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Separable modes*

A. B. ROMANOWSKA AND J. D. H. SMITH

1. Introduction

A *mode* (A, Ω) is an idempotent, entropic algebra, i.e. with each singleton a subalgebra and each operation a homomorphism [RS1, 145]. Examples of modes are furnished by affine spaces and their reducts (§2), semilattices, and convex sets (Example 5.1). There are also many interesting varieties of groupoid modes. A surprisingly large number of modes exhibit a certain similarity in their structure: they may be described as subreducts of Płonka sums of affine spaces. The main purpose of this paper is to present a result, Theorem 4.3, characterizing such modes by a separability condition (Definition 4.1). A sharper condition, finite separability (Definition 3.1), identifies those modes that are subreducts of affine spaces alone. The affine spaces appropriate to modes in a given variety $\mathfrak{V}$ of modes are selected by an "affinisation" process discussed in the second section. This process associates a commutative ring $R(\mathfrak{V})$ with each mode variety $\mathfrak{V}$. Affine spaces over $R(\mathfrak{V})$ then have $\mathfrak{V}$-reducts. For a cogenerating affine space F over $R(\mathfrak{V})$, finitely separable modes in $\mathfrak{V}$ are separated by $\mathfrak{V}$-homomorphisms into the $\mathfrak{V}$-reduct of F. If $\mathfrak{V}$ is the variety of barycentric algebras (Example 5.1), then F is the real line $\mathbb{R}$ and the finitely separable algebras are the convex sets. The extended reals $\mathbb{R}^\infty$, with an additional "point at infinity", carry a natural barycentric algebra structure. Any barycentric algebra may be separated by homomorphisms into $\mathbb{R}^\infty$ [Fl, Theorem 3.10]. For a general mode variety $\mathfrak{V}$, the separable modes in $\mathfrak{V}$ are separated by homomorphisms into an analogous extension F^∞ of the "finite points" (elements of F) by an additional "point at infinity". Examples are given (3.4 and §5) to show that general mode varieties may contain inseparable algebras. Nevertheless, Theorem 4.3 does represent the closest approach yet available to a general structure theory for modes.

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Most of the basic conventions and constructions used in the paper are as in [RS1], which serves as a reference for notations otherwise unexplained here. An exception is provided by the fundamental Płonka sum construction, which is based on join semilattices in this paper. Recall that a semilattice $(S, +)$ is a set S equipped with a commutative, idempotent, and associative binary *sum* $+$. The set S is partially ordered by

$$s \leq_+ t \quad \Leftrightarrow \quad s + t = t. \tag{1.1}$$

Let $\tau : \Omega \to \mathbb{N}$ be a *plural* type, i.e. such that $\forall \omega \in \Omega, \omega\tau > 1$. Then $(S, +)$ becomes an *Ω-semilattice* (S, Ω) on setting

$$s_1 \dots s_{\omega\tau}\omega = s_1 + \cdots + s_{\omega\tau} \tag{1.2}$$

for each basic operation ω. As a partially ordered set, $(S, \leq_+)$ may also be considered as a small category (S) with object set S, singleton morphism sets $(S)(s, t) = \{s \to t\}$ for $s \leq_+ t$, and empty morphism sets $(S)(s, t)$ for $s \nleq_+ t$. Let (Ω) denote the concrete category of Ω-algebras and homomorphisms between them. Let $G : (S) \to (\Omega)$ be a functor. Then the *Płonka sum SG over the semilattice* $(S, +)$ *by the functor* G is the disjoint union $SG = \dot{\bigcup}_{s \in S} sG$ of the underlying sets sG, equipped with the Ω-algebra structure given for an n-ary operation ω in Ω and $s_1, \dots, s_n, t = s_1 + \cdots + s_n$ in S by

$$\omega : s_1G \times \cdots \times s_nG \to tG; \qquad (x_1, \dots, x_n) \mapsto x_1(s_1 \to t)G \dots x_n(s_n \to t)G\omega. \tag{1.3}$$

The *projection* of the Płonka sum SG is the homomorphism $\pi : (SG, \Omega) \to (S, \Omega)$ with restrictions $sG \to \{s\}$.

Recall that an identity $w = w'$ is *regular* if the sets of variables involved on each side are equal [RS1, p. 13]. A variety is *regular* if it may be defined by regular identities. Thus regular varieties of plural type $\tau : \Omega \to \mathbb{N}$ contain the variety of $(\Omega$-) semilattices. For a given variety $\mathfrak{V}$, the *regularization* $\tilde{\mathfrak{V}}$ is the variety of algebras satisfying the regular identities satisfied by $\mathfrak{V}$-algebras. Płonka's Theorem [RS1, 239] describes the algebras in the regularization $\tilde{\mathfrak{V}}$ of an irregular variety $\mathfrak{V}$ as Płonka sums of $\mathfrak{V}$-algebras.

2. Affinisation

For a commutative ring R (with a unit element), *affine spaces* over R are defined algebraically as being either empty or idempotent reducts of unital R-modules.

Affine spaces over R form a variety $\underline{\underline{R}}$ [RS1, 255]. A *Mal'cev operation* is a ternary operation P on an algebra A satisfying the identities

$$(x, y, y)P = x = (y, y, x)P. \tag{2.1}$$

Then the variety $\underline{\underline{Z}}$ of affine spaces over the integers may be realised as the variety of algebras of type $\{(P, 3)\}$ for which P is an entropic Mal'cev operation, i.e. for which $P : (A^3, P) \to (A, P)$ is a homomorphism. The variety $\underline{\underline{R}}$ may be realised as a variety of algebras of type $(R \times \{2\}) \cup \{(P, 3)\}$. The action of P on an affine space A is as the entropic Mal'cev operation

$$(x, y, z)P = x - y + z, \tag{2.2}$$

while the action of an element r of R is as

$$xy\underline{r} = (1 - r)x + ry. \tag{2.3}$$

Given varieties $\mathfrak{V}_i$ of type $\tau_i : \Omega_i \to \mathbb{N}$, for $i = 1, 2$, the *tensor product* $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ is the variety of algebras A of type $\bigcup_{n=0}^{\infty} (\tau_1^{-1}(n) \cup \tau_2^{-1}(n)) \times \{n\}$ satisfying the identities that (A, Ω_1) be a $\mathfrak{V}_1$-algebra, that (A, Ω_2) be a $\mathfrak{V}_2$-algebra, and that each operation ω of Ω_1 is a homomorphism $(A^{\omega\tau_1}, \Omega_2) \to (A, \Omega_2)$ [RS1, 232]. There are reductions $\pi_i : \mathfrak{V}_1 \otimes \mathfrak{V}_2 \to \mathfrak{V}_i$ by means of which $\mathfrak{V}_1 \otimes \mathfrak{V}_2$-algebras may be considered just as $\mathfrak{V}_i$-algebras. For a variety $\mathfrak{V}$ of modes, the *affinisation* is defined to be the variety $\underline{\underline{Z}} \otimes \mathfrak{V}$. Now $\underline{\underline{Z}} \otimes \mathfrak{V}$ is also a variety of modes, but equipped with the Mal'cev operation (2.1). There is then a ring $R(\mathfrak{V})$ such that

$$\underline{\underline{Z}} \otimes \mathfrak{V} = \underline{\underline{R(\mathfrak{V})}} \tag{2.4}$$

[RS1, 254]. The $\mathfrak{V}$-reducts of $\underline{\underline{R(\mathfrak{V})}}$-algebras are often very useful and informative models of $\mathfrak{V}$-algebras.

EXAMPLE 2.5. If S is a commutative ring with a unit element, then $\underline{\underline{Z}} \otimes \underline{\underline{S}} = \underline{\underline{Z \otimes S}} = \underline{\underline{S}}$, so that $R(\underline{\underline{S}}) = S$. In other words, varieties of affine spaces over a ring are their own affinisations. □

EXAMPLE 2.6. Let (τ) denote the variety of all modes of given plural type $\tau : \Omega \to (\mathbb{N} - \{0, 1\})$ (cf. [RS1, p. 32]). Then $R(\tau)$ is the integral polynomial ring over a set $\{X_{\omega i} \mid \omega \in \Omega, 1 \leq i < \omega\tau\}$ of $\sum_{\omega \in \Omega} (\omega\tau - 1)$ indeterminates. For

$\omega \in \tau^{-1}(n+1)$, the corresponding operation on an affine space over $R(\tau)$ is

$$x_0 x_1 \ldots x_n \omega = x_0 \left(1 - \sum_{i=1}^{n} X_{\omega i}\right) + \sum_{i=1}^{n} x_i X_{\omega i} \tag{2.7}$$

for the indeterminates $X_{\omega 1}, \ldots, X_{\omega n}$ *pertaining to* ω. □

EXAMPLE 2.8. The variety $\mathfrak{D}$ of *differential groupoids* [RS4] or "LIR-groupoids" [Ro] [RR1] [RR2] is the variety of modes $(G, \cdot)$ with a single binary operation $\cdot$ satisfying the *reduction law*

$$x \cdot (y \cdot z) = x \cdot y. \tag{2.9}$$

In the notation of Example 2.6, let X be the indeterminate pertaining to groupoid multiplication. Then $R(\mathfrak{D})$ is a quotient of $\mathbb{Z}[X]$. Now (2.9) holds for $x \cdot y = x(1-X) + yX$ in $\underline{\underline{R(\mathfrak{D})}}$. Equating coefficients of z in $x(1-X) + (y(1-X) + zX)X = x \cdot (y \cdot z) = x \cdot y = x(1-X) + yX$ shows that $zX^2 = 0$, so that $R(\mathfrak{D})$ is a quotient of $\mathbb{Z}[X]/\langle X^2 \rangle = \mathbb{Z}[d]$, the ring of *integral dual numbers* with $d^2 = 0$. Conversely, affine spaces over $\mathbb{Z}[d]$ are differential groupoids under $x \cdot y = x(1-d) + yd$. Thus $R(\mathfrak{D}) = \mathbb{Z}[d]$. □

EXAMPLE 2.10. The variety $\underline{\underline{Ln}}$ of *left normal bands* is the variety of semigroup modes $(L, *)$ satisfying

$$x * y * z = x * z * y. \tag{2.11}$$

In the notation of Example 2.6, let X be the indeterminate pertaining to groupoid multiplication. Now in $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{Ln}}$, equating coefficients of y in $x * y = (x * y) * y$ gives $yX = yX(1-X) + yX$, or $yX = yX^2$. Then $x * (y * z) = x(1-X) + y(1-X)X + zX^2 = x(1-X) + zX = x * z$, whence $x * y = x * (z * y) = x * (y * z) = x * z$ using (2.11). Thus $x * y = x * x$, and equating coefficients of y gives $yX = 0$. This means that $R(\underline{\underline{Ln}})$ is a quotient of $\mathbb{Z}[X]/\langle X \rangle = \mathbb{Z}$. On the other hand, defining $x * y = xy\underline{0} = x$ in an integral affine space gives a $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{Ln}}$-algebra, so that $R(\underline{\underline{Ln}}) = \mathbb{Z}$. The variety $\underline{\underline{Lz}}$ of *left zero bands* is the variety of semigroups $(L, *)$ with

$$x * y = x. \tag{2.12}$$

Then $\underline{\underline{\mathbb{Z}}} \subseteq \mathbb{Z} \otimes \underline{\underline{Lz}} \subseteq \underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{Ln}} = \underline{\underline{\mathbb{Z}}}$, so that $R(\underline{\underline{Lz}})$ is also $\mathbb{Z}$. The variety of left normal bands is the regularisation of the variety of left zero bands, so one may paraphrase the above as

$$R(\underline{\underline{\tilde{Lz}}}) = R(\underline{\underline{Lz}}). \tag{2.13}$$

□

The equality (2.13) underlies and illustrates a more general equality between the affinisations of a variety and its regularisation.

THEOREM 2.14. *Let* $\mathfrak{V}$ *be an irregular plural variety of modes, with regularisation* $\tilde{\mathfrak{V}}$. *Then* $R(\mathfrak{V}) = R(\tilde{\mathfrak{V}})$.

Proof. By Płonka's Theorem [RS1, 239], there is a derived partition operation $*$ in $\tilde{\mathfrak{V}}$ satisfying the identities for a left normal band (cf. [RS1, 237(i)]). The corresponding reduction $\tilde{\mathfrak{V}} \to \underline{\underline{Ln}}$ commutes with the affinisation to give $\underline{\underline{\mathbb{Z}}} \otimes \tilde{\mathfrak{V}} \to \underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{Ln}}$. But by Example 2.10, $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{Ln}} = \underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{Lz}}$, so that the partition operation on $\underline{\underline{\mathbb{Z}}} \otimes \tilde{\mathfrak{V}}$-algebras is a left zero band. By Płonka's Theorem again, this means that the $\tilde{\mathfrak{V}}$-reducts of $\underline{\underline{\mathbb{Z}}} \otimes \tilde{\mathfrak{V}}$-algebras are $\mathfrak{V}$-algebras, whence $\underline{\underline{\mathbb{Z}}} \otimes \tilde{\mathfrak{V}} \subseteq \underline{\underline{\mathbb{Z}}} \otimes \mathfrak{V}$. Conversely, $\mathfrak{V} \subseteq \tilde{\mathfrak{V}}$ implies $\underline{\underline{\mathbb{Z}}} \otimes \mathfrak{V} \subseteq \underline{\underline{\mathbb{Z}}} \otimes \tilde{\mathfrak{V}}$. Thus $\underline{\underline{R(\mathfrak{V})}} = \underline{\underline{\mathbb{Z}}} \otimes \mathfrak{V} = \underline{\underline{\mathbb{Z}}} \otimes \tilde{\mathfrak{V}} = \underline{\underline{R(\tilde{\mathfrak{V}})}}$, from which $R(\mathfrak{V}) = R(\underline{\underline{R(\mathfrak{V})}}) = R(\underline{\underline{R(\tilde{\mathfrak{V}})}}) = R(\tilde{\mathfrak{V}})$ by Example 2.5. □

3. Finite separability

Let R-$\underline{\underline{Mod}}$ be the variety of modules over a commutative ring R with unit element. Let $\underline{\underline{Set_0}}$ be the variety of *pointed sets*, i.e. of all algebras of type $\{(0, 0)\}$. Then $\underline{\underline{Set_0}} \overline{\otimes} \underline{\underline{R}} = R$-$\underline{\underline{Mod}}$, and the reduction $U : R$-$\underline{\underline{Mod}} \to \underline{\underline{R}}$ forgets the non-idempotent derived operations on R-modules. Let F be a minimal cogenerator for R-$\underline{\underline{Mod}}$, so that each R-module M has an embedding $e_M : M \to F^I$ into a power of F depending on M [AF, §§8, 18], and F is uniquely specified up to isomorphism by minimality with respect to this property [AF, Prop. 18.16]. Let A be an affine space over R. Let AG be the free pointed set over A, which carries an R-module structure having 0 as zero element and the affine space A as a subreduct (improper unless A is empty). The embedding $e_{AG} : AG \to F^I$ then restricts to an $\underline{\underline{R}}$-morphism $A \to FU^I$ embedding A in a power of the affine space FU (or just F). Conversely, each subalgebra of each power of the affine space F is of course an affine space over R. Examples of these affine space reducts of module cogenerators are furnished by $\mathbb{R}$ in $\underline{\underline{\mathbb{R}}}$ and $(\mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Q}$ in $\underline{\underline{\mathbb{Z}}}$. The reducts of cogenerators underlie the two concepts of separability that are the main topic of the paper. The first is the concept of "finite separability".

DEFINITION 3.1. Let $\mathfrak{V}$ be a variety of modes. Let F be the $\mathfrak{V}$-reduct of the minimal cogenerator of the variety $R(\mathfrak{V})$-$\underline{\underline{Mod}}$. Then a $\mathfrak{V}$-algebra B is said to be *finitely separable in the variety* $\mathfrak{V}$ iff:

$$\forall a \neq b \in B, \quad \exists f \in \mathfrak{V}(B, F). \quad af \neq bf. \tag{3.2}$$

Algebras which are not finitely separable are called *finitely inseparable*.

Note that finitely separable algebras embed into powers of F, and thus are subreducts of affine spaces. Conversely, the $\mathfrak{V}$-reducts of $\underline{\underline{Z}} \otimes \mathfrak{V}$-algebras are all finitely separable.

PROPOSITION 3.3. *Let $\mathfrak{V}$ be an irregular plural variety of modes, with regularisation $\tilde{\mathfrak{V}}$. Then a $\mathfrak{V}$-algebra B is finitely separable in $\mathfrak{V}$ iff it is finitely separable in $\tilde{\mathfrak{V}}$.*

Proof. By Theorem 2.14, $R(\mathfrak{V}) = R(\tilde{\mathfrak{V}})$, so there is a common minimal cogenerator F. Then $\mathfrak{V}(B, F) = \tilde{\mathfrak{V}}(B, F)$. □

The following two examples show that finite separability is in some measure independent of the triviality of semilattice replicas.

EXAMPLE 3.4. *A mode with trivial semilattice replica may be finitely inseparable.* Consider the differential groupoid $(\mathbb{Z}_4, \underline{2})$, using the notation of (2.3). It has a congruence α partitioning $\mathbb{Z}_4$ as $\{\{0\}, \{2\}, \{1, 3\}\}$. The semilattice replica $(\mathbb{Z}_4^{\alpha\rho}, +)$ of $\mathbb{Z}_4^{\alpha}$ is trivial, since $0^{\alpha\rho} \leq_+ 0^{\alpha\rho} + 1^{\alpha\rho} = 2^{\alpha\rho} \leq_+ 1^{\alpha\rho} + 2^{\alpha\rho} = 1^{\alpha\rho} \leq_+ 2^{\alpha\rho} + 1^{\alpha\rho} = 0^{\alpha\rho}$. If $\mathbb{Z}_4^{\alpha}$ were finitely separable, it would embed as a subreduct of an affine space A. The diagonal $\hat{A}$ is a class of some congruence V on A^2. Then the groupoid operation applied to $(0^\alpha, 0^\alpha)V(1^\alpha, 1^\alpha)$ and $(0^\alpha, 1^\alpha)V(0^\alpha, 1^\alpha)$ would yield the contradiction $(0^\alpha, 2^\alpha)V(1^\alpha, 1^\alpha)$. □

EXAMPLE 3.5. *A mode with non-trivial semilattice replica may be finitely separable.* Consider the closed real unit interval $I = [0, 1]$ as a barycentric algebra (I, I°). It is finitely separable, as a subalgebra of the barycentric algebra (or convex set) reduct $(\mathbb{R}, I^\circ)$ of the real affine space $\mathbb{R}$. On the other hand, it has the non-trivial (join) semilattice replica $(\{\{0\}, \{1\}, I^\circ\}, +)$ with $\{0\} + I^\circ = I^\circ = \{1\} + I^\circ$. □

4. Separability

Let $\mathfrak{V}$ be a plural variety of modes, with regularisation $\tilde{\mathfrak{V}}$. Let F be the $\mathfrak{V}$-reduct of the (affine space reduct of the) minimal cogenerator of the equal varieties $R(\mathfrak{V})$-$\underline{\underline{Mod}}$ and $R(\tilde{\mathfrak{V}})$-$\underline{\underline{Mod}}$ (cf. Theorem 2.14). Let F^∞ be the Płonka sum over the 2-element semilattice $\overline{0 \to 1}$ by the functor to $\mathfrak{V}$ that sends the unique non-identity morphism to $F \to \{\infty\}$. Then F^∞ is a $\tilde{\mathfrak{V}}$-algebra [RS1, 238]. By Proposition 3.3, $\tilde{\mathfrak{V}}$-algebras are finitely separable if distinct points can be separated by homorphisms into F, i.e. by "finite homomorphisms". According to the following definition,

$\tilde{\mathfrak{V}}$-algebras are "separable" if distinct points can be separated by homomorphisms into F^∞, i.e. by homomorphisms which "may take infinite values".

DEFINITION 4.1. Let $\mathfrak{V}$ be a plural variety of modes, with regularisation $\tilde{\mathfrak{V}}$. Let F be the $\mathfrak{V}$-reduct of the minimal cogenerator of the variety $\underline{\underline{R(\mathfrak{V})}}$-$\underline{\underline{Mod}}$, with corresponding Płonka sum F^∞ in $\tilde{\mathfrak{V}}$. Then a $\mathfrak{V}$-algebra A is said to be *separable in the variety* $\mathfrak{V}$ iff:

$$\forall a \neq b \in A, \quad \exists f \in \tilde{\mathfrak{V}}(A, F^\infty). \quad af \neq bf. \tag{4.2}$$

If $\mathfrak{V}$ is an irregular plural variety of modes, then no $\mathfrak{V}$-algebra A can have a non-trivial semilattice quotient. Thus each f in (4.2) must map to F. In other words, separable algebras in irregular varieties are finitely separable. The structure of general separable algebras is described by the following theorem, the main result of the paper.

THEOREM 4.3. *A $\mathfrak{V}$-algebra A is separable in $\mathfrak{V}$ if and only if it is a subreduct of a Płonka sum of affine spaces over* $R(\mathfrak{V})$.

Proof. For a separable A, define $A^* = \tilde{\mathfrak{V}}(A, F^\infty)$. Note that A^* lies in $\tilde{\mathfrak{V}}$ [RS1, 159]. Define $A^{**} = \tilde{\mathfrak{V}}(A^*, F^\infty)$. The map

$$A \to A^{**}; \quad a \mapsto (a^{**} : A^* \to F^\infty; f \mapsto af) \tag{4.4}$$

is clearly well-defined, since for ω in Ω and $f_1, \dots, f_{\omega\tau}$ in A^*, one has $f_1 \dots f_{\omega\tau}\omega a^{**} = af_1 \dots f_{\omega\tau}\omega = af_1 \dots af_{\omega\tau}\omega = f_1 a^{**} \dots f_{\omega\tau}a^{**}\omega$. It is a $\tilde{\mathfrak{V}}$-homomorphism, since for ω in Ω, for $a_1, \dots, a_{\omega\tau}$ in A, and f in A^*, one has $fa_1^{**} \dots a_{\omega\tau}^{**}\omega = fa_1^{**} \dots fa_{\omega\tau}^{**}\omega = a_1 f \dots a_{\omega\tau}f\omega = a_1 \dots a_{\omega\tau}\omega f = fa_1 \dots a_{\omega\tau}\omega^{**}$. It embeds A into A^{**}, since for $a \neq b$ in A, the element f of A^* given by (4.2) satisfies $fa^{**} = af \neq bf = fb^{**}$. Thus A is a subalgebra of the $\tilde{\mathfrak{V}}$-algebra $A^{**} = \tilde{\mathfrak{V}}(A^*, F^\infty)$, which in turn is a reduct of the $\underline{\underline{R(\tilde{\mathfrak{V}})}}$-algebra A^{**}. By Płonka's theorem [RS1, 239], the $\underline{\underline{R(\tilde{\mathfrak{V}})}}$-algebra A^{**} is a Płonka sum of $\underline{\underline{R(\mathfrak{V})}}$-algebras, i.e. of affine spaces over $R(\mathfrak{V})$.

Conversely, suppose that a $\mathfrak{V}$-algebra A is a subreduct of the Płonka sum SG over a semilattice $(S, +)$ by the functor $G : (S) \to \underline{\underline{R(\mathfrak{V})}}$, with projection $\pi : SG \to S$. To show that A is separable in $\mathfrak{V}$, it suffices to verify that SG is separable in $\tilde{\mathfrak{V}}$. Fix an element s of S. Then a $\mathfrak{V}$-homomorphism $f : sG \to F$ may be extended to a $\tilde{\mathfrak{V}}$-homomorphism $\tilde{f} : SG \to F^\infty$ defined by

$$x\tilde{f} = \begin{cases} x(x\pi \to s)Gf & \text{if } x\pi \leq_+ s, \\ \infty & \text{otherwise.} \end{cases} \tag{4.5}$$

For ω in Ω and $x_1, \ldots, x_{\omega\tau}$ in SG, with $x_1\pi + \cdots + x_{\omega\tau}\pi = t$ in S, one must check the homomorphism condition

$$x_1\tilde{f} \ldots x_{\omega\tau}\tilde{f}\omega = x_1 \ldots x_{\omega\tau}\omega\tilde{f}. \tag{4.6}$$

There are two cases. If $(\forall 1 \leq i \leq \omega\tau, x_i\pi \leq_+ s)$ holds, then $x_1 \ldots x_{\omega\tau}\omega\pi = x_1\pi \ldots x_{\omega\tau}\pi\omega = x_1\pi + \cdots + x_{\omega\tau}\pi \leq_+ s + \cdots + s = s$. The map $\tilde{f}$ throughout (4.6) may then be replaced by $(x\pi \to s)Gf$. Since $(x\pi \to s)Gf$ is a $\mathfrak{B}$-homomorphism, equality holds in (4.6) in this case. Otherwise, $(\exists 1 \leq i \leq \omega\tau.\ x_i\pi \not\leq_+ s)$ holds, so $x_i\pi \leq_+ x_1\pi + \cdots + x_i\pi + \cdots + x_{\omega\tau}\pi = t \not\leq_+ s$. The right hand side of (4.6) is $x_1(x_1\pi \to t)G \ldots x_{\omega\tau}(x_{\omega\tau} \to t)G\omega\tilde{f} = \infty$, while the left hand side is $x_1\tilde{f} \ldots x_i\tilde{f} \ldots x_{\omega\tau}\tilde{f}\omega = x_1\tilde{f} \ldots \infty \ldots x_{\omega\tau}\tilde{f}\omega = \infty$, so that equality obtains again. Now consider a pair $a \neq b$ of distinct elements of SG. If $a\pi = b\pi = s$, then $af \neq bf$ for a $\mathfrak{B}$-homomorphism $f : sG \to F$, whence $a\tilde{f} \neq b\tilde{f}$ for the extending $\tilde{\mathfrak{B}}$-homomorphism $\tilde{f}: SG \to F^\infty$ given by (4.5). If $a\pi \neq b\pi$, then without loss of generality $a\pi \not\leq_+ b\pi = s$. Take $f : sG \to F$ to be any $\mathfrak{B}$-homorphism, e.g. a constant map. Then for the extending $\tilde{\mathfrak{B}}$-homomorphism $\tilde{f}: SG \to F^\infty$ given by (4.5), one has $b\tilde{f} \in F$ and $a\tilde{f} = \infty$, so $a\tilde{f} \neq b\tilde{f}$. Thus SG is separable in $\tilde{\mathfrak{B}}$. □

5. Examples and problems

This section begins with examples of the use of each direction of Theorem 4.3. The barycentric algebras discussed in Example 5.1 are known to be separable, so that Theorem 4.3 describes them structurally as subreducts of Płonka sums of affine spaces. On the other hand, the commutative binary modes of Example 5.3 are known to be subreducts of Płonka sums of quasigroups, i.e. of affine spaces. Theorem 4.3 then shows that they are separable.

EXAMPLE 5.1 (Barycentric algebras). Let E be a real vector space. For each real number r in the open unit interval $I^\circ =]0, 1[$, define a binary operation $\underline{r} : E^2 \to E$; $(x, y) \mapsto xy\underline{r} = x(1-r) + yr$ as in (2.3). The subalgebras (C, I°) of the algebras (E, I°) are exactly the convex subsets of real affine spaces. The variety generated by these convex sets (C, I°) is the variety $\mathfrak{B}$ of *barycentric algebras*. Convex sets form a subquasivariety $\mathfrak{C}$ of the variety $\mathfrak{B}$ (cf. [Fl], [Ig], [Ne], [RS1], [RS2]). The ring $R(\mathfrak{B})$ is the ring $\mathbb{R}$ of real numbers. First recall that by [RS1; 254, 255], the ring $R(\mathfrak{B})$ is in fact the free $\underline{\underline{R(\mathfrak{B})}}$-algebra on two generators. Then the identities

$$\begin{cases} xy\underline{p+r} = (xy\underline{p}, x, xy\underline{r})P, \\ xy\underline{(-r)} = (x, xy\underline{r}, x)P, \\ xy\underline{(pq)} = x\, xy\underline{p}\underline{q} \end{cases} \tag{5.2}$$

that hold in each $\underline{\underline{R(\mathfrak{B})}}$-algebra show that all the operations $\underline{r}$ for r in $\mathbb{R}$ belong to $R(\mathfrak{B})$. Now since each element of the free $\underline{\underline{R(\mathfrak{B})}}$-algebra on two generators has the normal form $xy\underline{r}$ for r in $\mathbb{R}$, $R(\mathfrak{B})$ consist exactly of these operations $\underline{r}$. They are pairwise distinct because the $\underline{\underline{R(\mathfrak{B})}}$-algebra $\mathbb{R}$ is obviously an element of $\underline{\underline{R(\mathfrak{B})}}$. It follows that $R(\mathfrak{B}) = \mathbb{R}$.

The vector space $\mathbb{R}$ is a minimal cogenerator for the variety $\underline{\underline{\mathbb{R}\text{-}Mod}}$ of real vector spaces. By [Fl], Theorem 3.10], each barycentric algebra (B, I°) embeds into a product of copies of $(\mathbb{R}^{\infty}, I^{\circ})$, and hence is separable in $\mathfrak{B}$. Theorem 4.3 thus identifies barycentric algebras as the I°-reducts of Płonka sums of real affine spaces. This description could also be obtained independently using results from [RS1], [RS2], [RS3], but the derivation would be longer and more complicated. □

EXAMPLE 5.3 (Commutative binary modes). *A commutative binary mode* $(G, \cdot)$ is a groupoid mode with commutative multiplication. Let $\underline{\underline{Cbm}}$ be the variety of commutative binary modes. By [JK, Theorem 5.3.1], each $\underline{\underline{Cbm}}$-groupoid $(G, \cdot)$ is a subreduct of a commutative quasigroup mode $(Q(G), \cdot, /, \backslash)$. By [RS1, 433], the variety of commutative quasigroup modes is the variety $\underline{\underline{\mathbb{D}}}$, where $\mathbb{D}$ is the ring of dyadic rational numbers. Since [RS3, Theorem 6.4] each $\underline{\underline{Cbm}}$-groupoid is a subreduct of a Płonka sum of commutative quasigroup modes, it may also be considered as a subreduct of a Płonka sum of affine spaces over $\mathbb{D}$. Now $R(\underline{\underline{Cbm}}) = \mathbb{D}$, by a calculation analogous to that of the previous example showing $R(\mathfrak{B}) = \mathbb{R}$. Theorem 4.3 then shows that each commutative binary mode is separable. □

Let $\mathfrak{B}$ be a plural variety of modes. The smallest mode variety contained in $\mathfrak{B}$ may be described as the variety $\underline{\underline{Tb}}$ of trivial bands [RS1, 224], whose algebras have less than 2 elements. Let $FS(\mathfrak{B})$ denote the class of finitely separable modes in $\mathfrak{B}$. Let $S(\mathfrak{B})$ denote the class of separable modes in $\mathfrak{B}$. There is then the chain of containments

$$\underline{\underline{Tb}} \subseteq FS(\mathfrak{B}) \subseteq S(\mathfrak{B}) \subseteq \mathfrak{B}. \tag{5.4}$$

Various examples exhibit proper and improper containments in (5.4). For $\underline{\underline{Tb}}$ itself, one of course has

$$\underline{\underline{Tb}} = FS(\underline{\underline{Tb}}) = S(\underline{\underline{Tb}}) = \underline{\underline{Tb}}. \tag{5.5}$$

For the variety $\underline{\underline{Lz}}$ of left zero bands (Example 2.10), one has

$$\underline{\underline{Tb}} \subset FS(\underline{\underline{Lz}}) = S(\underline{\underline{Lz}}) = \underline{\underline{Lz}}. \tag{5.6}$$

For the variety $\underline{\underline{Sl}}$ of semilattices, one has

$$\underline{\underline{Tb}} = FS(\underline{\underline{Sl}}) \subset S(\underline{\underline{Sl}}) = \underline{\underline{Sl}}. \tag{5.7}$$

For the variety $\mathfrak{D}$ of differential groupoids (Examples 2.8, 3.4), one has

$$\underline{\underline{Tb}} \subset FS(\mathfrak{D}) = S(\mathfrak{D}) \subset \mathfrak{D}. \tag{5.8}$$

For $\mathfrak{B}$ the variety of barycentric algebras (Example 5.1) or of commutative binary modes (Example 5.3), one has

$$\underline{\underline{Tb}} \subset FS(\mathfrak{B}) \subset S(\mathfrak{B}) = \mathfrak{B}. \tag{5.9}$$

Finally, for the regularisation $\tilde{\mathfrak{D}}$ of the variety of differential groupoids, one has

$$\underline{\underline{Tb}} \subset FS(\tilde{\mathfrak{D}}) \subset S(\tilde{\mathfrak{D}}) \subset \tilde{\mathfrak{D}}. \tag{5.10}$$

A plural mode variety $\mathfrak{B}$ would be said to be *separably trivial* if

$$\underline{\underline{Tb}} = FS(\mathfrak{B}) = S(\mathfrak{B}) \subset \mathfrak{B}, \tag{5.11}$$

i.e. if it were non-trivial, but all its separable algebras were trivial. A plural mode variety $\mathfrak{W}$ would be said to be *finitely separably trivial* if

$$\underline{\underline{Tb}} = FS(\mathfrak{W}) \subset S(\mathfrak{W}) \subset \mathfrak{W}, \tag{5.12}$$

i.e. if it properly contained the variety of semilattices as its class of separable algebras. Note that a separably trivial variety would necessarily be irregular, with a finitely separably trivial regularisation. The following problem arises.

PROBLEM 5.13. *Do separably trivial plural mode varieties exist? If not, do finitely separably trivial plural mode varieties exist?*

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Iowa State University
Ames, Iowa
USA

Instytut Matematyki P.W.
Warsaw, Poland