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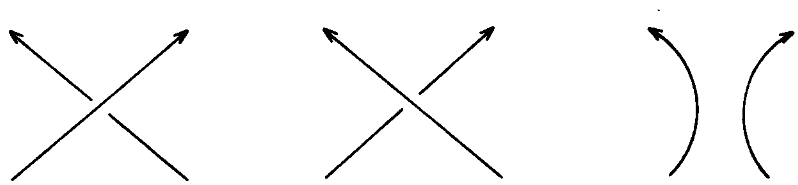
SKEIN POLYNOMIALS AND ENTROPIC RIGHT QUASIGROUPS

1. Introduction

The skein polynomial $K(l,m)$ of an oriented link K is an isotopy invariant introduced by various authors in the mid-1980s following the models of the Alexander and Jones polynomials [FH] [LM]. A surgery triple is an ordered triple (K_R, K_L, K_O) of links K_R, K_L, K_O having presentations (planar pictures) that coincide outside a ball, within which the representative presentations are as shown:



K_R K_L K_O

Thus within the ball, K_R has a right-handed crossing, K_L a left-handed crossing, and K_O no crossing at all. The integral group ring of the free abelian group on the two-element set $\{l,m\}$ is denoted by $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, and its elements are referred to as Laurent polynomials. Skein polynomials are then defined by the following theorem [LM, pp. 109, 112-3].

Theorem 1.1. To each oriented classical link K can be associated a Laurent polynomial $K(l,m)$ depending only on the isotopy class of K. The association is unique subject to the properties:

(1.2) if U is the unknot, then $U(l,m)=1$;

$$
(1.3) \quad \begin{cases} \text{if } (K_R, K_L, K_O) \text{ is a surgery triple, then} \\ lK_R(l,m) + l^{-1}K_L(l,m) + mK_O(l,m) = 0. \end{cases}
$$
□

Skein polynomials are often referred to by other names such as "oriented polynomials", "P-polynomials" (even by non-stutterers), and by various acronyms that are anagrams of HOMFLY. There are many open questions concerning skein polynomials (cf. [LM, § 4]). The purpose of the current note is to point out that two-variable polynomials in rings may not form the most appropriate algebraic environment for these link invariants. Instead, less familiar algebraic structures such as entropic right quasigroups seem better suited. Quasigroups and right quasigroups are described in the second section, while the third section deals with entropic quasigroups and piques. The fourth section then sets the skein polynomials into this new algebraic context.

2. Quasigroups and right quasigroups

A quasigroup $(Q,*)$ is a set Q equipped with a binary operation called multiplication, usua ly denoted by $*$ or juxtaposition, such that in the equation

$$(2.1) \qquad x * y = z$$

knowledge of any two of x, y, z in Q determines the third uniquely. For x in Q, define right multiplication

$$(2.2) \qquad R(x) : Q \longrightarrow Q \; ; \; y \mapsto y * x$$

and left multiplication

$$(2.3) \qquad L(x) : Q \longrightarrow Q \; ; \; y \mapsto x * y.$$

The right and left multiplications biject. From the standpoint of universal algebra, the definition (2.1) is awkward. A quasigroup may be defined alternatively as an algebra $(Q,*,/,\backslash)$ equipped with three binary operations, namely multiplication $*$, right division $/$, and left division $\backslash$, such that the identities

$$
(2.4) \quad \begin{cases} (x * y) / y = x \\ (x / y) * y = x \end{cases}
$$

and

$$(2.5)\qquad \begin{cases} x = y \backslash (y * x) \\ x = y * (y \backslash x) \end{cases}$$

are satisfied. The two definitions are equivalent [S3,117]. The first identity of (2.4) gives the injectivity of the right multiplication $R(y)$, while the second gives the surjectivity. The identities (2.5) give the bijectivity of the left multiplication $L(y)$. In a quasigroup $(Q,*,/,\backslash)$ the further identities

$$(2.6)\qquad \begin{cases} (y / x) \backslash y = x \\ y / (x \backslash y) = x \end{cases}$$

are satisfied.

A right quasigroup is an algebra $(Q,*,/)$ with binary operations of multiplication and right division such that the identities (2.4) are satisfied. Thus the right multiplications are required to biject, but no restriction is imposed on the left multiplications. Given a binary operation / on a set Q, define the opposite binary operation % on Q by

$$(2.7)\qquad x \,\%\, y = y \backslash x.$$

The identities (2.6) may then be rewritten as

$$(2.8)\qquad \begin{cases} (x \,\%\, y) \backslash y = x \\ (x \backslash y) \,\%\, y = x. \end{cases}$$

In particular,

Proposition 2.9. If $(Q,*,/,\backslash)$ is a quasigroup, then $(Q,\%,\backslash)$ is a right quasigroup. □

3. Entropic piques

A pointed idempotent quasigroup or pique $(Q,*,/,\backslash,0)$ is a quasigroup $(Q,*,/,\backslash)$ equipped with an additional nullary operation selecting an idempotent element 0 of Q (i.e. $\{0\}$ is a subquasigroup of Q). An algebra is entropic if each operation is a homomorphism [RS, 127]. A quasigroup is entropic iff it satisfies the identity

$$(3.1)\qquad (x * y) * (z * t) = (x * z) * (y * t).$$

In an entropic pique $(Q,*,/,\backslash,0)$, set $R(0)=R$ and $L(0)=L$. By (3.1), R and L commute. Murdoch's Second Structure Theorem

[Mu, Theorem 8] or the more general Structure Theorem for 3--algebras [S1, p. 76] [S2, § 6] then gives an abelian group structure (Q,+,0) such that

$$(3.2)\qquad \begin{cases} x * y = xR + yL, \\ x / y = xR^{-1} - yLR^{-1}, \\ x \backslash y = yL^{-1} - xRL^{-1}. \end{cases}$$

Conversely, (3.2) may be used to define an entropic pique structure on any module for the integral group ring $\mathbb{Z}[R^{\pm1},L^{\pm1}]$ of the free abelian group on the two-element set {R,L}. In particular, $(\mathbb{Z}[R^{\pm1},L^{\pm1}],*,/,\backslash,0)$ itself is the free entropic pique on the singleton {1}. This suggests replacing the complex analytical interpretation of "Laurent polynomials" implicit in their name by an algebraic interpretation as unary entropic pique operations.

4. Link invariants in entropic quasigroups

The basic formula (1.3) for the skein polynomials may be written as

$$(4.1)\qquad K_O(l,m) = -m^{-1}\, l\, K_R(l,m) - m^{-1}l^{-1}K_L(l,m).$$

The submonoid of $(\mathbb{Z}[l^{\pm1},m^{\pm1}],\cdot,1)$ generated by $\{(-m\, l)^{\pm1}, (-m^{-1}l^{-1})^{\pm1}\}$ is a free abelian group on $\{-m^{-1}l, -m^{-1}l^{-1}\}$. Set

$$(4.2)\qquad \begin{cases} R = -m^{-1}l, \\ L = -m^{-1}l^{-1}. \end{cases}$$

Use the same notation K both for an oriented link and for its skein polynomial. In particular U=1. The formulae (1.3) or (4.1) then appear as

$$(4.3)\qquad K_O = K_R R + K_L L.$$

In view of (3.2), this may rewritten as

$$(4.4)\qquad K_O = K_R * K_L$$

where the multiplication takes place in the free entropic pique $(\mathbb{Z}[R^{\pm1},L^{\pm1}],*,/,\backslash,0)$ on the singleton {U}.

In [LM, p. 113] a recursive procedure for calculating the skein polynomial of a link is discussed. Given a presentation K of a link, a reversal of a crossing denotes a transformation of the form $K = K_R \longmapsto K_L$ or $K = K_L \longmapsto K_R$. Similarly, <u>removal</u> of a crossing denotes a transformation of the form $K = K_R \longmapsto K_O$ or $K = K_L \longmapsto K_O$. Given a presentation K of a link, say with n crossings, there is a sequence

$$(4.5) \qquad K = K^0 \longmapsto K^1 \longmapsto \ldots \longmapsto K^{r-1} \longmapsto K^r$$

of reversals such that K^r is an unlink. Each presentation in (4.5) has n crossings. For each reversal $K^i \longmapsto K^{i+1}$ in (4.5), there is a corresponding removal $K^i \longmapsto J^i$ such that either (K^i, K^{i+1}, J^i) or (K^{i+1}, K^i, J^i) is a surgery triple. The link J^i is presented with n-1 crossings, so by induction on n its skein polynomial is known. The relations $J^i = K^i * K^{i+1}$ or $J^i = K^{i+1} * K^i$ given by (4.4) may be rewritten as

$$(4.6) \qquad \begin{cases} K^i = K^{i+1} \mathbin{\%} J^i = J^i R^{-1} - K^{i+1} L R^{-1} & \text{or} \\ K^i = K^{i+1} \backslash J^i = J^i L^{-1} - K^{i+1} R L^{-1} \end{cases}$$

respectively. Once K^r is known, the skein polynomial of $K^0=K$ may be computed by working backwards along (4.5) using (4.6). If U^c denotes the unlink with c components, it was noted in [LM, p. 113] that

$$(4.7) \qquad U^c = U^{c-1} * U^{c-1} = (R + L)^{c-1}.$$

Induction on n then proves the following.

Theorem 4.8. According to Proposition 2.9, the free entropic pique $(\mathbb{Z}[R^{\pm 1}, L^{\pm 1}], *, /, \backslash, 0)$ on the singleton $\{U\}$ determines an entropic right quasigroup $(\mathbb{Z}[R^{\pm 1}, L^{\pm 1}], \mathbin{\%}, \backslash)$. Let S be the sub-right-quasigroup generated by the set $\{U^c \mid c \in \mathbb{Z}^+\}$ of skein polynomials of unlinks. Then the skein polynomial of an arbitrary classical oriented link is an element of S. □

Theorem 4.8 may be summarized by the statement that skein polynomials of oriented links are entropic right quasigroup words in the skein polynomials of unlinks.

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