Algebra univers. **49** (2003) 305–319
0002-5240/03/030305 – 15
DOI 10.1007/s00012-003-1719-2

Algebra Universalis

Support functions of general convex sets

Dug-Hwan Choi and Jonathan D. H. Smith

Dedicated to the Memory of Gian-Carlo Rota

1. Introduction

The concept of the support function of a non-empty compact convex set was introduced by Minkowski at the end of the 19th century [3, pp. 106, 144, 231]. Since then it has played a vital rôle in many of the applications of convexity, from optimization theory to the geometry of numbers. Support functions of non-empty compact convex subsets of a finite-dimensional Euclidean space $\mathbb{R}^d$ are characterized as positively homogeneous convex real-valued functions on $\mathbb{R}^d$, and non-empty compact convex sets are determined uniquely by their support functions.

Work of Romanowska and the second author [5], [7, §11] extended the concept of the support function from compact non-empty convex sets to general bounded non-empty convex sets, thus to convex sets that are not necessarily closed. The method was to find a suitable codomain D_d, replacing the codomain $\mathbb{R}$ of Minkowski's support functions, so that non-empty bounded convex subsets of $\mathbb{R}^d$ are determined uniquely by their D_d-valued support functions defined on $\mathbb{R}^d$. Conditions were also found, analogous to Minkowski's positive homogeneity and convexity, characterizing the support functions amongst all the D_d-valued functions on $\mathbb{R}^d$. These were the so-called G-*conditions* of [5, Definition 4.11], [7, Definition 11.2].

The purpose of the current paper is to produce support functions (Definition 7.1) specifying general convex subsets of a finite-dimensional Euclidean space $\mathbb{R}^d$, essentially removing the restriction to the bounded, non-empty case that was imposed in [5]. A codomain $D_d^{\pm}$ is constructed so that arbitrary convex subsets of $\mathbb{R}^d$ are determined uniquely by their $D_d^{\pm}$-valued support functions defined on $\mathbb{R}^d$. These support functions are characterized amongst all $D_d^{\pm}$-valued functions on $\mathbb{R}^d$ by satisfaction of a set of conditions called the $G^{\pm}$-*conditions* (Section 6 below), analogues in turn of the G-conditions of [5] and of Minkowski's positive homogeneity and convexity conditions.

Presented by J. Kung.
Received September 2, 1999; accepted in final form December 14, 2000.
2000 *Mathematics Subject Classification*: 52A20; 06F99.
Key words and phrases: Support function, convex set, entropic algebra, mode, modal, barycentric algebra, semilattice, lexicographic order, ordinal product, ordinal sum, Płonka sum.

The correspondences between convex sets and their support functions go further. Algebraic structure on $\mathbb{R}$ induces algebraic structure on the set of real-valued functions on $\mathbb{R}^d$. The set of support functions of non-empty, compact convex sets should be closed under this algebraic structure, and should reflect comparable algebraic structure on the set of non-empty compact convex subsets of $\mathbb{R}^d$. The usual linear algebraic structure on $\mathbb{R}$ is unsuitable here: for example, the negative of a convex function is no longer convex. The key algebraic structure on $\mathbb{R}$ for use in the context of support functions comprises convex combinations forming a barycentric algebra (see [4]) and the maximum operation forming a join semilattice, the convex combinations distributing over the join so that the two structures combine to form a *modal* in the sense of [4]. The support functions then form a submodal of the induced modal structure on the full set of functions, and the modal structure on the support functions reflects exactly the modal structure on the compact convex sets given by convex combinations and convex hulls of unions. (See Section 3 below for an outline, and [4, 3.7] for details.) Given this algebraic approach to Minkowski's support functions, and the fact that the set of all convex subsets of $\mathbb{R}^d$ forms a modal under convex combinations and convex hulls of unions, it is then shown that the codomain $D_d^{\pm}$ carries a modal structure (Proposition 5.1) such that support functions form a submodal (Corollary 8.6) of the induced modal of functions from $\mathbb{R}^d$ to $D_d^{\pm}$, this submodal being isomorphic to the modal (3.3) of all convex subsets of $\mathbb{R}^d$ (Theorem 8.7).

2. Algebraic preliminaries

An algebra (A, Ω) of type $\tau\colon \Omega \to \mathbb{N}$ is *entropic* if each operation ω is a homomorphism

$$\omega\colon (A^{\omega\tau}, \Omega) \to (A, \Omega)\ ;\ (a_1, \dots, a_{\omega\tau}) \mapsto a_1 \cdots a_{\omega\tau}\omega. \tag{2.1}$$

An algebra is *idempotent* if each singleton subset is a subalgebra. An algebra is a *mode* if it is idempotent and entropic. An algebra $(A, +, \Omega)$ is a *modal* if $(A, +)$ is a join semilattice, (A, Ω) is a mode, and for each operation ω in Ω, the distributive law

$$a_1 \cdots (a_i + a_i') \cdots a_{\omega\tau}\omega = a_1 \cdots a_i \cdots a_{\omega\tau}\omega + a_1 \cdots a_i' \cdots a_{\omega\tau}\omega \tag{2.2}$$

holds for $1 \le i \le \omega\tau$.

For a real number r, define a binary operation $\underline{r}$ on $\mathbb{R}$ by

$$\underline{r}\colon \mathbb{R}^2 \to \mathbb{R}\ ;\ \ (x, y) \mapsto x(1 - r) + yr. \tag{2.3}$$

Let $I^\circ = [0, 1]^\circ$ denote the interior of the unit interval I. Let

$$D = (\mathbb{R}, +, I^\circ) \tag{2.4}$$

denote the algebra structure on $\mathbb{R}$ given by the join semilattice operation $x + y = \max\{x, y\}$ and the binary operations $\underline{p}$ of (2.3) for p in I°. Then D is a modal. Now let $\mathbb{R}^{\pm\infty} = \mathbb{R} \cup \{-\infty, \infty\}$ be the ordered set of *(doubly) extended reals*. For each element r of the set $\mathbb{R}^+$ of strictly positive reals, a binary operation $\underline{r}$ will be defined on $\mathbb{R}^{\pm\infty}$ to yield an algebra $(\mathbb{R}^{\pm\infty}, \mathbb{R}^+)$. Define $xy\underline{1} = y$. For real arguments x and y, define $xy\underline{r}$ using (2.3), so that $(\mathbb{R}, \mathbb{R}^+)$ is a subalgebra of $(\mathbb{R}^{\pm\infty}, \mathbb{R}^+)$. For $\{x, y\} \not\subseteq \mathbb{R}$ and $p \in I^\circ$, define

$$xy\underline{p} = \ \mathbf{if}\ -\infty \in x, y\ \mathbf{then}\ -\infty\ \mathbf{else}\ \infty \tag{2.5}$$

For $\{x, y\} \not\subseteq \mathbb{R}$ and $q \in (1, \infty)$, define

$$xy\underline{q} = \ \mathbf{if}\ x < \infty\ \mathbf{and}\ y = \infty\ \mathbf{then}\ \infty\ \mathbf{else}\ -\infty \tag{2.6}$$

Thus $\{-\infty\}$ is also a subalgebra of $(\mathbb{R}^{\pm\infty}, \mathbb{R}^+)$, and a sink of $(\mathbb{R}^{\pm\infty}, I^\circ)$. Bearing in mind that $(\mathbb{R}, \mathbb{R}^+)$ is a subalgebra of $(\mathbb{R}^{\pm\infty}, \mathbb{R}^+)$, one may summarize the algebraic structure on the doubly extended reals by the tables

I°	$-\infty$	$\mathbb{R}$	∞
$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\mathbb{R}$	$-\infty$	$\mathbb{R}$	∞
∞	$-\infty$	∞	∞

(2.7)

and

$(1, \infty)$	$-\infty$	$\mathbb{R}$	∞
$-\infty$	$-\infty$	$-\infty$	∞
$\mathbb{R}$	$-\infty$	$\mathbb{R}$	∞
∞	$-\infty$	$-\infty$	$-\infty$.

(2.8)

3. Extended real valued support functions

For a finite dimension d, let $\mathbb{R}^d$ denote the vector space $\mathbb{R}^d$ equipped with the Euclidean inner product

$$\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\ ;\ \ (x, y) \mapsto (x|y). \tag{3.1}$$

The vector space $\mathbb{R}^d$ also supports a mode structure $(\mathbb{R}^d, \mathbb{R}^+)$ of type $\mathbb{R}^+ \times \{2\}$ given as the d-th power of the mode $(\mathbb{R}, \mathbb{R}^+)$ using (2.3). [The d-th power semilattice structure on $\mathbb{R}^d$ will not be needed, so that $+$ will denote addition on Euclidean spaces $\mathbb{R}^d$, including $\mathbb{R}^1$, while $(\mathbb{R}, +)$ is the semilattice of (2.4).] Note that the convex subsets of $\mathbb{R}^d$ are precisely the subalgebras of $(\mathbb{R}^d, I^\circ)$. Let $\mathbb{R}^d T$ denote the set of all convex subsets of $\mathbb{R}^d$. This set becomes a mode $(\mathbb{R}^d T, I^\circ)$ under the *complex products*

$$XY\underline{p} = \{xy\underline{p} \mid x \in X, y \in Y\}. \tag{3.2}$$

Furthermore, $(\mathbb{R}^dT, +)$ is a join semilattice under the containment relation $\subseteq$; for convex subsets X and Y of $\mathbb{R}^d$, the join $X + Y$ is the convex hull of $X \cup Y$. One thus obtains a modal structure

$$(\mathbb{R}^dT, +, I^\circ) \tag{3.3}$$

on the set of convex subsets of $\mathbb{R}^d$ [7, §7].

Definition 3.1. Consider the function

$$H\colon \mathbb{R}^dT \times \mathbb{R}^d \to \mathbb{R}^{\pm\infty}\ ;\ \ (X, y) \mapsto \sup\{(x|y) \mid x \in X\}. \tag{3.4}$$

For a fixed convex subset X of $\mathbb{R}^d$, the *(extended real valued) support function* of X is the function

$$H_X\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}\ ;\ \ y \mapsto H(X, y). \tag{3.5}$$

For an element y of $\mathbb{R}^d$, the *supporting hyperplane* of X in the y-direction is

$$\big\{z \in \mathbb{R}^d \;\big|\; (x|y) = H_X(y)\big\}. \tag{3.6}$$

Note that (3.6) will be empty if $H_X(y)$ is not finite. Also, note

$$H_X(0) = \ \mathbf{if}\ X = \varnothing\ \mathbf{then}\ {-\infty}\ \mathbf{else}\ 0 \tag{3.7}$$

Thus the supporting hyperplane of $\varnothing$ in any direction is empty. For a non-empty convex subset of $\mathbb{R}^d$, the supporting hyperplane in the 0-direction is $\mathbb{R}^d$.

Proposition 3.2. *For a convex subset X of $\mathbb{R}^d$, the closure $\overline{X}$ of X is given by*

$$\overline{X} = \bigcup_{y \in \mathbb{R}^d} \big\{z \in \mathbb{R}^d \;\big|\; (z|y) \le H_X(y)\big\}. \tag{3.8}$$

Proof. For any element y of $\mathbb{R}^d$, one has $H_\varnothing(y) = -\infty$, so that (3.8) holds for empty X. Now suppose that X is non-empty. For each element (x, y) of $X \times \mathbb{R}^d$, one has $(x|y) \le \sup\{(z|y) \mid z \in X\} = H_X(y)$. Thus X is contained in the closed set on the right hand side of (3.8). Conversely, suppose that t does not lie in $\overline{X}$. Then 0 does not lie in the translated closed convex set $\overline{X} - t$. It follows that there is a vector y with $0 > H_{\overline{X}-t}(y) = \sup\{(x - t|y) \mid x \in \overline{X}\} = \sup\{(x|y) \mid x \in \overline{X}\} - (t|y)$, whence $(t|y) > \sup\{(x|y) \mid x \in \overline{X}\} \ge \sup\{(x|y) \mid x \in X\} = H_X(y)$ and t does not lie in the right hand side of (3.8). □

A function $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ is said to be *positively homogeneous* if $f(0) \in \{0, -\infty\}$ and

$$\forall r \in \mathbb{R}^+,\ \forall y \in \mathbb{R}^d,\ \ f(0y\underline{r}) = f(0)f(y)\underline{r}. \tag{3.9}$$

Proposition 3.3. *For each convex subset X of $\mathbb{R}^d$, the support function H_X is positively homogeneous.*

Proof. In view of (3.7), it suffices to verify

$$H_X(yr) = H_X(0y\underline{r}) = H_X(0)H_X(y)\underline{r} \tag{3.10}$$

for all y in $\mathbb{R}^d$ and r in $\mathbb{R}^+$. If X is empty, then all the terms of (3.10) are $-\infty$, since $\{-\infty\}$ is a subalgebra of $(\mathbb{R}^{\pm\infty}, \mathbb{R}^+)$. Otherwise, suppose that $H_X(y) = \infty$, so that $H_X(yr) = \infty$ also. Since X is non-empty, $H_X(0) = 0$, and the right hand side of (3.10) reduces to $0\infty\underline{r} = \infty$. Finally, if $H_X(y)$ is finite, then verification of (3.10) amounts to straightforward linear algebra. □

A function $f\colon (A,\Omega) \to (B,\Omega)$ between algebras of type $\tau\colon \Omega \to \mathbb{N}$, with ordered codomain, is said to be *convex* if its *epigraph*

$$\operatorname{epi} f = \{(a,b) \in A \times B \mid f(a) \le b\} \tag{3.11}$$

is a subalgebra of $(A \times B, \Omega)$ [4, (316)] [7, (5.3)]. For a function $f\colon (\mathbb{R}^d, I^\circ) \to (D, I^\circ)$ with D as in (2.4), note that this definition of convexity agrees with the usual one given by "Jensen's Inequality".

Proposition 3.4. *For each convex subset X of $\mathbb{R}^d$, the extended real valued support function $H_X\colon (\mathbb{R}^d, I^\circ) \to (\mathbb{R}^{\pm\infty}, I^\circ)$ is convex.*

Proof. Note that

$$\operatorname{epi} H_X = \{(y,s) \in \mathbb{R}^d \times \mathbb{R}^{\pm\infty} \mid \forall x \in X, (x|y) \le s\}. \tag{3.12}$$

Suppose that $(x|y) \le s$ and $(x|z) \le t$ for all x in X. Then $\forall x \in X, (x|yz\underline{p}) = (x|y)(x|z)\underline{p} \le st\underline{p}$, the latter inequality holding by the Monotonicity Lemma [4, 315] [7, 5.3] in the modal D of (2.4). □

4. Directed derivatives of convex functions

By Propositions 3.3 and 3.4, $H_X\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$, the extended real valued support function of a convex subset X of $\mathbb{R}^d$, is positively homogeneous and convex. In this section, directed derivatives of convex functions are defined, and are shown to preserve positive homogeneity and convexity of their primitives.

Lemma 4.1. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be convex. Then $f^{-1}\{-\infty\}$ is either $\varnothing$ or $\mathbb{R}^d$.*

Proof. Suppose $f(x) = -\infty$ for some x in $\mathbb{R}^d$. By the convexity of f, one has $f(0) = f(x(-x)\underline{1/2}) \le f(x)f(-x)\underline{1/2} = -\infty$, whence $f(0) = -\infty$. For each element y of $\mathbb{R}^d$, one then has $f(y) = f(0(2y)\underline{1/2}) \le f(0)f(2y)\underline{1/2} = -\infty$, whence $f(y) = -\infty$. □

Proposition 4.2. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be convex. Then for all x, y in $\mathbb{R}^d$, the limit $\lim_{r\to\infty} f(x)f(xy\underline{1/r})\underline{r}$ exists.*

Proof. If $f(x) = -\infty$, then by Lemma 4.1 $f(xy\ \underline{1/r}) = -\infty$ for all r in $\mathbb{R}^+$, whence $\lim_{r\to\infty} f(x)f(xy\ \underline{1/r})\underline{r} = -\infty$. If $f(x) = \infty$, then $f(x)f(xy\ \underline{1/r})\underline{r} = -\infty$ for all $r > 1$ by the bottom line of (2.8), whence $\lim_{r\to\infty} f(x)f(xy\ \underline{1/r})\underline{r} = -\infty$. Otherwise, $f(x)$ is finite. It will then be shown that

$$1 < q < r \Rightarrow f(x)f(xy\ \underline{1/r})\underline{r} \leq f(x)f(xy\ \underline{1/q})\underline{q}, \tag{4.1}$$

so that $f(x)f(xy\ \underline{1/r})\underline{r}$ ultimately decreases as $r \to \infty$. By Lemma 4.1, $f(xy\ \underline{1/q}) \neq -\infty$. If $f(xy\ \underline{1/q}) = \infty$, then the right hand side of (4.1) is ∞, so that the weak inequality holds. Otherwise, $f(xy\ \underline{1/q})$ is finite. Consider q/r in I°. By Lemma 4.1 and the convexity of f, one has $-\infty < f(xy\ \underline{1/r}) = f(x\ xy\ \underline{1/q}\ \underline{q/r}) \leq f(x)f(xy\ \underline{1/q})\underline{q/r} < \infty$, so that $f(xy\ \underline{1/r})$ is also finite. The Monotonicity Lemma [4, 315] in the modal D of (2.4) then implies

$$f(x)f(xy\ \underline{1/r})\underline{r} \leq f(x)f(x)f(xy\ \underline{1/q})\underline{q/r}\ \underline{r} = f(x)f(xy\ \underline{1/q})\underline{q},$$

as required. □

Definition 4.3. An element x of $\mathbb{R}^d$ is a *singular point* of a convex function $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ if there are elements y and z of $\mathbb{R}^d$ with $\lim_{r\to\infty} f(x)f(xy\ \underline{1/r})\underline{r} = \infty$ and $\lim_{r\to\infty} f(x)f(xz\ \underline{1/r})\underline{r} = -\infty$. The set of singular points of f is denoted by $M(f)$.

Definition 4.4. Let $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be convex. For elements x and y of $\mathbb{R}^d$, the *directed derivative* of f at x towards y is

$$f'(x;y) = \textbf{ if } x \in M(f) \textbf{ then } -\infty \textbf{ else } \lim_{r\to\infty} f(x)f(xy\ \underline{1/r})\underline{r}. \tag{4.2}$$

Cf. [1, 13], [5, (3,8)], [7, (11.4)].

Example. Let X be the epigraph of the exponential function $\exp\colon \mathbb{R} \to (\mathbb{R}, \leq)$. Consider $x = (0,-1), y = (-1,0)$, and $z = (1,0)$. Note $H_X(x) = 0$. Then $\lim_{r\to\infty} H_X(x)H_X(xy\ \underline{1/r})\underline{r} = \infty$, while $\lim_{r\to\infty} H_X(x)H_X(xz\ \underline{1/r})\underline{r} = -\infty$. Thus x is a singular point of $H_X : \mathbb{R} \to \mathbb{R}^{\pm\infty}$.

Proposition 4.5. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be convex. Then*

$$\forall x, y \in \mathbb{R}^d,\ \ f'(x;y) \leq f(y). \tag{4.3}$$

Proof. Note that (4.3) is immediate if $f(x) \in \{\pm\infty\}$, if $f(y) \in \{\pm\infty\}$, or if x is singular. Otherwise, it follows from $f(x)f(xy\ \underline{1/r})\underline{r} \leq f(x)f(x)f(y)\ \underline{1/r}\ \underline{r} = f(y)$ for $r > 1$. □

Proposition 4.6. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be convex. Then for each x in $\mathbb{R}^d$, the function $\mathbb{R}^d \to \mathbb{R}^{\pm\infty}$; $y \mapsto f'(x;y)$ is convex.*

Proof. For y, z in $\mathbb{R}^d$ and p in I°, the inequality

$$f'(x; yz\underline{p}) \le f'(x;y)f'(x;z)\underline{p} \tag{4.4}$$

must be proved. It is trivial if x is singular, if $\infty \in \{f'(x;y), f'(x;z)\} \subset \mathbb{R}^\infty$, or if $f(x)$ is not finite. Otherwise, it follows from

$$\begin{aligned} f(x)f(xyz\underline{p}\ \underline{1/r})\underline{r} &= f(x)f(xx\underline{p}yz\underline{p}\ \underline{1/r})\underline{r} \\ &= f(x)f(xy\ \underline{1/r}\ xz\ \underline{1/r}\ \underline{p})\underline{r} \\ &\le f(x)f(xy\ \underline{1/r})\underline{r}\ f(x)f(xy\ \underline{1/r})\underline{r}\ \underline{p}\ , \end{aligned}$$

in which idempotence is used for the first equality, the entropic law for the second, and the Monotonicity Lemma is used for the inequality. □

Lemma 4.7. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be positively homogeneous and convex. Then for each x in $\mathbb{R}^d$, one has $f'(x;0) \in \{0, -\infty\}$.*

Proof. The result is clear if x is singular or if $f(x)$ is not finite. Otherwise, one has

$$\begin{aligned} f'(x;0) &= \lim_{r\to\infty} f(x)f(x0\underline{1/r})\underline{r} = \lim_{x\to\infty} f(x)f(x[1-\tfrac{1}{r}])\underline{r} \\ &= \lim_{x\to\infty} \{f(x)[1-r] + f(x)[1-\tfrac{1}{r}]r\} = 0. \end{aligned}$$ □

Theorem 4.8. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be positively homogeneous and convex. Then for each x in $\mathbb{R}^d$, the function*

$$\mathbb{R}^d \to \mathbb{R}^{\pm\infty}\ ;\ y \mapsto f'(x;y) \tag{4.5}$$

is also positively homogeneous and convex.

Proof. In view of Proposition 4.6 and Lemma 4.7, it remains to be shown that

$$f'(x; 0y\ \underline{q}) = f'(x;0)f'(x;y)\underline{q} \tag{4.6}$$

for y in $\mathbb{R}^d$ and $q > 0$. By Proposition 4.6 and Lemma 4.1, the equality (4.6) is immediate if $f'(x;0) = -\infty$. Otherwise, Lemma 4.7 shows that (4.6) reduces to

$$f'(x; 0y\underline{q}) = 0f'(x;y)\underline{q} \tag{4.7}$$

with $f(x)$ finite. By the positive homogeneity of f, one has

$$f(x0y\underline{q}\ \underline{1/r}) = 0f\big(xy\underline{(0q\ \underline{1/r})/(1q\ \underline{1/r})}\big)\ \underline{(1q\ \underline{1/r})}. \tag{4.8}$$

Case (a): $f'(x;y) = \infty$. In this case $f(xy\ \underline{1/r}) = \infty$ for $r > 1$. By (4.8), one then has $f(x0y\underline{q}\ \underline{1/r}) = \infty$ for $r > 1$. Thus both sides of (4.7) are ∞.

Case (b): $f'(x;y)$ finite. In this case (4.8) yields

$$\begin{aligned} f'(x;0y\underline{q}) &= \lim_{r\to\infty} f(x)f(x0y\underline{q}\ \underline{1/r})\underline{r} \\ &= \lim_{r\to\infty} f(x)0f\big(xy\ \underline{(0q\ \underline{1/r})/(1q\ \underline{1/r})}\big)\ \underline{(1q\ \underline{1/r})}\ \underline{r} \\ &= \lim_{r\to\infty} q\big\{f(x)f\big(xy\underline{(0q\ \underline{1/r})/(1q\ \underline{1/r})}\big)\ \underline{(1q\ \underline{1/r})/(0q\ \underline{1/r})}\big\} \\ &= q\lim_{s\to\infty} f(x)f(xy\ \underline{1/s})\underline{s} = qf'(x;y), \end{aligned}$$

verifying (4.7). □

Proposition 4.9. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be convex and positively homogeneous, with $f(0) = 0$. Then for each y in $\mathbb{R}^d$,*

$$f'(0;y) = f(y). \tag{4.9}$$

Proof. If $f(0) = -\infty$, one has $f(y) = -\infty$ by Lemma 4.1, whence $f'(0;y) = -\infty$ by Proposition 4.5. Otherwise, for y in $\mathbb{R}^d$, one has $\lim_{r\to\infty} 0f(y/r)\underline{r} = \lim_{r\to\infty} f(y) = f(y)$. By Lemma 4.1, there is no y in $\mathbb{R}^d$ with $f(y) = -\infty$. Thus 0 is not a singular point of f, and (4.8) follows. □

5. Codomains for support functions

This section describes the construction, inductive over the dimension d, of the codomain modal $D_d^{\pm}$ for the support functions of convex subsets of $\mathbb{R}^d$. Along with the series of codomain modals $D_d^{\pm}$ for $d \geq 1$, there will be three other series of inductively defined modals $E_d^{\pm}$ (for $d \geq 0$), $F_d^{\pm}$ (for $d \geq 1$), and $G_d^{\pm}$ (for $d \geq 0$). The induction basis is a modal $(G_0^{\pm}, +, I^\circ)$ identified with the modal $(\mathbb{R}^0T, +, I^\circ)$ as in (3.3). Thus $G_0^{\pm} = \{\varnothing, \{0\}\}$ with join semilattice order $\varnothing \leq_+ \{0\}$ and with $\{\varnothing\}$ as a sink for the I°-reduct $(G_0^{\pm}, I^\circ)$, in accordance with (3.2). The induction step begins with a modal $G_{d-1}^{\pm}$ identified with the modal $(\mathbb{R}^{d-1}T, +, I^\circ)$ as in (3.3). Note that the modal D of (2.4) is *cancellative*, in the sense that

$$xy\underline{p} = xz\underline{p} \Rightarrow y = z \tag{5.1}$$

for p in I° and x, y, z in $\mathbb{R}$. Moreover, $(D, +)$ is a chain, and also $\mathbb{R}^{d-1}T$ has $\varnothing$ as a zero element. The *ordinal product*

$$E_d^{\pm} = D \circ G_{d-1}^{\pm} \tag{5.2}$$

is the Cartesian product $(D, I^\circ) \times (G_{d-1}^{\pm}, I^\circ)$ equipped with the *lexicographic order*

$$(d, g) \leq (d', g') \Leftrightarrow d < d' \textbf{ or } (d = d' \textbf{ and } g \leq_+ g'). \tag{5.3}$$

By a theorem of Slatinský [6, 3.14], the lexicographic order yields a join semilattice $(D \times G_{d-1}^{\pm}, +)$. By (5.1), it then follows that $(D \circ G_{d-1}^{\pm}, +, I^\circ)$ is a modal $E_d^{\pm}$ [5, Theorem 2.5] [7, Theorem 11.1].

Using the modal $(E_d^{\pm}, +, I^\circ)$ of (5.2), define a partial order on the disjoint union

$$D_d^{\pm} = \{(-\infty, \varnothing)\} \cup\ E_d^{\pm} \cup\ \{(\infty, \varnothing)\} \tag{5.4}$$

as the ordinal sum of the uniands in (5.4), i.e. with $(-\infty, \varnothing) < e < (\infty, \varnothing)$ for e in $E_d^{\pm}$. Define a contravariant functor J from the meet semilattice with Hasse diagram $0 \to 1 \to 2$ to the category of algebras of type $I^\circ \times \{2\}$ by $0J = \{(-\infty, \varnothing)\},\ 2J = (E_d^{\pm}, I^\circ)$, and $1J = \{(\infty, \varnothing)\}$. Define $(D_d^{\pm}, I^\circ)$ as the Płonka sum of the functor J [4, 236]. In other words, $(E_d^{\pm}, I^\circ)$ is a subalgebra of $(D_d^{\pm}, I^\circ)$, while the projection $(x, y) \mapsto x$ yields a homomorphism from $(D_d^{\pm}, I^\circ)$ to the reduct $(\mathbb{R}^{\pm\infty}, I^\circ)$ define by (2.7) on the doubly extended reals. The partial order on $D_d^{\pm}$ yields a join semilattice $(D_d^{\pm}, +)$.

Proposition 5.1. *The algebra* $(D_d^{\pm}, +, I^\circ)$ *is a modal.*

Proof. The distributive laws (2.2) remain to be verified. First note that the modal $(E_d^{\pm}, +, I^\circ)$ is a subalgebra of $(D_d^{\pm}, +, I^\circ)$. Then for $(\infty, \varnothing) \in \{x, y, z\} \not\ni (-\infty, \varnothing)$ and p in I°, both $(x + y)z\underline{p}$ and $xz\underline{p} + yz\underline{p}$ coincide with $(\infty, \varnothing)$. For $(-\infty, \varnothing) \in \{x, y, z\}$ and p in I°, both $(x + y)z\underline{p}$ and $xz\underline{p} + yz\underline{p}$ coincide with $(-\infty, \varnothing)$. □

The modal $D_d^{\pm}$ will become the codomain modal for support functions of convex subsets X of $\mathbb{R}^d$. The first components of ordered pairs in $D_d^{\pm}$ lie in $\mathbb{R}^{\pm\infty}$, and accommodate the extended real valued support functions defined in Section 3. Let $F_d^{\pm}$ be the set of all functions $f\colon \mathbb{R}^d \to D_d^{\pm}$, with the modal structure inherited from the modal $(D_d^{\pm}, +, I^\circ)$ of Proposition 5.1. Since $D_d^{\pm}$ is a subset of $\mathbb{R}^{\pm\infty} \times G_{d-1}^{\pm}$, and by the induction hypotheses $G_{d-1}^{\pm}$ has been identified with the set $\mathbb{R}^{d-1}T$ of convex subsets of $\mathbb{R}^{d-1}$, one may write each element f of $F_d^{\pm}$ in the form

$$f\colon \mathbb{R}^d \to D_d^{\pm}\ ;\ \ x \mapsto (H_f(x), C_f(x)) \tag{5.5}$$

with $H_f(x) \in \mathbb{R}^{\pm}$ and $C_f(x)$ in $\mathbb{R}^{d-1}T$. The function

$$H_f\colon \mathbb{R}^d \to \mathbb{R}^{\pm}\ ;\ \ x \mapsto H_f(x) \tag{5.6}$$

is called the *extended real function part* of f. For x in $\mathbb{R}^d$, the convex subset $C_f(x)$ of $\mathbb{R}^{d-1}$ is called the *crust shadow* of f (*in the x-direction*). In the following section, conditions are imposed on functions f in $F_d^{\pm}$ for them to lie in a subset $G_d^{\pm}$ of $F_d^{\pm}$. The inductive construction is completed in the rest of the paper by the proof (Theorem 8.7) that $G_d^{\pm}$ is a submodal of $F_d^{\pm}$ isomorphic with the modal $(\mathbb{R}^dT, +, I^\circ)$ of convex subsets of $\mathbb{R}^d$.

6. The $G^{\pm}$-conditions

This section presents the so-called $G^{\pm}$-conditions on a function

$$f\colon \mathbb{R}^d \to D_d^{\pm}\ ;\ \ x \mapsto (H_f(x), C_f(x)),$$

qualifying it for membership in the subset $G_d^{\pm}$ of $F_d^{\pm}$. Some definitions are required for the formulation of the $G^{\pm}$-conditions.

Definition 6.1. Fix a function $H\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$.

(a) Suppose $H(x)$ is finite for some non-zero x in $\mathbb{R}^d$. Define $\pi_x\colon \{z \in \mathbb{R}^d \mid (z|x) = H(x)\} \to \mathbb{R}^{d-1}$ to be the restriction of the projection from $\mathbb{R}^d$ along $x\mathbb{R}$ onto the orthogonal complement of $x\mathbb{R}$.

(b) If x is zero or $H(x)$ is not finite, define $\pi_x\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ to be the zero map.

Definition 6.2. Let $H\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be positively homogeneous and convex. An element x of $\mathbb{R}^d$ is a *linearity point* of H if

$$H(x)H(-x)\underline{1/2} = 0. \tag{6.1}$$

Define $L(H) = \{x \in \mathbb{R}^d \mid H(x)H(-x)\underline{1/2} = 0\}$ as the set of linearity points of H.

Proposition 6.3. *Let $H\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ be positively homogeneous and convex.*

(a) $H(0) = 0 \Rightarrow 0 \in L(H)$.

(b) $x \in L(H) \Rightarrow H(x) \in \mathbb{R}$ *and* $H(-x) = -H(x)$.

(c) $L(H)$ *is a subspace of the vector space* $\mathbb{R}^d$.

Proof. (a) is immediate by the idempotence of $\underline{1/2}$.

(b): For x in $L(H)$, one has $0 = H(0) = H(\overline{x(-x)}\underline{1/2}) \leq H(x)H(-x)\underline{1/2} = 0$, so that $\{H(x), H(-x)\} \subset \mathbb{R}$ by (2.7) and then $0 = \frac{1}{2}H(x) + \frac{1}{2}H(-x)$.

(c): Given (b), (c) follows by the traditional argument [1, §13]. □

For a function $f\colon \mathbb{R}^d \to D_d^{\pm}$ with positively homogeneous and convex extended real function part H_f, and for x in $\mathbb{R}^d$, define the *supercrust*

$$K_f(x) = \{z \in \mathbb{R}^d \mid \forall y \in \mathbb{R}^d, (z|y) \leq H_f'(x;y)\}. \tag{6.2}$$

Define $K_f(x)^\circ$ to be the relative interior of $K_f(x)$. (Equivalently, one may define $(K_f(x)^\circ, I^\circ)$ as the smallest non-empty sink of the barycentric algebra $(K_f(x), I^\circ)$ [4, 386].)

Definition 6.4. The set $G_d^{\pm}$ is the set of elements

$$f\colon \mathbb{R}^d \to D_d^{\pm}\ ;\ x \mapsto (H_f(x), C_f(x))$$

satisfying the following $G^{\pm}$-*conditions*:

($G^{\pm}C1$) $H_f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ is positvely homogeneous;

($G^{\pm}C2$) $H_f\colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ is convex;

($G^{\pm}C3$) $\forall x \in \mathbb{R}^d, C_f(x) \subseteq \pi_x(K_f(x))$;

($G^{\pm}C4$) $\forall x \in L(H_f), C_f(x) \supseteq \pi_x(K_f(x)^\circ)$;

($G^{\pm}C5$) For all nonzero x, y in $\mathbb{R}^d$,

$$\pi_x^{-1}(C_f(x)) \cap \{z \in \mathbb{R}^d \mid (z|y) = H_f(y)\} = \pi_y^{-1}(C_f(y)) \cap \{z \in \mathbb{R}^d \mid (z|x) = H_f(x)\}.$$

7. Support functions of general convex sets

Let X be a convex subset of $\mathbb{R}^d$. In this section, the support function $h_X : \mathbb{R}^d \to D_d^{\pm}$ of X will be defined, and shown to be an element of $G_d^{\pm}$. The extended real function part of $h_X : \mathbb{R}^d \to D_d^{\pm}$ is just the extended real valued support function $H_X : \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ of Definition 3.1. By Propositions 3.3 and 3.4, this extended real function part of h_X is positively homogeneous and convex, so that h_X already satisfies the first two conditions of Definition 7.3.

For a vector x in R^d, the *crust* of X (*in the x-direction*) is the intersection of X with its supporting hyperplane in the x-direction (as in Definition 3.1). The *crust shadow* of X (*in the x-direction*) is defined as the image

$$C_X(x) = \pi_x\big(X \cap \{z \in \mathbb{R}^d \mid (z|x) = H_X(x)\}\big) \tag{7.1}$$

of the crust of X under the "projection" π_x obtained from $H_X : \mathbb{R}^d \to \mathbb{R}^{\pm\infty}$ using Definition 7.1.

Definition 7.1. The *support function* of X is the function

$$h_X : \mathbb{R}^d \to D_d^{\pm}\ ;\ \ x \mapsto (H_X(x), C_X(x)) \tag{7.2}$$

in $F_d^{\pm}$. The supercrusts $K_{h_X}(x)$ of h_X are denoted $K_X(x)$.

Proposition 7.2. *For each vector x in $\mathbb{R}^d$,*

$$K_X(x) = \overline{X} \cap \{z \in \mathbb{R}^d \mid (z|x) = H_X(x)\}. \tag{7.3}$$

Proof. The result is immediate if $X = \varnothing$. Otherwise, if $x \in M(H_X)$ or $H_X(x) = \infty$, then the empty set $K_X(x)$ is a subset of the right hand side of (7.3). Otherwise, one has $H'_X(x; x) = H_X(x)$ and $H'_X(x; -x) = -H_X(x)$, so that

$$\begin{aligned} K_X(x) &= \bigcap_{y \in \mathbb{R}^d} \{z \in \mathbb{R}^d \mid (z|y) \le H'_X(x; y)\} \\ &\subseteq \Big[\bigcap_{y \in \mathbb{R}^d} \{z \in \mathbb{R}^d \mid (z|y) \le H_X(y)\}\Big] \cap \Big[\bigcap_{y \in \{\pm x\}} \{z \in \mathbb{R}^d \mid (z|y) \le H'_X(x; y)\}\Big] \\ &= \overline{X} \cap \{z \in \mathbb{R}^d \mid (z|y) = H_X(x)\} \end{aligned}$$

using Proposition 4.5 and Proposition 3.2.

Conversely, it must be shown that $K_X(x)$ contains the right hand side of (7.3) for $X \neq \varnothing$. If this right hand side is empty, the containment is trivial. Otherwise, $H_X(x) \in \mathbb{R}$.

Consider an element t of the right hand side. By Proposition 3.2, one has $(t|y) = (t|x)(t|xy\ \underline{1/r})\underline{r} \le H_X(x)H_X(xy\ \underline{1/r})\underline{r}$ for all y in $\mathbb{R}^d$ and r in $(1, \infty)$. In particular, there is no point z in $\mathbb{R}^d$ with $\lim_{r\to\infty} H_X(x)H_X(xz\ \underline{1/r})\underline{r} = -\infty$, so that x is not a singular point of H_X. By (4.1), one then has $(t|y) \leq H'_X(x;y)$ for all y in $\mathbb{R}^d$, so that t lies in the supercrust $K_X(x)$. $\square$

Theorem 7.3. *The support function $h_X : \mathbb{R}^d \to D_d^{\pm}$ of a convex subset X of $\mathbb{R}^d$ satisfies the $G^{\pm}$-conditions.*

Proof. The conditions $(G^{\pm}C1)$ and $(G^{\pm}C2)$ have already been verified. Moreover, $(G^{\pm}C3)$ is immediate from (7.1) and (7.3). To verify $(G^{\pm}C4)$, consider a linearity point x of H_X. By (3.8), one has $X \subseteq \{z \in \mathbb{R}^d \mid (z|x) = H_X(x)\}$. By (7.3),

$$x \in L(H_X) \Rightarrow \overline{X} = K_X(x). \tag{7.4}$$

By [2, Theorem 3.4(d)], one has $X^{\circ} = (\overline{X})^{\circ}$. Thus $C_X(x) = \pi_x(X) \supseteq \pi_x(X^{\circ}) = \pi_x((\overline{X})^{\circ}) = \pi_x(K_X(x)^{\circ})$, verifying $(G^{\pm}C4)$. Finally, $(G^{\pm}C5)$ holds, since each side of the equality reduces to

$$X \cap \{z \in \mathbb{R}^d \mid (z|x) = H_X(x)\} \cap \{z \in \mathbb{R}^d \mid (z|y) = H_X(y)\}.$$

$\square$

8. Characterization of support functions

By Theorem 7.3, there is a well-defined function

$$\eta : \mathbb{R}^d T \to G_d^{\pm} ; \;\; X \mapsto h_X. \tag{8.1}$$

For a function $f : \mathbb{R}^d \to D_d^{\pm} ; \;\; x \mapsto (H_f(x), C_f(x))$ in $G_d^{\pm}$, define the convex subset

$$P_f(x) = \{z \in \mathbb{R}^d \mid (z|x) < H_f(x)\} \cup \pi_x^{-1}(C_f(x)) \tag{8.2}$$

of $\mathbb{R}^d$ for each x in $\mathbb{R}^d$ and

$$X_f = \bigcap_{x \in \mathbb{R}^d} P_f(x). \tag{8.3}$$

Define the function

$$\xi : G_d^{\pm} \to \mathbb{R}^d T ; \;\; f \mapsto X_f. \tag{8.4}$$

In this section, it will be shown that ξ and η are mutually inverse modal isomorphisms between $\mathbb{R}^d T$ and $G_d^{\pm}$.

For X in $\mathbb{R}^d T$, recall the notation $K_{h_X}(x) = K_X(x)$. Similarly, write $P_X(x)$ for $P_{h_X}(x)$. By (6.2), (4.8), and (3.8), note $K_X(0) = \overline{X}$.

Proposition 8.1. *For X in $\mathbb{R}^d T$ and each x in $\mathbb{R}^d$, one has $K_X(0)^{\circ} \subseteq P_X(x)$.*

Proof. For empty X, one has $K_X(0)^{\circ} = \varnothing = P_X(x)$ for each x in X. Now suppose that X is non-empty. Note $K_X(0)^{\circ} \subseteq \mathbb{R}^d = P_X(0)$, since $C_X(0) = \{0\}$. For $0 \neq x \in L(H_X)$, (7.4) and $(G^{\pm}C4)$ yield $\pi_x(K_X(0)^{\circ}) = \pi_x(K_X(x)^{\circ}) \subseteq C_f(x)$, whence $K_X(x)^{\circ} \subseteq \pi_x^{-1}(C_f(x))$ by (8.2). If x is not a linearity point of H_X, then $K_X(0)^{\circ} = (\overline{X})^{\circ} \subseteq \{z \in \mathbb{R}^d \mid (z|x) < H_f(x)\}$. $\square$

Proposition 8.2. *For X in $\mathbb{R}^dT$ and $f = h_X$, one has $X = X_f$.*

Proof. For $X = \varnothing$, (7.1) and (8.2) yield $P_f(x) = \varnothing$ for all x in $\mathbb{R}^d$, whence $X = X_f$. Now suppose that X is non-empty. By (3.8), (7.1), and (8.2), one certainly has $X \subseteq X_f$. It remains to be shown that $X_f \subseteq X$. By [2, Theorem 3.1], [2, Theorem 3.4(d)], and Proposition 8.1, one has $\varnothing \subset X^\circ = (\overline{X})^\circ = K_X(0)^\circ \subseteq X_f$. Let u be an element of X_f. It must be shown that $u \in X$.

By the positive homogeneity of H_X, one has $P_f(ax) = P_f(x)$ for x in S^{d-1} and $a > 0$. If $\exists x \in S^{d-1}, (u|x) \geq H_X(x)$, then $\exists x \in S^{d-1}, u \in \pi_x^{-1}(C_X(x)) \subseteq X$, whence $u \in X$. Otherwise,

$$\forall x \in S^{d-1},\ (u|x) < H_X(x). \tag{8.5}$$

In this case $L(H_X) = \{0\}$, for $y \in L(H_X) \cap S^{d-1}$ implies

$$\bigcap_{x \in S^{d-1}} \{z \in \mathbb{R}^d \mid (z|x) < H_X(x)\}$$
$$\subseteq \{z \in \mathbb{R}^d \mid (z|y) < H_X(y)\} \cap \{z \in \mathbb{R}^d \mid (z|-y) < H_X(-y)\} = \varnothing.$$

Define

$$g \colon \mathbb{R}^d \to \mathbb{R}^{\pm\infty}\,;\, x \mapsto \quad \mathbf{if}\, H_X(x) = \infty\, \mathbf{then}\, \infty\, \mathbf{else}\, H_X(x) - (u|x) \tag{8.6}$$

and

$$V = \{z \in \mathbb{R}^d \mid \forall x \in S^{d-1},\ (z|x) \leq 0g(x)\ \underline{1/2}\}. \tag{8.7}$$

Note that V is a closed, convex subset of $\mathbb{R}^d$. By (8.5), one has $0 \in V$. Now $L(g) \subseteq L(H_X) = \{0\}$, so the relative interior V° of V coincides with its interior. By (8.5), one has $O \in V^\circ$. Thus for some positive r, there is an open ball $B_r(0) = \{z \in \mathbb{R}^d \mid (z|z) < r\}$ such that $O \in B_r(0) \subseteq V^\circ$. Then $u \in u + B_r(0) \subseteq u + V$, whence $z \in V \Rightarrow \forall x \in S^{d-1}, \Rightarrow \forall x \in S^{d-1}, (u+z|x) = (u|x) + (z|x) <$ $\mathbf{if}\, H_X(x) = \infty\, \mathbf{then}\, \infty\, \mathbf{else} (u|x) + \frac{1}{2}[H_X(x) - (u|x)] \leq H_X(x)$. Thus $\forall z \in V$, $u + z \in X^\circ$. In particular, $u = u + 0 \in X$, as required. □

Proposition 8.3. *The maps $\xi \colon G_d^\pm \to \mathbb{R}^dT$ and $\eta \colon \mathbb{R}^dT \to G_d^\pm$ are mutually inverse.*

Proof. By Proposition 8.2, $\eta\xi = 1_{\mathbb{R}^dT}$. Conversely, consider $f \colon x \mapsto (H_f(x), C_f(x))$ in $G_d^\pm$. If $H_f(0) = -\infty$, each supercrust $K_f(x)$ is empty, whence each crust shadow $C_f(x)$ is empty by ($G^\pm C3$). By (8.2) and (8.3), it follows that $X_f = \varnothing$, and then $f\xi\eta = h_\varnothing = f$, as required. The remainder of the proof that $\xi\eta = 1_{G_d^\pm}$ follows the lines of [5, Lemma 5.10]. □

Proposition 8.4. *For* $0 \neq x \in \mathbb{R}^d$*, for* $p \in I^\circ$*, and for* $A, B \in \mathbb{R}^d T$*,*

$$\left[A \cap \{z \mid (z|x) = H_{\overline{A}}(x)\}\right]\left[B \cap \{z \mid (z|x) = H_{\overline{B}}(x)\}\right]\underline{p} = AB\underline{p} \cap \{z \mid (z|x) = H_{\overline{AB\underline{p}}}(x)\}. \quad (8.8)$$

Proof. If both A and B are non-empty, the proof proceeds as for [5, Lemma 5.12]. Otherwise, both sides of (8.8) are empty. □

Proposition 8.5. *The map*

$$\varphi\colon (\mathbb{R}^d T, +, I^\circ) \to (F_d^\pm, +, I^\circ)\ ;\ X \mapsto h_X \quad (8.9)$$

is a modal homomorphism.

Proof. For A, B in $\mathbb{R}^d T$ and x in $\mathbb{R}^d$, it must be shown that

$$(H_{A+B}(x), C_{A+B}(x)) = (H_A(x), C_A(x)) + (H_B(x), C_B(x)) \quad (8.10)$$

and

$$(H_{AB\underline{p}}(x), C_{AB\underline{p}}(x)) = (H_A(x), C_A(x))(H_B(x), C_B(x))\underline{p} \quad (8.11)$$

for p in I°. If A and B are both non-empty, the proof proceeds as for [5, Lemma 5.13]. Otherwise, suppose without loss of generality that A is empty. Then the left hand side of (8.10) is $(H_B(x), C_B(x))$, while the right hand side is $(-\infty, \varnothing) + (H_B(x), C_B(x)) = (H_B(x), C_B(x))$. The left side of (8.11) is $(-\infty, \varnothing)$, while the right hand side is $(-\infty, \varnothing)(H_B(x), C_B(x))\underline{p} = (-\infty, \varnothing)$, as required. □

Corollary 8.6. *The set* $G_d^\pm$ *is a submodal of* $(F_d^\pm, +, I^\circ)$*.*

Proof. Note that $\eta\colon \mathbb{R}^d T \to G_d^\pm$ is the corestriction to $G_d^\pm$ of $\varphi\colon \mathbb{R}^d T \to F_d^\pm$. By Proposition 8.3, the set $G_d^\pm$ is the image of the modal isomorphism φ. □

Combining Propositions 8.3 and 8.5 with Corollary 8.6, one achieves the completion of the inductive constructions of Section 5.

Theorem 8.7. *The modals* $(\mathbb{R}^d T, +, I^\circ)$ *of convex subsets of* $\mathbb{R}^d$ *and* $(G_d^\pm, +, I^\circ)$ *of support functions are isomorphic.*

References

[1] T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper*, Chelsea, New York 1948.

[2] A. Brønsted, *An Introduction to Convex Polytopes*, Springer, New York 1983.

[3] H. Minkowski, *Gesammelte Abhandlungen* 2. Band Teubner, Leipzig 1911.

[4] A. B. Romanowska and J. D. H. Smith, *Modal Theory – An Algebraic Approach to Order, Geometry and Convexity*, Heldermann Verlag, Berlin 1985.

[5] A. B. Romanowska and J. D. H. Smith, *Support functions and ordinal products*, Geom. Dedicata **30** (1989), 281–296.

[6] E. Slatinský, *Die arithmetische Operation der Summe*, Archivum Math. **20** (1984), 9–20.
[7] J. D. H. Smith, *Modes and modals*, Discuss. Math. Algebra Stochastic Methods **19** (1999), 9–40.

Dug-Hwan Choi
School of Electrical and Computer Engineering, Sungkyunkwan University,
300 ChunChun-Dong, JangAn-Gu, Suwon, Kyunggi-Do 440-746, Korea

Jonathan D. H. Smith
Iowa State University, Ames, Iowa 50011, USA