 1. The 9 complex monomial solution triples (x, y, z), corresponding to 9 monomial comultiplication matrices $L = C(-x, y, z)$. Here, $\omega = \exp(2\pi i/3)$ is a primitive cube root of unity.

in agreement with (4.1)–(4.3). Theorem 5.11 below confirms that Table 1 presents the only monomial complex solutions.

5. The affine geometry of comultiplications

Following the preceding discussion of monomial solutions to Problem 3.19 for the affine-geometrical Rho-matrix P of (4.9), this chapter investigates general solutions; certainly over the field $\mathbb{C}$ of complex numbers, and to a lesser extent over other fields K or even commutative, unital rings S. Thus, triples $(x, y, z) \in S^3$ are sought, such that the matrix $L = C(-x, y, z)$ lies in the commutant of P, with $L^3 = 1_{S^3}$. We extend the notation of (4.16) to these more general settings, where

$$L^3 = C(-xX - yY - zZ, xZ + yX + zY, xY + yZ + zX) = C(1, 0, 0) = 1_{S^3}$$

provides the appropriate interpretation of (4.15).

In the setting of a ring S, the element m of the Rho-matrix P was taken to be invertible. In fact, since this element does not influence the subsequent solution procedures, which are entirely based on the circulant notation that suppresses the involvement of the Q-automorphisms ρ_1 and ρ_2, it suffices to consider $m = 1$ until the point (§5.5) at which an explicit form for the matrix L is required. An explanation for the irrelevance of the specific constant m in a classical context is given in [15]. In terms of the brief summary presented in §2.4.5, isotopic quasigroups yield isomorphic semisymmetrizations, and the respective quasigroups of (4.7) related to varying choices of m are isotopic.

5.1. **General observations.** Consider a commutative, unital ring S. Take

$$H = \begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix}, \text{ so } \det H = 3xyz - x^3 - y^3 - z^3 .$$

The cubic $\det H$ is called the *discriminant* of L. The three quantities (4.16) may be rewritten in the form

$$X = (x^2 - yz)/\det H , \quad Y = (y^2 - zx)/\det H , \quad Z = (z^2 - xy)/\det H$$

as solutions to the linear system

$$\begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{5.1}$$

reformulating (4.15). The set (4.15) of equations may be interpreted as specifying the set $\mathcal{R}$ (or more specifically, $\mathcal{R}_S$) of common roots of the cubics

$$\begin{aligned} E(x, y, z) &:= x^3 + y^3 + z^3 + 6xyz + 1 \\ &= 9xyz - \det H + 1 = xX + yY + zZ + 1 , \end{aligned} \tag{5.2}$$

$$F(x, y, z) := x^2y + y^2z + z^2x = xZ + yX + zY , \text{ and} \tag{5.3}$$

$$G(x, y, z) := x^2z + y^2x + z^2y = xY + yZ + zX \tag{5.4}$$

over the given commutative, unital ring S. Collectively, the three cubics (5.2)–(5.4) feature in an important identity, whose verification is routine.

Lemma 5.1. *Consider a commutative, unital ring S.*

a) *The identity*

$$(x + y + z)^3 = E(x, y, z) - 1 + 3F(x, y, z) + 3G(x, y, z) \tag{5.5}$$

holds for $x, y, z \in S$.

(b) *A triple $(x, y, z) \in \mathcal{R}$ satisfies $(x + y + z)^3 = -1$ in S.*

Symmetries of the set $\mathcal{R}$ of common roots of the cubics (5.2)–(5.4) are obtained as follows.

Proposition 5.2. *For the set $\mathcal{R}_S$ of triples $(x, y, z) \in S^3$ such that (x, y, z) is a common root of the cubics* (5.2)*–*(5.4)*:*

(a) *If $(x, y, z) \in \mathcal{R}_S$, then $(y, z, x) \in \mathcal{R}_S$;*
(b) *If $(x, y, z) \in \mathcal{R}_S$, then $(x, z, y) \in \mathcal{R}_S$;*

(c) *If* $(x,y,z)\in\mathcal{R}_{\mathbb{C}}$*, then* $(\overline{x},\overline{y},\overline{z})\in\mathcal{R}_{\mathbb{C}}$*;*
(d) *If* $(x,y,z)\in\mathcal{R}_{\mathbb{C}}$*, then* $(\omega x,\omega y,\omega z)\in\mathcal{R}_{\mathbb{C}}$*.*

Here, as indicated by the suffices, the latter statements (c), (d) *apply to the case* $S=\mathbb{C}$*. As before,* ω *denotes a primitive cube root of unity.*

Proof. (a) Each of the polynomials (5.2)–(5.4) is left invariant by the cyclic change of variables $(x\ y\ z)$.

(b) The variable transposition $(y\ z)$ leaves the symmetric polynomial (5.2) intact. On the other hand, $F(x,z,y)=x^2z+z^2y+y^2x=G(x,y,z)$.

(c) The claim is immediate, since the polynomials (5.2)–(5.4) have real coefficients.

(d) The cubic monomials that appear as summands in the polynomials (5.2)–(5.4) are not changed when each argument is multiplied by ω. □

Definition 5.3. The group of symmetries of the set $\mathcal{R}_{\mathbb{C}}$ generated by the symmetries (a)–(d) in the complex case is called the *little Galois group* Γ.

Remark 5.4. (a) Note that these symmetries are all linear transformations of S^3, except for the case (c) which is only semilinear.

(b) For $S=\mathbb{C}$, each symmetry is an isometry of the Hilbert space $\mathbb{C}^3$.

(c) Theorem 5.18 below provides an abstract description of the little Galois group.

Considering (5.3) alone, we make the following observation.

Lemma 5.5. *Let* S *be an integral domain. If a common root* $(x,y,z)\in S^3$ *of* (5.2)–(5.4) *has at least one zero component, then it represents a monomial solution.*

Proof. By Proposition 5.2(a), it suffices to consider the case $x=0$ in $F(x,y,z)=0$. Here, $y^2z=0$, whence $y=0$ or $z=0$. □

5.2. **Non-monomial complex solutions.** Non-monomial solution triples $(x,y,z)\in\mathbb{C}^3$ are named and displayed as complex conjugate pairs (s_{ij},s'_{ij}) in Table 2. A typical example of a non-monomial L-matrix is given by

$$C\left(\tfrac{1}{3}(\omega^2-1),\tfrac{1}{3}(\omega^2-\omega),\tfrac{1}{3}(\omega^2-\omega)\right)=\tfrac{1}{\sqrt{3}}C\left(\zeta^{-5},\zeta^{-3},\zeta^{-3}\right), \tag{5.6}$$

obtained as $C(-x,y,z)$ from the solution triple $s'_{11}=(x,y,z)$ in Table 2. Theorem 5.11 below confirms that Table 2 presents the only non-monomial complex solutions.

Name	x	y	z	Σ
s_{11}	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	$-\omega^2$
s'_{11}	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	$-\omega$
s_{12}	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	$-\omega^2$
s'_{12}	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	$-\omega$
s_{13}	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$-\omega^2$
s'_{13}	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$-\omega$
s_{21}	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	-1
s'_{21}	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	-1
s_{22}	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	-1
s'_{22}	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	-1
s_{23}	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	$\frac{1}{3}(\omega-\omega^2)=\frac{\zeta^{3}}{\sqrt{3}}$	-1
s'_{23}	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-\omega)=\frac{\zeta^{-3}}{\sqrt{3}}$	-1
s_{31}	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$-\omega$
s'_{31}	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$-\omega^2$
s_{32}	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$-\omega$
s'_{32}	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$-\omega^2$
s_{33}	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega)=\frac{\zeta^{-1}}{\sqrt{3}}$	$\frac{1}{3}(\omega^2-1)=\frac{\zeta^{-5}}{\sqrt{3}}$	$-\omega$
s'_{33}	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$\frac{1}{3}(1-\omega^2)=\frac{\zeta^{1}}{\sqrt{3}}$	$\frac{1}{3}(\omega-1)=\frac{\zeta^{5}}{\sqrt{3}}$	$-\omega^2$

Table 2. The 18 non-monomial complex solution triples (x, y, z), corresponding to 18 non-monomial comultiplication matrices $L = C(-x, y, z)$. Here, $\zeta = \exp(\pi i/6)$ as a primitive 12-th root of unity. The final column records $\Sigma = x + y + z$.

Table 2 directly manifests the symmetry noted in Proposition 5.2(c), by priming or unpriming a solution label. In Table 1, this symmetry appears as the transposition $(t_{2j}\ t_{3j})$.

According to Proposition 5.2(a), the 3-cycle $(x\ y\ z)$ permutes solutions. This symmetry is reflected in the labels s_{ij} of Table 2 and t_{ij} of Table 1, adding 1 modulo 3 to the second indices j.

Again, according to Proposition 5.2(b), the variable transposition $(y\ z)$ permutes solutions. In the tables, this symmetry fixes the elements s_{i1}, s'_{i1}, and t_{i1}.

Finally, there is the symmetry obtained from scalar multiplication by ω, as noted in Proposition 5.2(d). The symmetry is reflected in the labels s_{ij} of Table 2 and t_{ij} of Table 1, adding 1 modulo 3 to the first indices i. Correspondingly, it adds 2 modulo 3 to the first indices i of the labels s'_{ij}. The orbits under the symmetry are

$$s_j := \{ s_{1j}, s_{2j}, s_{3j} \}, \ s'_j := \{ s'_{1j}, s'_{2j}, s'_{3j} \}, \text{ and } t_j := \{ t_{1j}, t_{2j}, t_{3j} \} \tag{5.7}$$

for each $j \in \mathbb{Z}/_3$. This observation may be summarized as follows.

Lemma 5.6. *In the complex vector space $\mathbb{C}^3$, each of the respective sets (5.7) of points is jointly collinear, and collinear with the origin. Thus, one may regard $\{ s_j, s'_j, t_j \mid j \in \mathbb{Z}/_3 \}$ as a set of representatives for 9 points in the complex projective plane $\mathbb{PC}^2$ modeled by lines through the origin in $\mathbb{C}^3$.*

Using the symmetries of Proposition 5.2, it may now be confirmed that all the triples presented in Tables 1 and 2 actually are solutions.

Theorem 5.7. *For $i, j \in \mathbb{Z}_3$, each of the nine rows t_{ij} in Table 1 and eighteen rows s_{ij}, s'_{ij} in Table 2 is an element of the solution set $\mathcal{R}_{\mathbb{C}}$.*

Proof. It was already confirmed by (4.10) that $t_{11} = (-1, 0, 0)$ provides a solution. Now consider the row

$$(x, y, z) = \tfrac{1}{\sqrt{3}} \left(\zeta^{-1}, \zeta^3, \zeta^3 \right) = s_{11} \tag{5.8}$$

which sums to $-\omega^2$. Since $\left(-\omega^2 \right)^3 = -\omega^6 = -1$, it suffices by (5.5) to show that both $F(x, y, z)$ and $G(x, y, z)$ are zero.

By Proposition 5.2(b) and its proof, it merely suffices to verify that $F(x, y, z)$ is zero. But by (5.8), the value of $F(x, y, z) = x^2 y + y^2 z + z^2 x$ is proportional to $\zeta^{-2+3} + \zeta^{6+3} + \zeta^{6-1} = \zeta(1 + \zeta^8 + \zeta^4) = \zeta(1 + \omega + \omega^2) = 0$. An application of Proposition 5.2(c) then shows that s'_{11} is also a solution. Thus $t_{11}, s_{11}, s'_{11} \in \mathcal{R}_{\mathbb{C}}$.

An application of Proposition 5.2(a) now shows that $t_{1j}, s_{1j}, s'_{1j} \in \mathcal{R}_{\mathbb{C}}$ for each $j \in \mathbb{Z}/_3$. Finally, application of Proposition 5.2(d) shows that $t_{ij}, s_{ij}, s'_{ij} \in \mathcal{R}_{\mathbb{C}}$ for each $i \in \mathbb{Z}/_3$ and $j \in \mathbb{Z}/_3$, as required. □

Corollary 5.8. *Each element (x, y, z) of $\mathcal{R}_{\mathbb{C}}$ is a unit vector in the Hilbert space $\mathbb{C}^3$.*

Proof. The observation is immediate for monomial solutions from Table 1. So, by Remark 5.4(b), it suffices to examine (5.8). Here, $\|(x, y, z)\|^2 = 1$. □

5.3. **A Wessel plane representation.** A direct visual presentation of the solution set $\mathcal{R}_{\mathbb{C}}$, which is a subset of the Hilbert space $\mathbb{C}^3$, would require six real dimensions. However, Figure 4 exhibits a two-dimensional real representation in the complex plane, as explained in the caption. Solutions are represented as unit vectors in Figure 4, in deference to Corollary 5.8.

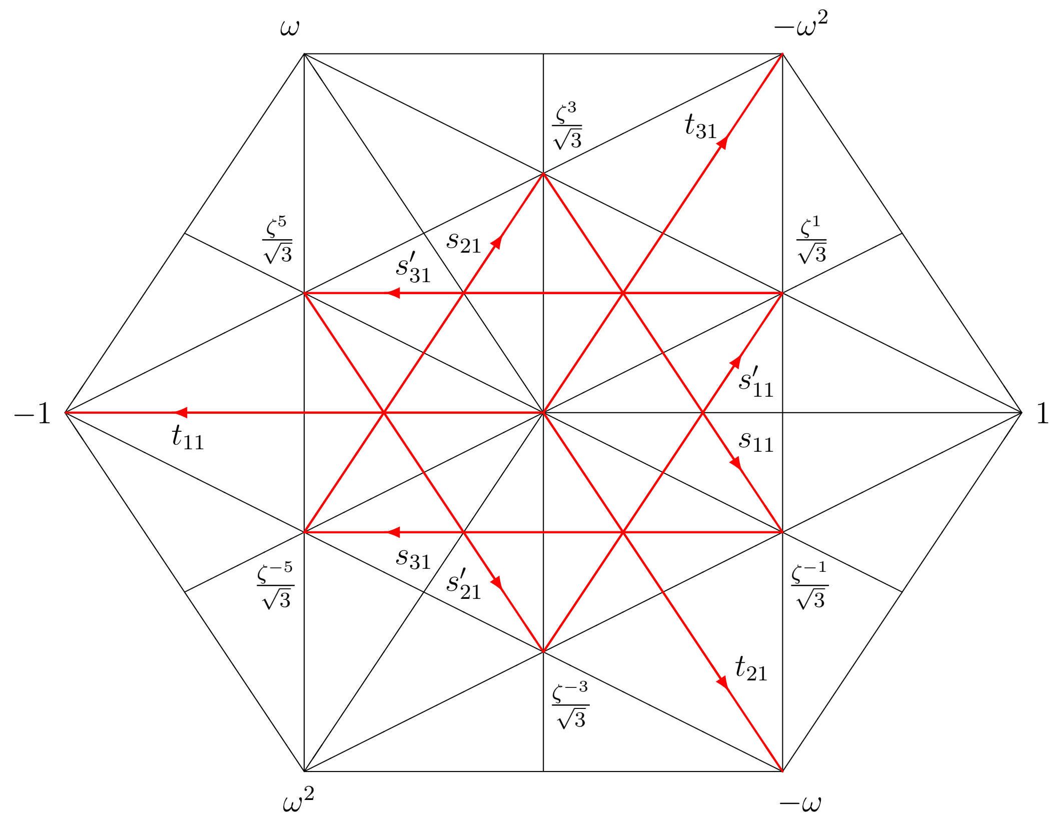


FIGURE 4. The solutions t_{k1}, s_{k1}, s'_{k1}, for $k \in \mathbb{Z}/_3$, viewed in the complex plane, with $\omega = \exp(2\pi i/3)$ and $\zeta = \exp(\pi i/6)$. A vector from z_1 to z_0 represents the solution $(z_0, z_1, z_1) \in \mathbb{C}^3$. The 18 remaining solutions may then be obtained by cyclic permutation of the coordinates, adding 1 or 2 modulo 3 to the second index j in t_{kj}, s_{kj}, s'_{kj}. (Color online.)

In geometric terms, the underlying figure may be summarized as follows. Take a regular hexagon (said to be *major*) of unit side. Use its barycenter as a seventh point, along with the six vertices of the hexagon itself, as the set of vertices for a triangulation of the major hexagon. Then, take the regular hexagon, designated as *minor*, that is the skeleton of the convex hull of the six barycenters of the triangles in the triangulation of the major hexagon. The unit vectors representing non-monomial solutions, as described in the figure caption, appear as directed short chords of the minor hexagon.

Note that, in order to coordinatize this geometry, and to obtain the full set of solutions that lie in $\mathcal{R}_{\mathbb{C}}$, it suffices by the following lemma to work with the cyclotomic field $K = \mathbb{Q}(\zeta)$.

Lemma 5.9. *The equation*

$$2\zeta^{-1} + \zeta^3 = \sqrt{3} \tag{5.9}$$

holds for the primitive twelfth root $\zeta = \exp(\pi i/6)$ of unity.

Proof. Note $\left(2\zeta^{-1} + \zeta^3\right)^2 = 4\zeta^{-2} + 4\zeta^2 + \zeta^6 = 8\Re(\zeta^2) - 1 = 8\Re(\omega) - 1 = 3$. On the other hand, $0 \leq \Re(2\zeta^{-1} + \zeta^3)$, since $0 \leq \Re(\zeta^{-1})$ and $0 \leq \Re(\zeta^3)$. This excludes $-\sqrt{3}$ as a possible value for the left hand side of (5.9). □

5.4. Affine geometry of the solution set.

The focus here in this section will progressively narrow down from a general commutative, unital ring S, to a general field K, and then to $\mathbb{C}$.

Proposition 5.10. *Consider the cubic surfaces*

$$\mathcal{F} = \{\, (x, y, z) \in S^3 \mid F(x, y, z) = 0 \,\} \tag{5.10}$$

and

$$\mathcal{G} = \{\, (x, y, z) \in S^3 \mid G(x, y, z) = 0 \,\} \tag{5.11}$$

in the affine space S^3. For each element d of S, define the affine plane

$$\mathcal{L}_d = \{\, (x, y, z) \in S^3 \mid x + y + z = d \,\} \,.$$

Then the solution set

$$\mathcal{R}_S = \biguplus_{d^3=-1} \mathcal{L}_d \cap \mathcal{F} \cap \mathcal{G} \tag{5.12}$$

in the affine space S^3.

Proof. By (5.5), the two equation systems

$$\begin{cases} E(x, y, z) = 0 \\ F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (x + y + z)^3 = -1 \\ F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

are equivalent in S^3. □

Now, consider a field K. For $0 \neq d \in K$, the plane $\mathcal{L}_d$ does not pass through the origin of the vector space K^3. Thus, the points of the affine plane $\mathcal{L}_d$ serve to model $\mathbb{P}K^2 \smallsetminus L_\infty$, the projective plane from which the line $L_\infty = \{\, (x : y : z) \mid x + y + z = 0 \,\}$ at infinity has been excised.

Theorem 5.11. *The* 27 *solutions to the system* (4.15) *presented in Tables 1 and 2 represent the full set* $\mathcal{R}_{\mathbb{C}}$ *of complex solutions. In particular, the* 3 *monomial solutions* (4.17) *are the only real solutions.*

Proof. Reinterpret (5.10) and (5.11) as cubic curves in $\mathbb{PC}^2$, with respective homogeneous equations $F(x,y,z)=0$ and $G(x,y,z)=0$. The respective tangents $T(u,v,w)$ at any point (x,y,z) of these curves are obtained by equating to zero the two forms

$$uZ(x,y,z)+vX(x,y,z)+wY(x,y,z) \tag{5.13}$$

and

$$uY(x,y,z)+vZ(x,y,z)+wX(x,y,z) \tag{5.14}$$

from the ends of (5.3) and (5.4).

If the curves $\mathcal{F}$ and $\mathcal{G}$ were to share a common component, this common component would necessarily be a tangent. In this case, certain triples (Z,X,Y) and (Y,Z,X) would be proportional, say $(Z,X,Y)=\lambda(Y,Z,X)$, with the proportionality constant λ as a cube root of unity. The linear system (5.1) would then yield the contradiction $0=xX+\lambda yX+\lambda^2 zX=-1$.

Therefore, since the cubic curves $\mathcal{F}$ and $\mathcal{G}$ have no common component, Bézout's Theorem [10, §5.3] shows that $|\mathcal{F}\cap\mathcal{G}|\leq 3^2=9$. In particular, returning to the affine setting of Proposition 5.10, $|\mathcal{L}_d\cap\mathcal{F}\cap\mathcal{G}|\leq 3^2=9$ for each d in the cyclic group $\langle\omega\rangle$, whence $|\mathcal{R}_{\mathbb{C}}|\leq 3^3=27$ by (5.12). However, Theorem 5.7 shows that $|\mathcal{R}_{\mathbb{C}}|\geq 27$, so all the inequalities are saturated. □

Corollary 5.12. *Interpreting* (5.10) *and* (5.11) *as cubic curves in* $\mathbb{PC}^2$, *each point on* $\mathcal{F}$ *and* $\mathcal{G}$ *is simple.*

Proof. The simplicity follows from the saturation of the inequalities in the last paragraph of the proof of the theorem [10, Cor. 5.3.2]. □

5.5. **Non-monomial comultiplications.** Section 4.4 gave a brief account of the monomial comultiplications that accompany the affine multiplications in the real case. This section provides an initial examination of the non-monomial comultiplications in the complex case, proposing a geometrical interpretation.

5.5.1. *Reduction to a unitary case.* Recalling the Rho-matrix

$$\mathrm{P}=C(0,0,\rho_3)=\begin{bmatrix}0&0&\rho_3\\ \rho_1&0&0\\ 0&\rho_2&0\end{bmatrix}=\begin{bmatrix}0&0&1\\ -m&0&0\\ 0&m^{-1}&0\end{bmatrix}$$

of (4.9), consider the matrix

$$L = \tfrac{1}{\sqrt{3}} C\left(\zeta^{-5}, \zeta^{-3}, \zeta^{-3}\right) = \frac{1}{\sqrt{3}} \begin{bmatrix} \zeta^{-5} & m^{-1}\zeta^{3} & \zeta^{-3} \\ m\zeta^{3} & \zeta^{-5} & m\zeta^{-3} \\ \zeta^{-3} & m^{-1}\zeta^{-3} & \zeta^{-5} \end{bmatrix} \tag{5.15}$$

of (5.6), obtained from the solution triple $s'_{11} = \frac{1}{\sqrt{3}}\left(\zeta^{1}, \zeta^{-3}, \zeta^{-3}\right)$. Bearing in mind the symmetries of Proposition 5.2, restricting to this particular matrix L brings no serious loss of generality. The subsequent discussion of the single case is a preliminary to the fuller study of all 27 comultiplications in §5.6.

Note that the characteristic polynomial $\det(T - L)$ of L is independent of m. (For example, on each forward or backward diagonal in the Sarrus' Rule for computing the determinant, an entry of m in one place is cancelled by an entry of m^{-1} in another.) Thus, it will suffice to consider the cases $m = \pm 1$, corresponding to addition or subtraction on $\mathbb{C}$. In these cases, L is unitary, as confirmed by the following lemma whose proof is a direct computation. Here and below, the Hermitian conjugate (conjugate transpose) of a square complex matrix N is written as $N^{\dagger}$.

Lemma 5.13. *For the matrix L of* (5.15)*, the equation*

$$3LL^{\dagger} = \begin{bmatrix} 2 + m^{-2} & m\omega + m^{-1}\omega^{2} + m & -\omega - m^{-2} - \omega^{2} \\ m\omega^{2} + m^{-1}\omega + m & 2m^{2} + 1 & -m - m^{-1}\omega - m\omega^{2} \\ -\omega^{2} - m^{-2} - \omega & -m - m\omega^{2} - m\omega & 2 + m^{-2} \end{bmatrix}$$

holds. In particular, L is unitary if $m = \pm 1$.

5.5.2. *Eigenvectors in the unitary case.* When L is unitary, $L^{2} = L^{-1} = L^{\dagger}$. In anticipation of the notation of §5.6, write L'_{11} for the matrix L of (5.15) with $m = 1$, reflecting its provenance from s'_{11}. The multiset spectrum of L'_{11} is $\langle \omega, \overline{\omega}, \overline{\omega} \rangle$, with

$$\begin{aligned} \mathbf{v}_0 &= \tfrac{1}{\sqrt{3}} \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}, \\ \mathbf{v}_1 &= \tfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & -\overline{\omega} \end{bmatrix}, \quad \text{and} \\ \mathbf{v}_2 &= \tfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & \overline{\omega} & -\omega \end{bmatrix} \end{aligned}$$

as respective row eigenvectors. Later, it will be helpful to consider the unitary matrix

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & -1 & 1 \\ 1 & \omega & -\overline{\omega} \\ 1 & \overline{\omega} & -\omega \end{bmatrix} \tag{5.16}$$

diagonalizing L'_{11}, so that $VL'_{11}V^\dagger = \omega \oplus \overline{\omega} \oplus \overline{\omega}$, the diagonal matrix $\mathrm{diag}(\omega, \overline{\omega}, \overline{\omega})$. Then

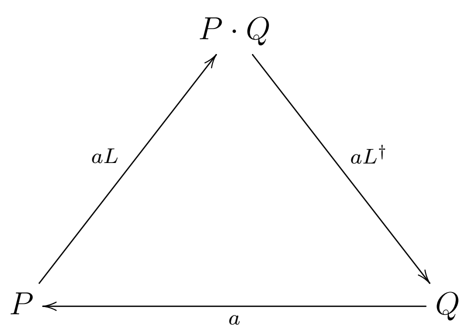
$$\begin{aligned}\Delta\colon \mathbf{v}_0 &\mapsto \omega\mathbf{v}_0 \oplus \omega^2\mathbf{v}_0\,;\\ \mathbf{v}_1 &\mapsto \omega^2\mathbf{v}_1 \oplus \omega\mathbf{v}_1\,;\\ \mathbf{v}_2 &\mapsto \omega^2\mathbf{v}_2 \oplus \omega\mathbf{v}_2\,.\end{aligned}$$

We may choose $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ as an ordered orthonormal basis for the Hilbert space $A = \mathbb{C}^3$ of qutrits.

5.5.3. *Geometrical description of the unitary non-monomial case.* In our unitary non-monomial case, the minimal polynomial is $(T - \omega)(T - \overline{\omega})$, so $1 + L + L^2 = 1 + L + L^\dagger = 0$ and $L^{-1} = L^\dagger = -1 - L$. We have

$$a\Delta = aL \oplus aL^{-1} = aL \oplus aL^\dagger = aL \oplus a(-1 - L)$$

with $-a = aL + aL^\dagger$ for each element $a \in A = \mathbb{C}^3$. Since the matrices L and $L^\dagger$ are unitary, $\|a\| = \|aL\| = \|aL^\dagger\|$. Thus, the comultiplication may be construed as erecting an equilateral triangle



on each vector $a \in \mathbb{C}^3$. Vertex names in the equilateral triangle refer to the dual situation that is provided by the multiplication in Eves' *equihoops* — idempotent entropic semisymmetric quasigroups — as displayed in [33, Figure 1]. Compare [9, 13], [25, Ex. 436], [26, Ex. 6.5B].

5.6. **The 27 complex comultiplications.** This section aims for a more comprehensive overview of the full set $\mathcal{M}$ or $\mathcal{M}_{\mathbb{C}}$ of all 27 comultiplications, extending the treatment of the comultiplication matrix L'_{11} presented in the preceding section. Here, it will also be convenient to write $\mathcal{M}_{\mathbb{R}}$ for the set of 3 real comultiplications, all of course monomial.

5.6.1. *Symmetries.* Proposition 5.2, specialized to the complex case, presents the following symmetries of $\mathcal{R}_{\mathbb{C}}$:

(a) If $(x, y, z) \in \mathcal{R}_{\mathbb{C}}$, then $(y, z, x) \in \mathcal{R}_{\mathbb{C}}$;
(b) If $(x, y, z) \in \mathcal{R}_{\mathbb{C}}$, then $(x, z, y) \in \mathcal{R}_{\mathbb{C}}$;

Name	$j=1$	$j=2$	$j=3$
M_{1j}	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix}$
M_{2j}	$\omega\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\omega\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}$	$\omega\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix}$
M_{3j}	$\overline{\omega}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\overline{\omega}\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}$	$\overline{\omega}\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix}$
L_{1j}	$\frac{1}{\sqrt{3}}\begin{bmatrix}\zeta^5&\zeta^{-3}&\zeta^3\\\zeta^{-3}&\zeta^5&\zeta^3\\\zeta^3&\zeta^3&\zeta^5\end{bmatrix}$	$\frac{1}{\sqrt{3}}\begin{bmatrix}\zeta^{-3}&\zeta^5&\zeta^3\\\zeta^{-3}&\zeta^{-3}&\zeta^{-1}\\\zeta^{-1}&\zeta^3&\zeta^{-3}\end{bmatrix}$	$\frac{1}{\sqrt{3}}\begin{bmatrix}\zeta^{-3}&\zeta^{-3}&\zeta^{-1}\\\zeta^5&\zeta^{-3}&\zeta^3\\\zeta^3&\zeta^{-1}&\zeta^{-3}\end{bmatrix}$
L'_{1j}	$\frac{1}{\sqrt{3}}\begin{bmatrix}\zeta^{-5}&\zeta^3&\zeta^{-3}\\\zeta^3&\zeta^{-5}&\zeta^{-3}\\\zeta^{-3}&\zeta^{-3}&\zeta^{-5}\end{bmatrix}$	$\frac{1}{\sqrt{3}}\begin{bmatrix}\zeta^3&\zeta^{-5}&\zeta^{-3}\\\zeta^3&\zeta^3&\zeta^1\\\zeta^1&\zeta^{-3}&\zeta^3\end{bmatrix}$	$\frac{1}{\sqrt{3}}\begin{bmatrix}\zeta^3&\zeta^3&\zeta^1\\\zeta^{-5}&\zeta^3&\zeta^{-3}\\\zeta^{-3}&\zeta^1&\zeta^3\end{bmatrix}$
L_{2j}	$\frac{\omega}{\sqrt{3}}\begin{bmatrix}\zeta^5&\zeta^{-3}&\zeta^3\\\zeta^{-3}&\zeta^5&\zeta^3\\\zeta^3&\zeta^3&\zeta^5\end{bmatrix}$	$\frac{\omega}{\sqrt{3}}\begin{bmatrix}\zeta^{-3}&\zeta^5&\zeta^3\\\zeta^{-3}&\zeta^{-3}&\zeta^{-1}\\\zeta^{-1}&\zeta^3&\zeta^{-3}\end{bmatrix}$	$\frac{\omega}{\sqrt{3}}\begin{bmatrix}\zeta^{-3}&\zeta^{-3}&\zeta^{-1}\\\zeta^5&\zeta^{-3}&\zeta^3\\\zeta^3&\zeta^{-1}&\zeta^{-3}\end{bmatrix}$
L'_{2j}	$\frac{\overline{\omega}}{\sqrt{3}}\begin{bmatrix}\zeta^{-5}&\zeta^3&\zeta^{-3}\\\zeta^3&\zeta^{-5}&\zeta^{-3}\\\zeta^{-3}&\zeta^{-3}&\zeta^{-5}\end{bmatrix}$	$\frac{\overline{\omega}}{\sqrt{3}}\begin{bmatrix}\zeta^3&\zeta^{-5}&\zeta^{-3}\\\zeta^3&\zeta^3&\zeta^1\\\zeta^1&\zeta^{-3}&\zeta^3\end{bmatrix}$	$\frac{\overline{\omega}}{\sqrt{3}}\begin{bmatrix}\zeta^3&\zeta^3&\zeta^1\\\zeta^{-5}&\zeta^3&\zeta^{-3}\\\zeta^{-3}&\zeta^1&\zeta^3\end{bmatrix}$
L_{3j}	$\frac{\overline{\omega}}{\sqrt{3}}\begin{bmatrix}\zeta^5&\zeta^{-3}&\zeta^3\\\zeta^{-3}&\zeta^5&\zeta^3\\\zeta^3&\zeta^3&\zeta^5\end{bmatrix}$	$\frac{\overline{\omega}}{\sqrt{3}}\begin{bmatrix}\zeta^{-3}&\zeta^5&\zeta^3\\\zeta^{-3}&\zeta^{-3}&\zeta^{-1}\\\zeta^{-1}&\zeta^3&\zeta^{-3}\end{bmatrix}$	$\frac{\overline{\omega}}{\sqrt{3}}\begin{bmatrix}\zeta^{-3}&\zeta^{-3}&\zeta^{-1}\\\zeta^5&\zeta^{-3}&\zeta^3\\\zeta^3&\zeta^{-1}&\zeta^{-3}\end{bmatrix}$
L'_{3j}	$\frac{\omega}{\sqrt{3}}\begin{bmatrix}\zeta^{-5}&\zeta^3&\zeta^{-3}\\\zeta^3&\zeta^{-5}&\zeta^{-3}\\\zeta^{-3}&\zeta^{-3}&\zeta^{-5}\end{bmatrix}$	$\frac{\omega}{\sqrt{3}}\begin{bmatrix}\zeta^3&\zeta^{-5}&\zeta^{-3}\\\zeta^3&\zeta^3&\zeta^1\\\zeta^1&\zeta^{-3}&\zeta^3\end{bmatrix}$	$\frac{\omega}{\sqrt{3}}\begin{bmatrix}\zeta^3&\zeta^3&\zeta^1\\\zeta^{-5}&\zeta^3&\zeta^{-3}\\\zeta^{-3}&\zeta^1&\zeta^3\end{bmatrix}$

TABLE 3. The full set $\mathcal{M}_{\mathbb{C}}$ of all 27 complex comultiplication matrices.

(c) If $(x,y,z)\in\mathcal{R}_{\mathbb{C}}$, then $(\overline{x},\overline{y},\overline{z})\in\mathcal{R}_{\mathbb{C}}$;
(d) If $(x,y,z)\in\mathcal{R}_{\mathbb{C}}$, then $(\omega x,\omega y,\omega z)\in\mathcal{R}_{\mathbb{C}}$.

Here, (a) and (b) provides symmetries of $\mathcal{R}_{\mathbb{R}}$. The linearity of the circulant notation (compare Definition 3.12) enables the map

$$\mathcal{C}\colon \mathcal{R}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}; (x, y, z) \mapsto C(-x, y, z) \tag{5.17}$$

to be used to transfer symmetries of $\mathcal{R}_{\mathbb{C}}$ across to symmetries of $\mathcal{M}_{\mathbb{C}}$. It restricts appropriately to a map $\mathcal{R}_{\mathbb{R}} \to \mathcal{M}_{\mathbb{R}}$. To facilitate the transfer, it is convenient to introduce a notation for elements of $\mathcal{M}_{\mathbb{C}}$ that reflects the earlier notation for elements of the solution set $\mathcal{R}_{\mathbb{C}}$:

(i) For $(x, y, z) = t_{ij} \in \mathcal{R}_{\mathbb{C}}$, set $M_{ij} = C(-x, y, z) \in \mathcal{M}_{\mathbb{C}}$;
(ii) For $(x, y, z) = s_{ij} \in \mathcal{R}_{\mathbb{C}}$, set $L_{ij} = C(-x, y, z) \in \mathcal{M}_{\mathbb{C}}$;
(iii) For $(x, y, z) = s'_{ij} \in \mathcal{R}_{\mathbb{C}}$, set $L'_{ij} = C(-x, y, z) \in \mathcal{M}_{\mathbb{C}}$.

These specifications apply for $i, j \in \mathbb{Z}/_3$. They transfer the solution labels from Tables 1 and 2 across to the comultiplication matrix labels in Table 3. As a mnemonic, the M_{ij} are monomial matrices. The notation L'_{11} used in §5.5.2 is a special case of (iii). Also, note that $\mathcal{M}_{\mathbb{R}}$ appears as the set $\{ M_{1j} \mid j \in \mathbb{Z}/_3 \}$ (affording a real representation of the cyclic group C_3) in this notation.

The respective symmetries (a)–(d) of Proposition 5.2 were reflected in the element labels of $\mathcal{R}_{\mathbb{C}}$ as follows:

(a) $(x, y, z) \mapsto (y, z, x)$ invokes $j \mapsto j + 1$ in t_{ij}, s_{ij}, s'_{ij};
(b) $(x, y, z) \mapsto (x, z, y)$ fixes the elements s_{i1}, s'_{i1}, and t_{i1}, implementing the transposition $(2\ 3)$ on the indices $j \neq 1$ in s_{ij}, s'_{ij}, and t_{ij};
(c) $(x, y, z) \mapsto (\overline{x}, \overline{y}, \overline{z})$ interchanges s_{ij} with s'_{ij} and t_{2j} with t_{3j}, while fixing t_{1j};
(d) $(x, y, z) \mapsto (\omega x, \omega y, \omega z)$ invokes $i \mapsto i + 1$ in t_{ij}, s_{ij} and $i \mapsto i - 1$ in s'_{ij}.

The actual transfer of symmetries of $\mathcal{R}_{\mathbb{C}}$ to symmetries of $\mathcal{M}_{\mathbb{C}}$ under $\mathcal{C}$ appears as follows.

Proposition 5.14. *Consider solutions (x, y, z) in $\mathcal{R}_{\mathbb{C}}$.*

(a) *The symmetry $C(-x, y, z) \mapsto C(-y, z, x)$ of $\mathcal{M}_{\mathbb{C}}$, invoking $j \mapsto j+1$ in M_{ij}, L_{ij}, L'_{ij}, acts as right or left multiplication by M_{13}.*
(b) *The symmetry $C(-x, y, z) \mapsto C(-x, z, y)$ of $\mathcal{M}_{\mathbb{C}}$, fixing the elements L_{i1}, L'_{i1}, and M_{i1} while implementing the transposition $(2\ 3)$ on the indices $j \neq 1$ in L_{ij}, L'_{ij}, and M_{ij}, acts as matrix transposition. Thus, the matrices L_{i1}, L'_{i1}, and M_{i1} are symmetric.*
(c) *The symmetry $C(-x, y, z) \mapsto C(-\overline{x}, \overline{y}, \overline{z})$ of $\mathcal{M}_{\mathbb{C}}$, which fixes M_{1j} while interchanging L_{ij} with L'_{ij} and M_{2j} with M_{3j}, acts as complex conjugation. In particular, the matrices M_{1j} are real.*

(d) *The symmetry* $C(-x,y,z)\mapsto C(-\omega x,\omega y,\omega z)$ *of* $\mathcal{M}_{\mathbb{C}}$*, which invokes the index shifts* $i\mapsto i+1$ *in* M_{ij},L_{ij} *and* $i\mapsto i-1$ *in* L'_{ij}*, acts as right or left multiplication by* M_{21}*.*

Proof. Since $\rho_1=-1$ and $\rho_2=\rho_3=1$, we have $C(-x,y,z)\overset{(3.6)}{=}$

$$\begin{bmatrix}-x & \rho_3 y\rho_1^{-1} & z\\ \rho_1 z\rho_3^{-1} & -\rho_1 x\rho_1^{-1} & \rho_2^{-1}y\rho_3\\ y & \rho_3^{-1}z\rho_2 & -\rho_3^{-1}x\rho_3\end{bmatrix}=\begin{bmatrix}-x & -y & z\\ -z & -x & y\\ y & z & -x\end{bmatrix}. \tag{5.18}$$

(a): Note

$$\begin{bmatrix}-x & -y & z\\ -z & -x & y\\ y & z & -x\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & -1\\ 1 & 0 & 0\\ 0 & -1 & 0\end{bmatrix}=\begin{bmatrix}-y & -z & x\\ -x & -y & z\\ z & x & -y\end{bmatrix}.$$

Left multiplication by M_{13} acts similarly.

(b) is immediate from the right hand side of (5.18).

(c) is also immediate from the right hand side of (5.18).

(d): Note that $M_{21}=C(-\omega,0,0)$ is the scalar matrix ω. □

5.6.2. *The algebra* $\mathcal{M}$*.* The simultaneous diagonalization of each matrix N in $\mathcal{M}_{\mathbb{C}}$ that is displayed in Table 4 establishes the following result.

Theorem 5.15. (a) *The set* $\mathcal{M}$ *or* $\mathcal{M}_{\mathbb{C}}$ *of all* 27 *complex comultiplication matrices forms an elementary abelian subgroup of exponent* 3 *generated by* $\{L_{31},L_{22},L_{23}\}$ *inside the unitary group* $\mathsf{U}(3)$*.*

(b) *The set* $\{M_{ij}\mid i,j\in\mathbb{Z}/_3\}$ *of monomial matrices is the kernel of the determinant map* $\det\colon\mathcal{M}_{\mathbb{C}}\to\mathbb{C}$*. It is generated by* $\{M_{13},M_{21}\}$*, and its cosets are* $\{L_{ij}\mid i,j\in\mathbb{Z}/_3\}$ *and* $\{L'_{ij}\mid i,j\in\mathbb{Z}/_3\}$*.*

The group structure on $\mathcal{M}_{\mathbb{C}}$ is not homogeneous, for example having the identity element M_{11} as a fixed point of the group automorphisms. Homogeneity requires relaxation of the associativity. For the following, compare Example 2.7(b).

Corollary 5.16. *An idempotent, totally symmetric quasigroup is provided by the operation* $(N_1,N_2)\mapsto(-N_1-N_2)$ *on the set* $\mathcal{M}_{\mathbb{C}}$*, endowing it with the structure of an affine geometry* $\mathsf{AG}(3,3)$ *of dimension* 3 *over* $\mathsf{GF}(3)$*.*

Corollary 5.16 realizes an action of the full affine group $\mathrm{Aut}\big(\mathsf{AG}(3,3)\big)\cong\big(\mathbb{Z}/_3,+\big)^3\rtimes\mathsf{GL}(3,3)$ on $\mathcal{M}_{\mathbb{C}}$, and also on $\mathcal{R}_{\mathbb{C}}$ by pullback along the map $\mathcal{C}$ of (5.17). Within the action, the symmetries of $\mathcal{M}_{\mathbb{C}}$ that are described in

Name	$j=1$	$j=2$	$j=3$
M_{1j}	$\omega^0 \oplus \omega^0 \oplus \omega^0$	$\omega^0 \oplus \omega^2 \oplus \omega^1$	$\omega^0 \oplus \omega^1 \oplus \omega^2$
M_{2j}	$\omega^1 \oplus \omega^1 \oplus \omega^1$	$\omega^1 \oplus \omega^0 \oplus \omega^2$	$\omega^1 \oplus \omega^2 \oplus \omega^0$
M_{3j}	$\omega^2 \oplus \omega^2 \oplus \omega^2$	$\omega^2 \oplus \omega^1 \oplus \omega^0$	$\omega^2 \oplus \omega^0 \oplus \omega^1$
L_{1j}	$\omega^2 \oplus \omega^1 \oplus \omega^1$	$\omega^2 \oplus \omega^2 \oplus \omega^0$	$\omega^2 \oplus \omega^0 \oplus \omega^2$
L'_{1j}	$\omega^1 \oplus \omega^2 \oplus \omega^2$	$\omega^1 \oplus \omega^0 \oplus \omega^1$	$\omega^1 \oplus \omega^1 \oplus \omega^0$
L_{2j}	$\omega^0 \oplus \omega^2 \oplus \omega^2$	$\omega^0 \oplus \omega^0 \oplus \omega^1$	$\omega^0 \oplus \omega^1 \oplus \omega^0$
L'_{2j}	$\omega^0 \oplus \omega^1 \oplus \omega^1$	$\omega^0 \oplus \omega^2 \oplus \omega^0$	$\omega^0 \oplus \omega^0 \oplus \omega^2$
L_{3j}	$\omega^1 \oplus \omega^0 \oplus \omega^0$	$\omega^1 \oplus \omega^1 \oplus \omega^0$	$\omega^1 \oplus \omega^0 \oplus \omega^1$
L'_{3j}	$\omega^2 \oplus \omega^0 \oplus \omega^0$	$\omega^2 \oplus \omega^1 \oplus \omega^2$	$\omega^2 \oplus \omega^2 \oplus \omega^1$

TABLE 4. Diagonalized versions $VNV^\dagger$ of the 27 complex comultiplication matrices N from Table 3, using the common diagonalizing matrix V of (5.16). Here, each triple $\omega^i \oplus \omega^j \oplus \omega^k$ denotes the corresponding diagonal matrix $\mathrm{diag}(\omega^i, \omega^j, \omega^k)$.

Proposition 5.14 may be translated into affine terms. For this purpose, it is convenient to write a diagonalized matrix $\omega^i \oplus \omega^j \oplus \omega^k$ simply as ijk with $i, j, k \in \mathbb{Z}/_3$.

Proposition 5.17. *Consider points ijk of the affine geometry $\mathcal{M}_\mathbb{C}$.*

(a) $C(-x, y, z) \mapsto C(-y, z, x)$ *acts as* $ijk \overset{a}{\mapsto} (i+0)(j+1)(k+2)$.
(b) $C(-x, y, z) \mapsto C(-x, z, y)$ *acts as* $ijk \overset{b}{\mapsto} ikj$.
(c) $C(-x, y, z) \mapsto C(-\overline{x}, \overline{y}, \overline{z})$ *acts as* $ijk \overset{c}{\mapsto} (-i)(-j)(-k)$.
(d) $C(-x, y, z) \mapsto C(-\omega x, \omega y, \omega z)$ *acts as* $ijk \overset{d}{\mapsto} (i+1)(j+1)(k+1)$.

Theorem 5.18. *The little Galois group Γ of Definition 5.3, as a common symmetry group of the solution set $\mathcal{R}_\mathbb{C}$ and comultiplication set $\mathcal{M}_\mathbb{C}$ from Propositions 5.2, 5.14 and 5.17, is isomorphic to $C_3^3 \rtimes C_2^2$.*

Proof. By Proposition 5.17(a) and (d) respectively, translations a by 012 and d by 111 lie in G. Furthermore, using

$$ijk \xmapsto{b} ikj \xmapsto{a} (i+0)(k+1)(j+2) \xmapsto{b} (i+0)(j+2)(k+1)\,,$$

translation by 021 lies in Γ. Since the set $\{\,012, 111, 021\,\}$ spans the vector space $(\mathbb{Z}/_3)^3$, the full set C_3^3 of translations of the affine geometry $\mathsf{AG}(3,3)$

lies in Γ. Finally, the involutions b and c (appearing in Proposition 5.17(b) and (c) respectively) generate the linear part C_2^2 of Γ. □

Corollary 5.19. *The common symmetry groups of the solution set* $\mathcal{R}_{\mathbb{R}}$ *and comultiplication set* $\mathcal{M}_{\mathbb{R}}$ *are isomorphic to the triality group* $S_3 \cong C_3 \rtimes C_2$.

The juxtaposition of Corollary 5.19 with Theorem 5.18 emphasizes that, while the triality group S_3 is a symmetry of multiplications, the complex affine case studied in this chapter admits the somewhat richer symmetry Γ of comultiplications.

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