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Subalgebra Systems of Idempotent Entropic Algebras

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1. INTRODUCTION

This paper deals with some of the theory of a class of algebras, the idempotent entropic algebras, that enjoy a number of remarkable properties. Predominant among these is the structure of their subalgebra systems, described here by the concept of an IEO-semilattice. Examples of idempotent entropic algebras, detailed in the next section, include sets, semilattices, normal bands, vector spaces under the formation of centroids, and convex sets under the filling in of convex hulls. Corresponding examples of IEO-semilattices, detailed in Section 3, include semilattices, meet-distributive bisemilattices, semilattice-normal semirings, certain semilattice-ordered groupoids, and algebras modelling the semilattice-ordered sets of utilities of game theory and mathematical economics. The fundamental lemmas governing IEO-semillatices are derived in Section 3. The next two sections examine the relationship between the system of finitely generated subalgebras and of all subalgebras, carrying this relationship over to abstract IEO-semilattices and whole classes of IEO-semilattices. Section 6 then brings the main theorem on the abstract algebraic significance of the systems of subalgebras, including representations for IEO-semilattices. Finally, Section 7 has a short discussion of possible extensions of the topics treated.

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2. Idempotent Entropic Algebras

An *algebra* (A, Ω) consists of an *underlying set* A and a set Ω of *operations* ω, for each of which a natural number $\operatorname{ar}(\omega)$ called the *arity* of ω is assigned together with an *action* $\omega: A^{\operatorname{ar}(\omega)} \to A$; $(a_1, ..., a_{\operatorname{ar}(\omega)}) \mapsto a_1 \cdots a_{\operatorname{ar}(\omega)}\omega$ *of* ω *on* A. The arity of ω will be generically denoted by n, ω being referred to as an n-ary operation. The algebra (A, Ω) is said to be *idempotent* if each singleton subset of A is a subalgebra of (A, Ω), i.e., if $x \cdots x\omega = x$ for each x in A, ω in Ω. Note that idempotence of (A, Ω) implies that there are no nullary or non-identical unary operations. The algebra (A, Ω) is said to be *entropic* if, for each n-ary operation ω and n'-ary operation ω', the action $\omega: (A^n, \omega') \to (A, \omega')$ is a homomorphism of ω'-algebras, i.e., if

$$x_{11} \cdots x_{1n'}\omega' \cdots x_{n1} \cdots x_{nn'}\omega'\omega = x_{11} \cdots x_{n1}\omega \cdots x_{1n'} \cdots x_{nn'}\omega\omega' \tag{2.1}$$

for elements x_{ij} $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n')$ of A. This paper is concerned with algebras that are both idempotent and entropic.

Example 2.0. *If Ω is empty, the algebra (A, Ω) is vacuously idempotent and entropic. The algebra (A, Ω) in this case is essentially just the set A.*

Example 2.1. If $\Omega = \{\cdot\}$ consists of just one binary idempotent commutative associative operation, then the algebra $(A, \cdot)$ is a *semilattice*. Writing the product of elements a, b of A in the usual way as $a \cdot b$, condition (2.1) takes the form $a \cdot b \cdot c \cdot d = a \cdot c \cdot b \cdot d$, so that semilattices are idempotent entropic algebras.

Example 2.2. More generally, the idempotent entropic semigroups are precisely the *normal bands* [4, p. 127, ex. 13].

Example 2.3. Idempotent entropic quasigroups $(A, \cdot)$ are precisely the *distributive barycentric spaces* of Soublin [10, Example I.1.3]. The set A has an abelian group structure $(A, +)$ together with an endomorphism δ of $(A, +)$ such that both δ and $(1 - \delta)$ are invertible, the product $\cdot$ on A then being defined by $x \cdot y = x\delta + y(1 - \delta)$.

Example 2.4. Let A be a convex subset of $\mathbb{R}^m$, and let the unit interval $I = [0, 1]$ be a set of binary operations on A via the action $\lambda: A^2 \to A$; $(a, b) \mapsto \lambda a + (1 - \lambda)b$ of "λ-weighted averaging" for λ in I. Then (A, I) is an idempotent entropic algebra.

Let (A, Ω) be an idempotent entropic algebra, ω an n-ary operation, and $S_1, ..., S_n$ subsets of A. The *complex product* $S_1 \cdots S_n\omega$ is the subset $\{s_1 \cdots s_n\omega \mid s_i \in S_i\}$ of A. If the subsets S_i are subalgebras of (A, Ω), then $S_1 \cdots S_n\omega$ is itself a subalgebra of A, since for each n'-ary operation ω' and elements $x_1, ..., x_{n'} \in S_1 \cdots S_n\omega$, say with $x_j = x_{1j} \cdots x_{nj}\omega$, (2.1) shows that $x_1 \cdots x_{n'}\omega' = (x_{11} \cdots x_{1n'}\omega') \cdots (x_{n1} \cdots x_{nn'}\omega')\omega$, which lies in $S_1 \cdots S_n\omega$ since $x_{i1} \cdots x_{in'}\omega' \in S_i$ for $i = 1, ..., n$. Thus Ω operates on the set $\mathrm{Sub}(A, \Omega)$ of non-empty subalgebras of (A, Ω). (The notation $\mathrm{Sub}\, A$ is also used occasionally: similar notations suffer similar abbreviations in cases where there is no danger of ambiguity.) The action of Ω is idempotent, since for a subalgebra S of (A, Ω) one has $S \cdots S\omega \subseteq S$, while conversely for each x in S the idempotence of ω on A implies that $x = x \cdots x\omega$, so that $S \subseteq S \cdots S\omega$. Now an identity on an algebra is said to be *regular and linear* if the variables involved in the identity appear precisely once on each side of the identity. For example, the entropic identities (2.1) are regular and linear. Regular linear identities satisfied by (A, Ω) are also satisfied by $(\mathrm{Sub}(A, \Omega), \Omega)$ (cf. [9]), and thus Ω is entropic on $\mathrm{Sub}(A, \Omega)$. One may summarise as follows.

PROPOSITION 2.1. *For an idempotent entropic algebra (A, Ω), the algebra $(\mathrm{Sub}(A, \Omega), \Omega)$ is also idempotent and entropic. Further, $(\mathrm{Sub}(A, \Omega), \Omega)$ satisfies all regular linear identities satisfied by (A, Ω).*

A subalgebra S of (A, Ω) is said to be *generated* by a subset X of S, written $S = \langle X \rangle$, if S is the smallest subalgebra of (A, Ω) containing X. In the universal algebra terminology of [2, p. 145], S is the set of Ω-words $w = w(x_1, ..., x_n)$ in elements $x_1, ..., x_n$ of X. Note, too, in this terminology, that the idempotence and entropicity of the (basic) operations Ω are inherited by the derived operations $\bar{w}$ determined by the Ω-words w. A subalgebra S of (A, Ω) is said to be *finitely generated* by a subset X if $S = \langle X \rangle$ and X is finite.

LEMMA 2.1. *Let ω be an n-ary operation of (A, Ω). If, for each $1 \leqslant i \leqslant n$, the subalgebra S_i of A is finitely generated by the set X_i, then the complex product $S_1 \cdots S_n\omega$ is finitely generated by the set $X_1 \cdots X_n\omega$.*

Proof. The typical element of $S_1 \cdots S_n\omega$ is $w_1 \cdots w_n\omega$, where $w_i = w_i(x_{i1}, ..., x_{in_i})$ is an Ω-word in elements $x_{i1}, ..., x_{in_i}$ of X_i. It will be proved by downward induction that for each $1 \leqslant j \leqslant n+1$, the elements $x_1 \cdots x_{j-1} w_j w_{j+1} \cdots w_n\omega$ with $x_i \in X_i$ for $1 \leqslant i < j$ lie in $\langle X_1 \cdots X_n\omega \rangle$. This is certainly true for $j = n+1$, and the required result follows if it can be established for $j = 1$. Suppose, then, that the hypothesis is established for $j+1$. Let m denote the arity n_j of the derived operation $\bar{w}_j$. Now

$$\begin{aligned}
&x_1 \cdots x_{j-1} w_j w_{j+1} \cdots w_n \omega \\
&\quad = x_1 \cdots x_{j-1}(x_{j1} \cdots x_{jm} \bar{w}_j)\, w_{j+1} \cdots w_n \omega \\
&\quad = (x_1 \cdots x_1 \bar{w}_j) \cdots (x_{j-1} \cdots x_{j-1} \bar{w}_j)(x_{j1} \cdots x_{jm} \bar{w}_j)(w_{j+1} \cdots w_{j+1} \bar{w}_j) \cdots \\
&\qquad (w_n \cdots w_n \bar{w}_j)\omega \qquad \text{by idempotence} \\
&\quad = (x_1 \cdots x_{j-1} x_{j1} w_{j+1} \cdots w_n \omega) \cdots \\
&\qquad (x_1 \cdots x_{j-1} x_{jm} w_{j+1} \cdots w_n \omega)\, \bar{w}_j \qquad \text{by entropicity.}
\end{aligned}$$

The latter term already lies in $\langle X_1 \cdots X_n \omega \rangle$ by the induction assumption; thus the first term does, thereby establishing the hypothesis for j and completing the inductive proof. ∎

Lemma 2.1 may be strikingly illustrated in the context of Example 2.4. The subalgebras here are just the convex subsets of A. Taking $A = \mathbb{R}^3$, $\lambda = 0.5$, S_1 a (closed convex) triangle, and S_2 a (closed) line segment piercing the triangle, the complex product, the set of midpoints between points of the triangle and points of the line segments, is obtained as a prism sliced by the triangle.

Let $\mathrm{Fg}(A, \Omega)$ denote the set of finitely generated (non-empty) subalgebras of (A, Ω). Lemma 2.1 has the following proposition as an immediate corollary.

PROPOSITION 2.2. *For an idempotent entropic algebra* (A, Ω), $(\mathrm{Fg}(A, \Omega), \Omega)$ *forms an idempotent entropic algebra, a subalgebra of* $(\mathrm{Sub}(A, \Omega), \Omega)$. *In particular* $(\mathrm{Fg}(A, \Omega), \Omega)$ *satisfies all regular linear identities satisfied by* (A, Ω).

Finally, note that there is a homomorphism

$$\iota: (A, \Omega) \to (\mathrm{Fg}(A, \Omega), \Omega);\ a \mapsto \langle a \rangle \qquad (2.2)$$

of Ω-algebras embedding the algebra (A, Ω) inside $\mathrm{Fg}(A, \Omega)$. The homomorphism ι is known as the *canonical embedding.*

3. IEO-SEMILATTICES

Propositions 2.1 and 2.2 of the previous section showed that for an idempotent entropic algebra (A, Ω), the sets $\mathrm{Sub}(A, \Omega)$ of subalgebras and $\mathrm{Fg}(A, \Omega)$ of finitely generated subalgebras have an idempotent entropic algebra structure Ω under the complex products. These sets also have an additional (join-) semilattice structure $+$ obtained by setting $S_1 + S_2 = \langle S_1 \cup S_2 \rangle$. For S_i finitely generated by X_i, $i = 1, 2$, $S_1 + S_2$ is finitely generated by $X_1 \cup X_2$. The Ω-algebra structure on these sets is related to the semilattice structure by distributive laws. In general, the operator domain Ω on a set B is said to be *distributive* over $+$ on B if, for each n-ary ω in Ω and $1 \leqslant j \leqslant n$,

$$x_1 \cdots (x_j + x_j') \cdots x_n \omega = x_1 \cdots x_j \cdots x_n \omega + x_1 \cdots x_j' \cdots x_n \omega \tag{3.1}$$

for elements $x_1, ..., x_n, x_j'$ of B.

LEMMA 3.1. *For an idempotent entropic algebra* (A, Ω), *the complex products* Ω *on* $\mathrm{Sub}(A, \Omega)$ *are distributive over the semilattice* $(\mathrm{Sub}(A, \Omega), +)$.

Proof. Let $\bar{w}$ denote the generic (m-ary) operator derived from Ω. Let ω be an n-ary operator in Ω, $1 \leqslant j \leqslant n$, and $S_1, ..., S_n$, S_j' subalgebras of (A, Ω). Then

$$\begin{aligned}
&S_1 \cdots S_j \cdots S_n \omega + S_1 \cdots S_j' \cdots S_n \omega \\
&\quad = \{(s_{11} \cdots x_1 \cdots s_{1n} \omega) \cdots (s_{m1} \cdots x_m \cdots s_{mn} \omega) \bar{w} \\
&\qquad \mid s_{il} \in S_i, x_l \in S_j \cup S_j'\} \\
&\quad = \{(s_{11} \cdots s_{m1} \bar{w}) \cdots (x_1 \cdots x_m \bar{w}) \cdots (s_{1n} \cdots s_{mn} \bar{w}) \omega \\
&\qquad \mid s_{il} \in S_i, x_l \in S_j \cup S_j'\} \\
&\quad \subseteq S_1 \cdots (S_j + S_j') \cdots S_n \omega \\
&\quad = \{s_1 \cdots (x_1 \cdots x_m \bar{w}) \cdots s_n \omega \mid s_i \in S_i, x_l \in S_j \cup S_j'\} \\
&\quad = \{(s_1 \cdots s_1 \bar{w}) \cdots (x_1 \cdots x_m \bar{w}) \cdots (s_n \cdots s_n \bar{w}) \omega \mid s_i \in S_i, x_l \in S_j \cup S_j'\} \\
&\quad = \{(s_1 \cdots x_1 \cdots s_n \omega) \cdots (s_1 \cdots x_m \cdots s_n \omega) \bar{w} \mid s_i \in S_i, x_l \in S_j \cup S_j'\} \\
&\quad \subseteq S_1 \cdots S_j \cdots S_n \omega + S_1 \cdots S_j' \cdots S_n \omega. \quad \blacksquare
\end{aligned}$$

The total structures $(\mathrm{Sub}(A, \Omega), +, \Omega)$ and $(\mathrm{Fg}(A, \Omega), +, \Omega)$ thus motivate the following definition.

DEFINITION 3.1. An *idempotent-entropic operator semilattice* or IEO-*semilattice* $(B, +, \Omega)$ is an idempotent entropic algebra (B, Ω) with a semilattice structure $(B, +)$ such that Ω is distributive over $+$.

EXAMPLE 3.0. If Ω is empty, the IEO-semilattices $(B, +, \Omega)$ are just the semilattices.

EXAMPLE 3.1. If $\Omega = \{\cdot\}$ consists of just one binary semilattice operation, IEO-semilattices $(B, +, \Omega)$ are precisely the *meet-distributive* *bisemilattices* [8]. In particular, distributive lattices are such IEO-semilattices, as are semilattices with the binary operation taken twice, once as $+$ and once in Ω.

EXAMPLE 3.2. If the Ω-structure of an IEO-semilattice is a normal band, the IEO-semilattices are the so-called *semilattice-normal semirings.* (Cf. [7], where the doubly distributive members of this class are considered.)

EXAMPLE 3.3. More generally, if Ω consists of a single idempotent entropic binary operation, an IEO-semilattice $(B, +, \Omega)$ is a semilattice-ordered groupoid. This follows from the Monotonicity Lemma below.

EXAMPLE 3.4. If Ω consists of the unit interval of weighted averaging operators of Example 2.4, the corresponding IEO-semilattices provide models of semilattice-ordered sets of utilities in the sense of mathematical economics and game theory, as in [6, 3.6.1] with the complete ordering $(3:A)$ generalised to a semilattice ordering. As in Example 3.3, the von Neumann–Morgenstern axioms $(3:B)$ follow from the Monotonicity Lemma below.

The considerations at the beginning of this section may be summarised by the following proposition.

PROPOSITION 3.1. *For an idempotent entropic algebra* (A, Ω), *the algebras* $(\mathrm{Sub}(A, \Omega), +, \Omega)$ *and* $(\mathrm{Fg}(A, \Omega), +, \Omega)$ *are IEO-semillatices.*

In subsequent sections of this paper certain key properties of IEO-semilattices play an important role. These properties are collected in the following propositions. Throughout, $(B, +, \Omega)$ denotes an IEO-semilattice. The (join-)semilattice ordering $\leqslant$ on B is the partial order on B defined by $a \leqslant b$ iff $a + b = b$.

PROPOSITION 3.2 (the Monotonicity Lemma). *Let* ω *be an* n*-ary operation of* Ω, *and* $a_1, ..., a_n$, $b_1, ..., b_n$ *elements of* B. *If* $a_i \leqslant b_i$ *for each* $1 \leqslant i \leqslant n$, *then* $a_1 \cdots a_n \omega \leqslant b_1 \cdots b_n \omega$. *In other words, the mapping* $\omega\colon (B^n, \leqslant) \to (B, \leqslant)$ *is monotone.*

Proof. For each integer j from 1 to n, one has that $a_j \leqslant b_j \Rightarrow a_j + b_j = b_j \Rightarrow a_1 \cdots a_j \cdots a_n \omega + a_1 \cdots b_j \cdots a_n \omega$

$$= a_1 \cdots (a_j + b_j) \cdots a_n \omega \text{ by distributivity}$$

$= a_1 \cdots b_j \cdots a_n \omega$, whence $a_1 \cdots a_j \cdots a_n \omega \leqslant a_1 \cdots b_j \cdots a_n \omega$. Then $a_1 a_2 \cdots a_{n-1} a_n \omega \leqslant b_1 a_2 \cdots a_{n-1} a_n \omega \leqslant b_1 b_2 \cdots a_{n-1} a_n \omega \leqslant \cdots \leqslant b_1 b_2 \cdots b_{n-1} a_n \omega \leqslant b_1 b_2 \cdots b_{n-1} b_n \omega$. ∎

The next proposition, relating a sum of products of elements of B to a product of sums of these elements, derives its name from the observation that distributivity expresses a product of sums of elements of B as a sum of products of those elements.

PROPOSITION 3.3 (the Reversed Distributivity Lemma). *Let ω be an n-ary operation of Ω, and $a_{ij} \in B$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r$. Then*

$$a_{11} \cdots a_{n1}\omega + \cdots + a_{1r} \cdots a_{nr}\omega \leqslant (a_{11} + \cdots + a_{1r}) \cdots (a_{n1} + \cdots + a_{nr})\omega.$$

Proof. For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r$, $a_{ij} \leqslant a_{i1} + \cdots + a_{in}$. Thus $a_{1j} \cdots a_{nj}\omega \leqslant (a_{11} + \cdots + a_{1r}) \cdots (a_{n1} + \cdots + a_{nr})\omega$ by the Monotonicity Lemma, whence $a_{11} \cdots a_{n1}\omega + \cdots + a_{1r} \cdots a_{nr}\omega \leqslant (a_{11} + \cdots + a_{1r}) \cdots (a_{n1} + \cdots + a_{nr})\omega$, as required. ∎

PROPOSITION 3.4 (the Sum-superiority Lemma). *For an n-ary operation ω of Ω and elements $a_1, ..., a_n$ of B, $a_1 \cdots a_n\omega \leqslant a_1 + \cdots + a_n$.*

Proof. $a_1 + \cdots + a_n = (a_1 + \cdots + a_n) \cdots (a_1 + \cdots + a_n)\omega = \sum_{\{x_1, ..., x_n\} = \{a_1, ..., a_n\}} x_1 \cdots x_n\omega \geqslant a_1 \cdots a_n\omega$. ∎

4. IDEALS AND COMPACT ELEMENTS IN IEO-SEMILATTICES

For an idempotent entropic algebra (A, Ω), Proposition 3.1 supplies two IEO-semilattices, namely $(\mathrm{Sub}(A, \Omega), +, \Omega)$ and $(\mathrm{Fg}(A, \Omega), +, \Omega)$. These IEO-semilattices are related in that each subalgebra of (A, Ω) is the sum of the finitely generated subalgebras contained in it, while conversely of course the finitely generated subalgebras are just a special kind of subalgebra. The current section develops the concepts necessary to express this relationship abstractly within the language of IEO-semilattices.

DEFINITION 4.1. A non-empty subset I of an IEO-semilattice $(B, +, \Omega)$ is called an *ideal* of B if

(i) $a \leqslant b \in I \Rightarrow a \in I$ and

(ii) $a, b \in I \Rightarrow a + b \in I$.

By the Sum-superiority Lemma, it follows that ideals of $(B, +, \Omega)$ are subalgebras of (B, Ω). Defining the *down-set* $\downarrow Y := \{x \in X \mid \exists y \in Y.\ x \leqslant y\}$ for a subset Y of a partially ordered set X, the ideals of B are precisely those subalgebras I of $(B, +, \Omega)$ for which $I = \downarrow I$ holds.

The set $\mathrm{Id}(B, +, \Omega)$ of ideals of the IEO-semilattice $(B, +, \Omega)$ has a partial order $\leqslant$ defined by set-theoretic inclusion. With respect to this partial order $\mathrm{Id}(B, +, \Omega)$ has a complete (join-) semilattice structure $+$, the supremum of a non-empty family $\{I_\xi \mid \xi \in \Xi\}$ of ideals I_ξ being $\sup_{\xi \in \Xi} I_\xi = \downarrow\{b_{\xi_1} + \cdots + b_{\xi_r} \mid \exists r.\ \exists \xi_1, ..., \xi_r \in \Xi.\ \forall 1 \leqslant i \leqslant r,\ b_{\xi_i} \in I_{\xi_i}\}$. In particular the sum of two ideals I, I' is $I + I' = \downarrow\{b + b' \mid b \in I,\ b' \in I'\}$. For an n-ary operation ω of Ω, and ideals $I_1, ..., I_n$ of $(B, +, \Omega)$, one may define a new ideal $(I_1, ..., I_n)\omega$ called the *ideal product* to be the down-set of the complex product $I_1 \cdots I_n \omega$ of the sets $I_1, ..., I_n$ considered as subalgebras of (B, Ω). Condition (i) of Definition 4.1 for $(I_1, ..., I_n)\omega$ is immediate. To verify condition (ii), suppose $b \leqslant b_1 \cdots b_n \omega$ and $b' \leqslant b'_1 \cdots b'_n \omega$ for elements b_i, b'_i of I_i, b, b' of B. Then $b + b' \leqslant b_1 \cdots b_n \omega + b'_1 \cdots b'_n \omega \leqslant (b_1 + b'_1) \cdots (b_n + b'_n)\omega \in I_1 \cdots I_n \omega$, the latter inequality holding by the Reversed Distributivity Lemma. The ideal product is clearly idempotent.

For an m-ary derived operation $\bar{w}$, the Monotonicity Lemma implies that $(I_1, ..., I_m)\bar{w} = \{b \mid \exists b_i \in I_i.\ b \leqslant b_1 \cdots b_m \bar{w}\}$. It is then immediate that regular linear identities satisfied by (B, Ω), such as the entropic laws, are also satisfied by $(\mathrm{Id}(B, +, \Omega), \Omega)$. Now suppose that ω is an n-ary operation of Ω, $1 \leqslant j \leqslant n$, and that $I_1, ..., I_{j-1}$, $I_{j+1}, ..., I_n$, I_ξ $(\xi \in \Xi)$ are ideals of $(B, +, \Omega)$. Then $b \in \sup_{\xi \in \Xi}(I_1, ..., I_\xi, ..., I_n)_\omega \Rightarrow \exists r.$ $\forall 1 \leqslant s \leqslant r$, $\forall 1 \leqslant i \neq j \leqslant n$, $\exists \xi_s \in \Xi$. $\exists b_{\xi_s} \in I_{\xi_s}$, $b_{is} \in I_i$. $b \leqslant b_{11} \cdots b_{\xi_1} \cdots b_{n1}\omega + \cdots + b_{1r} \cdots b_{\xi_r} \cdots b_{nr}\omega \leqslant (b_{11} + \cdots + b_{1r}) \cdots (b_{\xi_1} + \cdots + b_{\xi_r}) \cdots (b_{n1} + \cdots + b_{nr})\omega \Rightarrow b \in (I_1, ..., \sup_{\xi \in \Xi} I_\xi, ..., I_n)\omega$, the latter inequality holding by the Reversed Distributivity Lemma. Conversely $b \in (I_1, ..., \sup_{\xi \in \Xi} I_\xi, ..., I_n)\omega \Rightarrow \exists r. \forall 1 \leqslant s \leqslant r, \forall 1 \leqslant i \neq j \leqslant n, \exists \xi_s \in \Xi. \exists b_{\xi_s} \in I_{\xi_s}, b_i \in I_i$. $b \leqslant b_1 \cdots (b_{\xi_1} + \cdots + b_{\xi_r}) \cdots b_n \omega = b_1 \cdots b_{\xi_1} \cdots b_n \omega + \cdots + b_1 \cdots b_{\xi_r} \cdots b_n \omega$ $\Rightarrow b \in \sup_{\xi \in \Xi}(I_1, ..., I_\xi, ..., I_n)\omega$. Thus ω distributes over arbitrary sums. Altogether, one has the following result.

PROPOSITION 4.1. *For an IEO-semilattice* $(B, +, \Omega)$, *the algebra* $(\mathrm{Id}(B, +, \Omega), +, \Omega)$ *is again an IEO-semilattice, in which* $(\mathrm{Id}(B, +, \Omega), +)$ *is a complete semilattice and distributes over arbitrary sums. Further, the algebra* $(\mathrm{Id}(B, +, \Omega), \Omega)$ *satisfies all regular linear identities satisfied by* (B, Ω).

The significance of the IEO-semilattice of ideals for the subalgebras of an idempotent entropic algebra is given by the following proposition.

PROPOSITION 4.2. *For an idempotent entropic algebra* (A, Ω), *the IEO-semilattices* $(\mathrm{Sub}(A, \Omega), +, \Omega)$ *and* $(\mathrm{Id}(\mathrm{Fg}\, A, +, \Omega), +, \Omega)$ *are isomorphic via the mappings*

$$\delta\colon \mathrm{Sub}\, A \to \mathrm{Id}(\mathrm{Fg}\, A);\ S \mapsto \{F \subseteq S \mid F \in \mathrm{Fg}\, A\}$$

and

$$\gamma\colon \mathrm{Id}(\mathrm{Fg}\, A) \to \mathrm{Sub}\, A;\ I \mapsto \langle\{a \in A \mid \exists F \in I.\ a \in F\}\rangle.$$

Proof. It is routine to check that γ and δ are well-defined mutually inverse semilattice homomorphisms. Let ω be an n-ary operation of Ω, and $S_1, ..., S_n \in \operatorname{Sub} A$. Then $(\delta(S_1), ..., \delta(S_n))\omega = \downarrow\{F_1 \cdots F_n\omega \mid S_i \supseteq F_i \in \operatorname{Fg} A\}$, which is certainly contained in $\delta(S_1 \cdots S_n\omega) = \{F \subseteq S_1 \cdots S_n\omega \mid F \in \operatorname{Fg} A\}$. Conversely, suppose $\langle\{a_1, ..., a_r \mid a_i \in A\}\rangle \subseteq S_1 \cdots S_n\omega$. Each a_i lies in $S_1 \cdots S_n\omega$, say $a_i = a_{i1}, ..., a_{in}\omega$ with $a_{ij} \in S_j$. Let $F_j = \langle\{a_{ij} \mid 1 \leqslant i \leqslant n\}\rangle$. Then each a_i lies in $F_1 \cdots F_n\omega$. Thus $\langle\{a_1, ..., a_r \mid a_i \in A\}\rangle \leqslant F_1 \cdots F_n\omega$ with $S_j \supseteq F_j \in \operatorname{Fg} A$, so that $\delta(S_1 \cdots S_n\omega) \subseteq (\delta(S_1), ..., \delta(S_n))\omega$. Equality follows, proving that δ, and hence γ also, are homomorphisms of Ω-algebras. ▌

The notion of ideal thus enables one to pass abstractly from $(\operatorname{Fg} A, +, \Omega)$ to $(\operatorname{Sub} A, +, \Omega)$. To go in the reverse direction, the concept of compactness is appropriate. An element c of a complete semilattice $(C, +)$ is said to be *compact* if each family $\{c_\xi \mid \xi \in \Xi\}$ of elements of C *covering* c (i.e., such that $c \leqslant \sup_{\xi \in \Xi} c_\xi$) contains a finite subcover $\{c_1, ..., c_r\}$, so that $c \leqslant c_1 + \cdots + c_r$. The set $\operatorname{Cp}(C, +)$ of compact elements of C forms a subsemilattice of $(C, +)$ (cf. [1, Sect. VIII.5]).

DEFINITION 4.2. An IEO-semilattice $(C, +, \Omega)$ is said to be *arithmetical* if it satisfies the following condition:

(i) $(C, +)$ is a complete semilattice;

(ii) Ω is distributive over arbitrary sums;

(iii) each element of C is the supremum of the compact elements beneath it;

(iv) $\operatorname{Cp} C$ is a subalgebra of (C, Ω).

The name comes by analogy with a corresponding notion for lattices (cf. [3, Definition I.4.6]).

DEFINITION 4.3. A mapping $f\colon (C, +, \Omega) \to (C', +, \Omega)$ between arithmetical IEO-semilattices is said to be an *arithmorphism* if it is a homomorphism of IEO-semilattices with $f(\operatorname{Cp} C) \subseteq \operatorname{Cp} C'$.

Not every homomorphism of IEO-semilattices between arithmetical IEO-semilattices is an arithmorphism. For example, consider $f\colon (C, +, \Omega) \to (C', +, \Omega)$ with $f(C) = \{c'\}$ for a non-compact element c' of C'. The next proposition, showing how to pass abstractly from $(\operatorname{Sub} A, +, \Omega)$ to $(\operatorname{Fg} A, +, \Omega)$, supplies examples of arithmetical IEO-semilattices having non-compact elements (take A infinitely generated there).

PROPOSITION 4.3. *For an idempotent entropic algebra (A, Ω), the IEO-semilattice $(\operatorname{Sub} A, +, \Omega)$ is arithmetical. Its compact elements are the finitely generated subalgebras.*

Proof. Conditions (i) and (ii) of Definition 4.2 for (Sub A, $+$, Ω) follow from Propositions 4.1 and 4.2. Condition (iii) and the fact that the compact elements are the finitely generated subalgebras are standard results (cf. [1, Sects. VIII. 4–5]), while condition (iv) then follows by Lemma 2.1. ∎

EXAMPLE 4.1. If (A, Ω) is as in Example 2.4, then the set of convex subsets of A forms an arithmetical IEO-semilattice having the set of closed polytopes contained in A as its set of compact elements.

5. EQUIVALENCE OF GENERAL AND ARITHMETICAL IEO-SEMILATTICES

Sections 2 to 4 have mainly been concerned with individual idempotent entropic algebras, IEO-semilattices, and arithmetical IEO-semilattices. Attention now turns to complete classes of such algebras, particularly the *varieties* in the sense of universal algebra [2, p. 162] satisfying a given set of identities. Among the identities the regular linear ones have a special significance indicated by Propositions 2.1, 2.2, and 4.1. For a fixed domain Ω of idempotent entropic operators, let **B** be the variety of IEO-semilattices $(B, +, \Omega)$ whose Ω-reducts (B, Ω) satisfy a given set of regular linear identities. As examples where this set of identities is non-empty one might consider the meet-distributive bisemilattices of Example 3.1 or the semilattice-normal semirings of Example 3.2. Let **C** be the subclass of **B** consisting of the arithmetical IEO-semilattices. The elements of **B**, **C** respectively may be taken to be the objects of categories whose morphisms are respectively the homomorphisms of IEO-semilattices and the arithmorphisms between the arithmetical IEO-semilattices. These categories will also be denoted by **B** and **C**. The aim of this section is to prove that the categories **B** and **C** are equivalent.

THEOREM 5.1. *Let* $J\colon \mathbf{B} \to \mathbf{C}$ *be the assignment of* $(\mathrm{Id}(B, +, \Omega), +, \Omega)$ *to each IEO-semilattice* $(B, +, \Omega)$ *in* **B** *and* $\mathrm{Id}\, B \to \mathrm{Id}\, B'$; $I \mapsto \sup_{b \in I} \downarrow f(b)$ *to each morphism* $f\colon B \to B'$ *in* **B**. *Let* $K\colon \mathbf{C} \to \mathbf{B}$ *be the assignment of* $(\mathrm{Cp}(C, +), +, \Omega)$ *to each arithmetical IEO-semilattice* $(C, +, \Omega)$ *in* **C** *and the restriction* $g|_{\mathrm{Cp}\, C}\colon (\mathrm{Cp}\, C, +, \Omega) \to (\mathrm{Cp}\, C', +, \Omega)$ *to each arithmorphism* $g\colon (C, +, \Omega) \to (C', +, \Omega)$ *in* **C**. *Then J and K are functors for which there are natural transformations* $\eta\colon 1_{\mathbf{B}} \dot{\to} KJ$ *with*

$$\eta_B\colon B \to \mathrm{Cp}\,\mathrm{Id}\, B;\ b \mapsto \downarrow b \qquad \textit{for each object } B \textit{ of } \mathbf{B}$$

and $\varepsilon\colon JK \dot{\to} 1_{\mathbf{C}}$ *with*

$$\varepsilon_C\colon \mathrm{Id}\,\mathrm{Cp}\, C \to C;\ I \mapsto \sup_{k \in I} k \qquad \textit{for each object } C \textit{ of } \mathbf{C}$$

such that $(J, K; \eta, \varepsilon): \mathbf{B} \to \mathbf{C}$ *is an adjoint equivalence of categories* (*in the notation of* [5, IV.4]).

Proof. The first task is to check the conditions of Definition 4.2 on $(\mathrm{Id}\, B, +, \Omega)$ for $(B, +, \Omega)$ in $\mathbf{B}$. Conditions (i) and (ii) hold by Proposition 4.1. By standard arguments as for [3, Proposition I.4.12], condition (iii) holds and the compact elements of $\mathrm{Id}\, B$ are just the principal ideals $\downarrow b$ for b in B. Condition (iv) then follows, since for n-ary ω in Ω and $b_1, ..., b_n$ in B,

$$(\downarrow b_1, ..., \downarrow b_n)\omega = \downarrow(\downarrow b_1) \cdots (\downarrow b_n)\omega = \downarrow b_1 \cdots b_n \omega, \tag{5.1}$$

the latter equality holding by the Monotonicity Lemma and the transitivity of $\leqslant$ on B. Thus $(\mathrm{Id}\, B, +, \Omega)$ is an arithmetical IEO-semilattice. It is an object of $\mathbf{C}$ by the last part of Proposition 4.1.

For a morphism $f: B \to B'$ in $\mathbf{B}$, it follows directly from the definitions that Jf preserves suprema. For n-ary ω in Ω and ideals $I_1, ..., I_n$ of $(B, +, \Omega)$, $Jf((I_1, ..., I_n)\omega) = \downarrow\{b'_1 + \cdots + b'_r \mid \exists b_i \in B, b_{ij} \in I_j.\ b'_i \leqslant f(b_i), b_i \leqslant b_{i1} \cdots b_{in}\omega\}$. Call this set X. Let Y denote the set $(Jf(I_1), ..., Jf(I_n))\omega = \downarrow\{c'_1 \cdots c'_n\omega \mid \exists c_j \in I_j.\ c'_j \leqslant f(c_j)\}$. For the typical element y of Y, one has that $y \leqslant c'_1 \cdots c'_n\omega \leqslant f(c_1) \cdots f(c_n)\omega = f(c_1 \cdots c_n\omega)$ with c_j in I_j, so that $y \in X$, i.e. $Y \subseteq X$. Conversely, for the typical element x of X, one has $x \leqslant b'_1 + \cdots + b'_r \leqslant f(b_1) + \cdots + f(b_r) = f(b_1 + \cdots + b_r) \leqslant f(b_{11} \cdots b_{1n}\omega + \cdots + b_{r1} \cdots b_{rn}\omega) \leqslant f((b_{11} + \cdots + b_{r1}) \cdots (b_{1n} + \cdots + b_{rn})\omega) = f(b_{11} + \cdots + b_{r1}) \cdots f(b_{1n} + \cdots + b_{rn})\omega$, the last inequality holding by the monotonicity of the homomorphism f and by the Reversed Distributivity Lemma. It follows that x lies in Y, whence $X \subseteq Y$. The equality of X and Y then shows that Jf is an Ω-homomorphism. Finally, for each element a of B, one has that $Jf(\downarrow a) = \sup_{b \leqslant a} \downarrow f(b) = \downarrow f(a)$, so that Jf maps compact elements of $\mathrm{Id}\, B$—namely principal ideals—onto compact elements of $\mathrm{Id}\, B'$. Thus Jf is an arithmorphism. The other requirements of functoriality for J follow easily.

The rest of the proof is straightforward, mainly using the standard ideas of [3, Proposition I.4.12]. Equality (5.1) shows that η_B is an Ω-homomorphism, while to show that ε_C is an Ω-homomorphism, note that for an n-ary ω in Ω and ideals $I_1, ..., I_n$ of $\mathrm{Cp}\, C$, one has $\sup_{k \in (I_1, ..., I_n)\omega} k = \sup_{k_i \in I_i} k_1 \cdots k_n\omega = (\sup_{k_1 \in I_1} k_1) \cdots (\sup_{k_n \in I_n} k_n)\omega$, the latter equality holding by condition (ii) of Definition 4.2 for the arithmetical IEO-semilattice C. ∎

6. Freeness of IEO-Semilattices of Subalgebras

Let $\mathbf{A}$ be a variety of idempotent entropic algebras with operation Ω. Let $\mathbf{B}$ be the corresponding variety of IEO-semilattices whose Ω-reducts satisfy the regular linear identities of **A**. Let **C** be the corresponding category of arithmetical IEO-semilattices and arithmorphisms. There is a forgetful functor from **B** (taken as a category with the homomorphisms of IEO-semilattices as morphisms) to the category **A** obtained by forgetting the join structure. Theorem 6.1 below shows that the construction of the IEO-semilattice $(\mathrm{Fg}(A, \Omega), +, \Omega)$ of finitely generated subalgebras of an algebra (A, Ω) of **A** furnishes a left adjoint to this forgetful functor. In other words, $(\mathrm{Fg}(A, \Omega), +, \Omega)$ is the free IEO-semilattice in **B** over the algebra (A, Ω) of **A**.

THEOREM 6.1. *Let (A, Ω) be an idempotent entropic algebra in* **A**, *and $(B, +, \Omega)$ an IEO-semilattice in* **B**. *Then each Ω-homomorphism $f\colon (A, \Omega) \to (B, \Omega)$ can be extended to a unique homomorphism $\bar{f}\colon (\mathrm{Fg}(A, \Omega), +, \Omega) \to (B, +, \Omega)$ of IEO-semilattices whose composite $\bar{f} \circ \iota$ with the canonical embedding* (2.2) *is f.*

Proof. For a subalgebra S of A finitely generated by a set X, $\bar{f}(S)$ is defined to be $\sum_{x \in X} f(x)$. The mapping $\bar{f}$ is well-defined, since if S is also finitely generated by the set Y, then for each element y of Y, say $y = w(x_1, ..., x_m)$ for an Ω-word w and elements $x_1, ..., x_m$ of X, one has that $f(y) = f(x_1 \cdots x_m \bar{w}) = f(x_1) \cdots f(x_m) \bar{w} \leqslant f(x_1) + \cdots + f(x_m) \leqslant \sum_{x \in X} f(x)$, the penultimate inequality coming from the Sum-superiority Lemma. Thus $\sum_{y \in Y} f(y) \leqslant \sum_{x \in X} f(x)$, and the reverse inequality obtains by symmetry, whence equality and the well-definition of the mapping $\bar{f}$.

To show that $\bar{f}$ is an Ω-homomorphism, consider an n-ary operation ω in Ω and subalgebras $S_1, ..., S_n$ of A finitely generated by $X_1, ..., X_n$, respectively. Then

$$\begin{aligned}
\bar{f}(S_1) \cdots \bar{f}(S_n)\omega &= \Big(\sum_{x_1 \in X_1} f(x_1)\Big) \cdots \Big(\sum_{x_n \in X_n} f(x_n)\Big)\omega \\
&= \sum_{(x_1, ..., x_n) \in X_1 \times \cdots \times X_n} f(x_1) \cdots f(x_n)\omega \\
&= \sum_{(x_1, ..., x_n) \in X_1 \times \cdots \times X_n} f(x_1 \cdots x_n \omega) \\
&= \sum_{y \in X_1 \cdots X_n \omega} f(y) = \bar{f}(S_1 \cdots S_n \omega)
\end{aligned}$$

as required, the second equality coming from the distributivity of $(B, +, \Omega)$ and the last from Lemma 2.1. To show that $\bar{f}$ is a semilattice homomorphism, take subalgebras of A as above with $n = 2$. Then $\bar{f}(S_1) + \bar{f}(S_2) = \sum_{x_1 \in X_1} f(x_1) + \sum_{x_2 \in X_2} f(x_2) = \sum_{x \in X_1 \cup X_2} f(x) = \bar{f}(S_1 + S_2)$ since $S_1 + S_2 = \langle X_1 \cup X_2 \rangle$. The uniqueness of $\bar{f}$ is immediate, since for a mapping $\bar{f}$ satisfying the same conditions, and for a subalgebra S of A finitely generated by $\{x_1, ..., x_r\}$, one has that $\bar{f}(S) = \bar{f}(\langle x_1 \rangle + \cdots + \langle x_r \rangle) = \bar{f}(\langle x_1 \rangle) + \cdots + \bar{f}(\langle x_1 \rangle) = \bar{f}\iota(x_1) + \cdots + \bar{f}\iota(x_r) = f(x_1) + \cdots + f(x_r) = \bar{f}(S)$. ∎

Taking Ω empty, Theorem 6.1 yields the usual description of the free semilattice on a set X as the set of finite non-empty subsets of X under union. Indeed, an immediate corollary of Theorem 6.1 describes the free algebras in the variety **B** of IEO-semilattices.

COROLLARY 6.1. *The free IEO-semilattice in the variety* **B** *over a set X is the IEO-semilattice* $(\mathrm{Fg}(\mathbf{A}(X), \Omega), +, \Omega)$ *of finitely-generated subalgebras of the free algebra* $(\mathbf{A}(X), \Omega)$ *in the variety* **A** *over the set X.*

Proof. For an IEO-semilattice $(B, +, \Omega)$ in **B** and a set mapping $\hat{f}: X \to B$, the freeness of $(\mathbf{A}(X), \Omega)$ yields a unique Ω-homomorphism $f: (\mathbf{A}(X), \Omega) \to (B, \Omega)$ extending $\hat{f}$. Theorem 6.1 then yields a unique homomorphism $\bar{f}: (\mathrm{Fg}(\mathbf{A}(X), \Omega), +, \Omega) \to (B, +, \Omega)$ of IEO-semilattices extending f, and thus extending $\hat{f}$. ∎

As a second corollary of Theorem 6.1, one obtains the following Representation Theorem for IEO-semilattices in **B**.

COROLLARY 6.2. *Every IEO-semilattice* $(B, +, \Omega)$ *in* **B** *is a quotient of the IEO-semilattice* $(\mathrm{Fg}(B, \Omega), +, \Omega)$ *of finitely generated subalgebras of its* Ω*-reduct* (B, Ω). *Each element b of B is represented by the congruence class of the singleton subalgebra* $\{b\}$ *containing it.*

Proof. Taking the identity mapping 1_B on B as the Ω-homomorphism $1_B: (B, \Omega) \to (B, \Omega)$ for f in Theorem 6.1, the surjective IEO-semilattice morphism

$$\bar{1}_B: (\mathrm{Fg}(B, \Omega), +, \Omega) \to (B, +, \Omega)$$

produces $(B, +, \Omega)$ as the required quotient. Since $\bar{1}_B \circ \iota = 1_B$, each element b of B is the image under $\bar{1}_B$ of the singleton $\{b\}$.

One may display the Representation Theorem diagrammatically:

$$\begin{array}{ccc} (B, \Omega) & \xrightarrow{\ \iota\ } & (\mathrm{Fg}(B, \Omega), +, \Omega) \\ {\scriptstyle 1_B}\downarrow & & \downarrow{\scriptstyle \bar{1}_B} \\ (B, \Omega) & \xrightarrow[1_B]{} & (B, +, \Omega). \end{array} \tag{6.1}$$

Regarded as a diagram of Ω-algebras, this shows that for an IEO-semilattice $(B, +, \Omega)$, the Ω-reduct (B, Ω) is a retract of $(\mathrm{Fg}(B, \Omega), \Omega)$.

Since the variety of semilattices is locally finite—finitely generated semilattices are finite—Theorem 6.1 and Corollary 6.2 here are generalisations of Theorem 3.1 and Corollary 3.3 of [8], corresponding to taking **A** to be the variety of semilattices, with commutativity and associativity as regular linear identities.

Corresponding to Theorem 6.1 for **B** and $\mathrm{Fg}(A, \Omega)$, Theorem 6.2 below shows that the construction of the arithmetical IEO-semilattice $(\mathrm{Sub}(A, \Omega), +, \Omega)$ of subalgebras of an algebra (A, Ω) of **A** furnishes a left adjoint to the functor from **C** to **A** assigning the Ω-algebra of compact elements to each arithmetical IEO-semilattice.

THEOREM 6.2. *Let (A, Ω) be an idempotent entropic algebra in* **A**, *and $(C, +, \Omega)$ an arithmetical IEO-semilattice in* **C**. *Then each Ω-homomorphism $f: (A, \Omega) \to (\mathrm{Cp}(C, +), \Omega)$ can be extended to a unique arithmorphism $f': (\mathrm{Sub}(A, \Omega), +, \Omega) \to (C, +, \Omega)$ such that the following diagram commutes*:

$$\begin{array}{ccccc}
(A, \Omega) & \xrightarrow{\iota} & (\mathrm{Fg}\, A, +, \Omega) & \hookrightarrow & (\mathrm{Sub}\, A, +, \Omega) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f'} \\
(\mathrm{Cp}\, C, \Omega) & \xrightarrow[1_{\mathrm{Cp}\, C}]{} & (\mathrm{Cp}\, C, +, \Omega) & \hookrightarrow & (C, +, \Omega).
\end{array} \qquad (6.2).$$

Proof. Define $f' = \varepsilon_C \circ J\bar{f} \circ \delta$, where ε_C and J are as in Theorem 5.1, $\bar{f}$ is as in Theorem 6.1, and δ is as in Proposition 4.2. Using further notation from these results and [5, Sect. IV.4], commutation of (6.2) means that $K(f' \circ \gamma) = \bar{f} \circ \eta^{-1}_{\mathrm{Fg}A}$, whence $JK(f' \circ \gamma) = J(\bar{f} \circ \eta^{-1}_{\mathrm{Fg}A})$. Since JK is naturally isomorphic to 1_C this specifies $f' \circ \gamma$ uniquely. Since γ is an isomorphism it follows that f' is uniquely specified. ∎

Corresponding to Corollary 6.2, one immediately has the following Representation Theorem for arithmetical IEO-semilattices in **C**.

COROLLARY 6.3. *Every arithmetical IEO-semilattice $(C, +, \Omega)$ in* **C** *is a quotient of the arithmetical IEO-semilattice* $(\mathrm{SubCp}\, C, +, \Omega)$ *of subalgebras of its Ω-algebra $(\mathrm{Cp}\, C, \Omega)$ of compact elements. Each element c of C is represented by the congruence class of the subalgebra of compact elements below c.*

Proof. Taking $f = 1_{\mathrm{Cp}C}: (\mathrm{Cp}\, C, \Omega) \to (\mathrm{Cp}\, C, \Omega)$ in Theorem 6.2, $1'_{\mathrm{Cp}C}: (\mathrm{SubCp}\, C, +, \Omega) \to (C, +, \Omega)$ produces $(C, +, \Omega)$ as the required quotient. Note that for c in C, $\{k \in \mathrm{Cp}\, C \mid k \leqslant c\}$ is a subalgebra of $(\mathrm{Cp}\, C, \Omega)$ by the Monotonicity Lemma, mapping to c under $1'_{\mathrm{Cp}C} = \varepsilon_C \circ J\bar{1}_{\mathrm{Cp}C} \circ \delta$. ∎

Corollary 6.3 may be illustrated by the arithmetical IEO-semilattices of Example 4.1. Each convex subset of A is represented by the congruence class of the algebra of closed polytopes contained within it.

7. Concluding Remarks

This last section discusses some possibilities for and limitations on extending results of the paper. Perhaps the most obvious question for a reader steeped in the traditions of [1] is whether it is possible to add consideration of intersections of subalgebras to the constructions presented here, thus having, say, IEO-lattices instead of IEO-semilattices. Certainly one would then have to adjoin the empty subalgebra as the intersection of two unequal one-element subalgebras. However, the analogues of Theorem 6.1 and Corollary 6.2 would then fail in general. Consider an IEO-lattice $(B, +, \wedge, \Omega)$ with elements $b_1 \neq b_2$ and $b_3 \neq b_4$ such that $b_1 \wedge b_2 \neq b_3 \wedge b_4$. The mapping $\bar{1}_B$ cannot be a full lattice homomorphism, since if it were one could deduce the contradiction $\bar{1}_B(\emptyset) = \bar{1}_B(\langle b_1 \rangle \cap \langle b_2 \rangle) = \bar{1}_B(\langle b_1 \rangle) \wedge \bar{1}_B(\langle b_2 \rangle) = b_1 \wedge b_2 \neq b_3 \wedge b_4 = \bar{1}_B(\emptyset)$. If one is to enjoy the full advantages of the Ω-structure one has to forgo the intersections.

One modification which presents no difficulty is to consider "pointed" versions of the constructions and results of this paper. This would proceed in the same way that Proposition 4.2 and Corollary 4.3 of [8] were derived from Theorem 3.1 and Corollary 3.3 there, and would yield direct generalisations of the pointed bisemilattice results. It is hoped to apply these "pointed" versions in a future paper dealing with the fine structure of various IEO-semilattices.

Finally, one should note that despite the naturalness of the entropic laws, they are not necessary for closure of the set of subalgebras of an idempotent algebra under the operations of complex product. To see this, consider the case of semigroups, for which the idempotent entropic algebras are the normal bands as in Example 2.2. Let L be a left zero semigroup [4, (5.13)], i.e., with $lm = l$ for l, m in L, and let L' denote the semigroup obtained from L by adjoining an identity element 1 such that $1.1 = 1$, $1.l = l = l.1$ for l in L. Now although L itself is normal, L' is not if L has at least two distinct elements l, m, since $(1.l)(m.1) = l \neq m = (l.m)(l.1)$. However, a simple case analysis establishes that the set of subsemigroups of L' is closed under the complex product. There are three kinds of subsemigroup of L', namely subsemigroups S, T of L, the singleton $\{1\}$, and unions $S' = S \cup \{1\}$, $T' = T \cup \{1\}$. Then $S'T = S \cup T$ with $(S \cup T)(S \cup T) = S \cup T$, $ST' = S$, and $S'T' = (S \cup T)'$ with $(S \cup T)'(S \cup T)' = (S \cup T)'$. This example raises the problem of classifying those idempotent algebras, or at least those varieties of idempotent algebras, for which the sets of subalgebras are closed under the complex products.

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