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Subdifferentiation of Monotone Functions from Semilattices to Distributive Lattices

J. D. H. SMITH
Department of Mathematics, Iowa State University, Ames, IA 50011, U.S.A.

and

LOIS THUR
Department of Mathematics and Physics, University of Montevallo, Montevallo, AL 35115, U.S.A.

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Abstract. An abstract algebraic interpretation of subgradients of real-valued convex functions is presented, and the definition is extended to modal theory. The main result is that a convex (i.e., monotone) function from a semilattice to a *complete* distributive lattice is the join of its set of subgradients.

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1. Introduction

The rich theory of convexity in real vector spaces (cf. [1], [7]) has been abstracted in a number of different ways (e.g. [3], [4], [5], [6]), many of which emphasize the connection with the theory of ordered sets. One such approach [8] involves modes (universal algebras which are idempotent and entropic) and modals (modes distributive over a semilattice). Examples of modes are furnished by convex sets under convex combinations, and by semilattices. Examples of modals are furnished by the real line under convex combinations and maximization, and by distributive lattices. The dissertation [9] initiated the program of extending the theory of subdifferentiation of convex real-valued functions [7] to convex functions from a mode to a modal (in the sense of [8, p. 58]). As noted below (Proposition 3), a function from a semilattice to a distributive lattice is convex in the sense of modal theory if and only if it is monotone. Thus the current paper is concerned with subdifferentiation of monotone functions from semilattices to distributive lattices. The main result (Theorem 7) expresses such monotone functions (with complete codomain) as the join of a set of semilattice homomorphisms. These semilattice homomorphisms are analogous to the subgradients of a classical convex function. Detailed analysis of an illustrative test case (Section 4) suggests that the order structure of these sets of subgradient homomorphisms will warrant further investigation.

2. Algebraic Preliminaries

Recall that an *operation* on a set A is a function

$$\omega\colon A^n \to A; \quad (a_1, \ldots, a_n) \mapsto a_1 \cdots a_n\omega,$$

where $n = \omega\tau$ is the *arity* of ω. An *algebra* $A = (A, \Omega)$ is a set A along with a set Ω of *basic* operations on A. The map $\tau\colon \Omega \to \mathbb{N}$ is the *type* of A. An algebra (M, Ω) is called a *mode* iff every singleton set is a subalgebra (M is *idempotent*) and every operation is an Ω-homomorphism (M is *entropic*). The entropicity condition guarantees that, for two Ω-modes A and B, the set $\mathrm{Hom}(A, B)$ of homomorphisms from A to B forms another Ω-mode, where operations on functions are defined componentwise.

For any algebra (A, Ω), $A\mathbf{S}$ denotes the set of nonempty subalgebras of A, and if $\Omega\tau \subseteq \mathbb{N}^+$, then $A\mathbf{S}_\varnothing := A\mathbf{S} \cup \{\varnothing\}$ denotes the set of all subalgebras of A.

Again, recall that a semilattice $(S, \cdot)$ is a commutative, idempotent semigroup. Entropicity of S follows from associativity and commutativity: $(w \cdot x) \cdot (y \cdot z) = (w \cdot y) \cdot (x \cdot z)$, so semilattices are modes. There are two different ways to order a semilattice $(S, \cdot)$ as a poset:

$$x \leqslant_\wedge y \iff x \cdot y = x, \quad \text{or} \tag{MS}$$

$$x \leqslant_\vee y \iff x \cdot y = y. \tag{JS}$$

The poset $(S, \leqslant_\wedge)$ is called a *meet semilattice*, while $(S, \leqslant_\vee)$ is called a *join semilattice*. The corresponding algebras with implicit orderings are denoted $(S, \wedge)$ and $(S, \vee)$, respectively.

An algebra $(M, \vee, \Omega)$ is called a *modal* iff (M, Ω) is a mode, $(M, \vee)$ is a join semilattice, and the operations in Ω distribute over $\vee$. Explicitly:

$$\begin{gathered}\forall \omega \in \Omega,\ \forall 1 \leqslant j \leqslant \omega\tau, \quad \forall x_1, \ldots, x_{\omega\tau}, x'_j \in A, \\ x_1 \cdots (x_j \vee x'_j) \cdots x_{\omega\tau}\omega = (x_1 \cdots x_j \cdots x_{\omega\tau}\omega) \vee (x_1 \cdots x'_j \cdots x_{\omega\tau}\omega).\end{gathered} \tag{D}$$

EXAMPLE 1. Distributive lattices $(D, \vee, \wedge)$ are modals.

EXAMPLE 2. For each $\lambda \in \mathrm{I}^\circ = (0, 1)$, define an operation on $\mathbb{R}^n$:

$$\underline{\lambda}\colon (\mathbb{R}^n)^2 \to \mathbb{R}^n; \quad (x, y) \mapsto (1 - \lambda)x + \lambda y =: xy\underline{\lambda}.$$

The algebra $(\mathbb{R}^n, \mathrm{I}^\circ)$ is a mode, where $\lambda \in \mathrm{I}^\circ$ is construed as the operation $\underline{\lambda}$. As $(\mathbb{R}, \max)$ is a join semilattice and each $\underline{\lambda}$ distributes over max, $(\mathbb{R}, \max, \mathrm{I}^\circ)$ forms a modal. Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *convex* if its epigraph is convex, equivalently if for every $\lambda \in \mathrm{I}^\circ$ and every $x, y \in \mathbb{R}^n$, it satisfies

$$xy\underline{\lambda}f \leqslant xfyf\underline{\lambda}.$$

Call a mode (M, Ω) and a modal $(D, \vee, \Omega)$ *compatible* iff the modes (M, Ω) and (D, Ω) have the same type. Let (M, Ω) be a mode and $(D, \vee, \Omega)$ a compatible modal. A function $f\colon M \to D$ is *Ω-convex* iff for every $\omega \in \Omega$, and every $x_1, \dots, x_{\omega\tau} \in M$, it satisfies

$$x_1 \cdots x_{\omega\tau}\omega f \leqslant_\vee x_1 f \cdots x_{\omega\tau} f\omega. \tag{CX}$$

This agrees with the definition above for $(M, \Omega) = (\mathbb{R}^n, \mathrm{I}^\circ)$ and $(D, \vee, \Omega) = (\mathbb{R}, \max, \mathrm{I}^\circ)$. Also, for a meet semilattice $(S, \wedge)$ and distributive lattice $(D, \vee, \wedge)$, a function $f\colon S \to D$ is convex iff for every $x, y \in S$, $(x \wedge y)f \leqslant xf \wedge yf$. Note that the set $\mathrm{Conv}(M, D)$ of convex functions from a mode M to a compatible modal D forms a modal of the same type as D when operations are defined componentwise.

The following proposition generalizes [2, Prop. 5.11(i)].

PROPOSITION 3. *A function from a meet semilattice to a distributive lattice is convex if and only if it is monotone.*

Proof. Recall that the orders $\leqslant_\wedge$ and $\leqslant_\vee$ coincide in a distributive lattice. Let S be a meet semilattice and D a distributive lattice. Let $g\colon S \to D$ be $\wedge$-convex, and $x, y \in S$. Then g is monotone since

$$x \leqslant y \iff x = x \wedge y \Longrightarrow xg = (x \wedge y)g \leqslant xg \wedge yg \leqslant yg.$$

Conversely, let g be monotone, and $x, y \in S$. Since $x \wedge y \leqslant x$ and $x \wedge y \leqslant y$, g satisfies $(x \wedge y)g \leqslant xg$ and $(x \wedge y)g \leqslant yg$. Thus $(x \wedge y)g \leqslant xg \wedge yg$. □

3. Subgradients

For a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$, the *subgradient of f at x* is

$$\partial f(x) = \{h \in \mathrm{Hom}((\mathbb{R}^n, \mathrm{I}^\circ), (\mathbb{R}, \mathrm{I}^\circ)) \mid h \leqslant f, xh = xf\},$$

the set of affine functions approximating f from below at x. This definition agrees with [7 (3.20), p. 33]. It turns out that ∂f is a function from $\mathbb{R}^n$ into $(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}), \mathrm{I}^\circ)\mathbf{S}$, because $\partial f(x)$ is closed under the operations in I° (cf. [9, Def. 4.2]). The function ∂f is called the *subdifferential* of f. This example gives the motivation for more general definitions in modal theory.

In the following, let $(S, \wedge)$ be a semilattice, $(D, \vee, \wedge)$ a distributive lattice, and $f\colon S \to D$ a convex function.

DEFINITION 4. For $c \in S$, the *subgradient of f at c* is the set of meet homomorphisms approximating f from below at c, specifically,

$$\partial f(c) := \{k \in \mathrm{Hom}((S, \wedge), (D, \wedge)) \mid k \leqslant f, ck = cf\}.$$

Note that as $(D, \vee, \wedge)$ is a distributive lattice, the set D^S of functions from S to D forms a distributive lattice $(D^S, \vee, \wedge)$. Recall that $(\mathrm{Hom}(S, D), \wedge)$ is a meet semilattice whose underlying set is a subset of D^S.

PROPOSITION 5. *The set $\partial f(c)$ forms a (possibly empty) subsemilattice of* $(\mathrm{Hom}(S, D), \wedge)$.

Proof. Let $h, k \in \partial f(c)$. By definition $h \leqslant f$, so that $h \wedge k \leqslant f$. Also $ch = ck = cf$, so $c(h \wedge k) = ch \wedge ck = cf \wedge cf = cf$. Thus $h \wedge k \in \partial f(c)$. Therefore $\partial f(c)$ is a subsemilattice of $\mathrm{Hom}(S, D)$. □

PROPOSITION 6. *If D is a complete lattice, then for every $c \in S$, $\partial f(c)$ is nonempty and has a maximal element.*

Proof. We show that the hypotheses of Zorn's lemma hold. Thus we need $\partial f(c)$ nonempty and the existence of upper bounds of chains in $\partial f(c)$. Let T be a maximal chain in $(S, \wedge)$ containing c. Define a function

$$h_T\colon\ S \to D\colon\ b \mapsto \bigvee\{xf \mid x \wedge b = x \in T\}.$$

We claim that h_T is a meet homomorphism and is in $\partial f(c)$. Let $a, b \in S$. Then

$$\begin{aligned}
&ah_T \wedge bh_T \\
&\quad = \left(\bigvee\{xf \mid x = x \wedge a \in T\}\right) \wedge \left(\bigvee\{yf \mid y = y \wedge b \in T\}\right) \\
&\quad = \bigvee\{xf \wedge yf \mid x = x \wedge a,\ y = y \wedge b \in T\} && \text{[distributivity in } D\text{]} \\
&\quad = \bigvee\{(x \wedge y)f \mid x = x \wedge a,\ y = y \wedge b \in T\} && [f \text{ monotone}] \\
&\quad = \bigvee\{tf \mid t = x \wedge a \wedge y \wedge b = x \wedge y \in T\} && [T \text{ a chain}] \\
&\quad = \bigvee\{tf \mid t = t \wedge a \wedge b \in T\} \\
&\quad = (a \wedge b)h_T.
\end{aligned}$$

Clearly, $h_T \leqslant f$ and $ch_T = cf$ since $c \in T$, so $h_T \in \partial f(c) \neq \varnothing$. Now let Θ be any nonempty chain of homomorphisms in $\partial f(c)$. Define a function $h = \bigvee\{k \in \mathrm{Conv}(S, D) \mid k \in \Theta\}$. We need $h \in \partial f(c)$. If we can show h is a homomorphism, we are done since then

$$(\forall k \in \Theta,\ ck = cf \,\&\, k \leqslant f) \Longrightarrow (ch = cf \,\&\, h \leqslant f) \Longrightarrow h \in \partial f(c).$$

Let $x, y \in S$. Note first that since every $k \in \Theta$ is convex, h is convex (because arbitrary joins exist in $\mathrm{Conv}(S, D)$). So $(x \wedge y)h \leqslant xh \wedge yh$, and also,

$$
\begin{aligned}
xh \wedge yh &= \Big(\bigvee\{xp \mid p \in \Theta\}\Big) \wedge \Big(\bigvee\{yq \mid q \in \Theta\}\Big) \\
&= \bigvee\{xp \wedge yq \mid p, q \in \Theta\} && \text{[distributivity in } D\text{]} \\
&\leqslant \bigvee\{xk \wedge yk \mid p, q \in \Theta,\ k = p \vee q\} && [\wedge \text{ is monotone}] \\
&= \bigvee\{xk \wedge yk \mid k \in \Theta\} && [\Theta \text{ is a chain}] \\
&= \bigvee\{(x \wedge y)k \mid k \in \Theta\} && \text{[homomorphism]} \\
&= (x \wedge y)h.
\end{aligned}
$$

Thus h is a homomorphism. By Zorn's Lemma, $\partial f(c)$ has maximal elements.□

THEOREM 7. *A convex function from a meet semilattice to a complete distributive lattice is a join of semilattice homomorphisms.*

Proof. Let $f\colon (S, \wedge) \to (D, \vee, \wedge)$ be convex. It has been shown that for every $c \in S$, there is an $h \in \partial f(c)$ such that $ch = cf$. Thus $\bigvee \partial f(S) \geqslant f$, and since clearly $\bigvee \partial f(S) \leqslant f$, the theorem is proved. □

For f, S, and D as above with D complete, define the *subdifferential of* f to be the function

$$\partial f\colon\ S \to (\mathrm{Hom}(S, D))\mathbf{S};\quad c \mapsto \partial f(c).$$

By Propositions 5 and 6, this is well defined.

Note: If the distributive lattice is not complete, we can define the differential of f with larger codomain, $\partial f\colon\ S \to (\mathrm{Hom}(S, D))\mathbf{S}_{\varnothing}$, since the empty set is also a submode.

4. An Illustration

A meet semilattice S and a completely distributive lattice D are shown by their Hasse diagrams in Figure 1. Also shown is the position in D of the images of the elements of S under a (convex) function $f\colon\ S \to D$. The subgradient

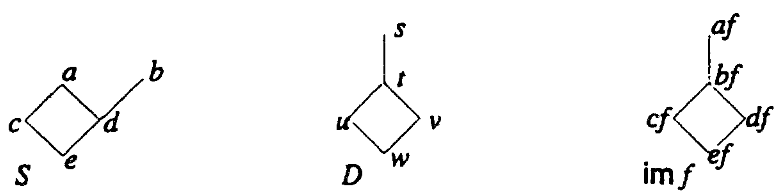

Fig. 1. Hasse diagrams of $(S, \wedge)$, $(D, \vee, \wedge)$, and $\mathrm{im}\, f$.

homomorphisms for f are shown separately in Figure 2, while Figure 3 shows all of them as a sub-poset of $(\mathrm{Hom}(S, D), \leqslant \wedge)$, labelled by the row and column headings of Figure 2.

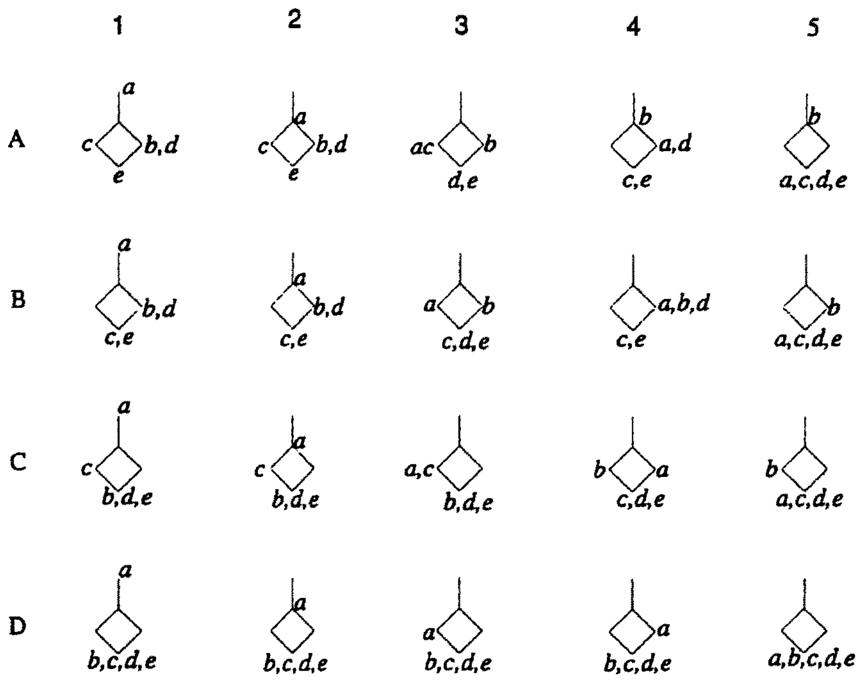

Fig. 2. Hasse diagrams of im h for subgradient homomorphisms h.

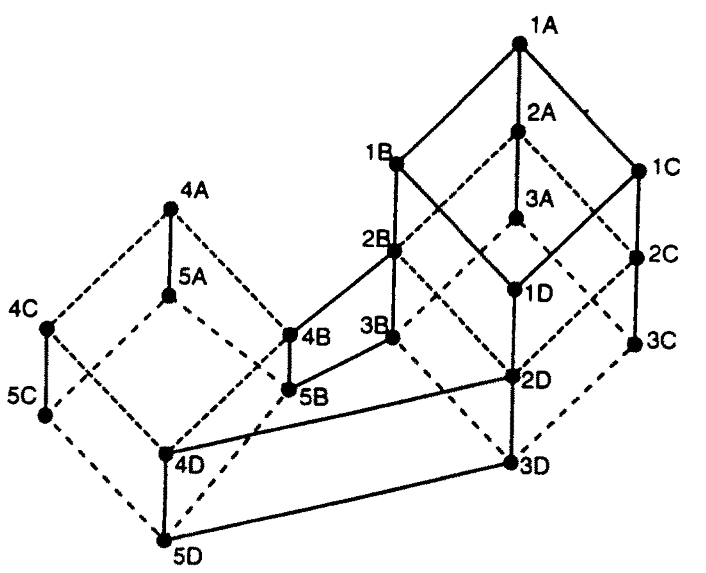

Fig. 3. Hasse diagrams of the poset of subgradient homomorphisms for f.

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