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Semilattice-Based Dualities

Abstract. The paper discusses "regularisation" of dualities. A given duality between (concrete) categories, e.g. a variety of algebras and a category of representation spaces, is lifted to a duality between the respective categories of semilattice representations in the category of algebras and the category of spaces. In particular, this gives duality for the regularisation of an irregular variety that has a duality. If the type of the variety includes constants, then the regularisation depends critically on the location or absence of constants within the defining identities. The role of schizophrenic objects is discussed, and a number of applications are given. Among these applications are different forms of regularisation of Priestley, Stone and Pontryagin dualities.

Key words: Priestley duality, Stone duality, Pontryagin duality, character group, program semantics, Bochvar Logic, regular identities.

1. Introduction

Priestley duality [26] [27] between bounded distributive lattices and ordered Stone spaces may be recast as a duality between unbounded distributive lattices and bounded ordered Stone spaces [6, p.170]. In joint work with Gierz [7], and using a range of ad hoc techniques, the first author combined this form of Priestley duality with the Płonka sum construction to obtain duality for distributive bisemilattices without constants. Such algebras have recently found application in the theory of program semantics [15] [28]. The variety of distributive bisemilattices without constants is the regularization of the variety of unbounded distributive lattices, i.e. the class of algebras, of the same type as unbounded lattices, satisfying each of the regular identities satisfied by all unbounded distributive lattices. In this sense, the duality of [7] may be described as the regularization of Priestley duality without constants.

For a type with constants, there are various forms of regularization according to the location or absence of the constants within identities. Part of the motivation for the current work is the desire to apply these versions of regularization to Priestley duality in its original form, i.e. as a duality for bounded distributive lattices. Indeed the resulting algebras, distributive bisemilattices with constants, are more readily suited to the applications in program semantics. Another motivation for the current work is the desire

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to extend traditional Pontryagin duality by regularization. Since Pontryagin duality (between discrete abelian groups and compact Hausdorff abelian groups) involves constants, such extensions have to involve constants as well.

The ad hoc techniques of [7] did not lend themselves well to further development. As a first step, the present authors gave a general and more conceptual treatment of the regularization of a duality for algebras without constants [34]. There were two key ingredients in this treatment. The first (which was also used independently by Idziak in a different context [12]) was the interpretation of the Płonka sum as an equivalence between semilattice representations or sheaves and (total spaces of) bundles. The second was a purely categorical theorem showing how an initial duality could be enlarged to a duality between categories of representations of semilattices in the categories of the initial duality. (The initial duality was recovered from the enlarged duality as the duality for representations of trivial semilattices.) The present paper recalls the key concepts of [34], but the reader is referred there for details such as the proof of the duality theorem for semilattice representations. The proof remains valid for the improved version of this duality theorem (Theorem 4.4) formulated here with less restrictive completeness assumptions.

For algebras with constants, there are three main forms of regularization. These are known as regularization, symmetrization, and symmetric regularization. They correspond to three possible locations of a constant object in a meet semilattice viewed as a poset category: terminal, initial, or arbitrary. There are three corresponding varieties of semilattices with constant. The second section of the paper discusses duality theorems for semilattices without constants and for these three varieties of semilattices with constant. These semilattice dualities may be viewed as regularizations of (trivial) duality for trivial algebras. Semilattice duality is the basis for regularization of other dualities.

The third section starts with the analysis of identities with and without constants that leads to the various forms of regularization. It then presents the various forms of the Płonka sum construction that describe algebras in the various regularizations of a strongly irregular variety $\mathfrak{V}$. For the symmetrization and symmetric regularization, the appropriate Płonka sums represent semilattices in a new category ${}^2\mathfrak{V}$. This new category is itself a sort of Płonka sum of the category $\mathfrak{V}$ (with homomorphisms as morphisms) and the "non-constant reduction" of $\mathfrak{V}$, the variety $\mathfrak{V}_+$ generated by the non-constant reducts of algebras from $\mathfrak{V}$.

The fourth section begins with the concept of compatibility between dualities for $\mathfrak{V}$ and $\mathfrak{V}_+$. Given such compatible dualities, there is a duality for the Płonka sum category ${}^2\mathfrak{V}$ (Theorem 4.2). The section then leads up to the main Theorem 4.6 presenting dualities for all the various regularizations of a strongly irregular variety $\mathfrak{V}$, with or without constants. Subsequent sections give a selection of examples illustrating how Theorem 4.6 may be applied. In the applications, one is often interested in axiomatizing semilattice representations as sets with structure (algebraic, relational, or topological). This may be achieved by the use of schizophrenic objects as discussed in Proposition 5.1, and by the basic Theorem 5.2. The first applications, discussed in the fifth section, are the regularizations of Priestley, Stone, and Pontryagin duality. These applications are relatively straightforward.

The sixth section shows how Priestley duality may be symmetrically regularized. In order to obtain compatible dualities for distributive lattices with and without lower bounds, the standard Priestley duality for distributive lattices with lower bounds has to undergo a subtle modification involving "creation operators" and "destruction operators". The key issue is that the underlying set functor from a concrete category of sets with structure is not always a functor forgetting that structure. (A standard example of this phenomenon occurs when the underlying set of an object in a category of Banach spaces is the unit disc of the Banach space, not the whole space [35, p. 47].) The seventh section presents various regularizations of Lindenbaum-Tarski duality, based on a new version of this duality covering the case of pointed sets. The final section deals with varieties such as Boolean algebras and abelian groups having definable constants. Although it may not be possible to find a compatible duality for the non-constant reduction of such a variety, a simple modification to Theorem 4.6 enables it to provide symmetric regularizations of dualities for these varieties. Stone and Pontryagin dualities are analyzed from this standpoint.

The topic of this paper involves concepts from a number of branches of mathematics. Readers unfamiliar with any one of these branches may often be able to suspend disbelief temporarily in order not to lose the thread of the discussion. For category theory, one may refer to [16] or [35]; for sheaf theory, to [13] or [17]; for universal algebra, to [31]; for duality, to [5] or [13]. Note that mappings and functors are generally written to the right of their arguments so that concatenations such as (4.11) or (6.6) may be read in natural order from left to right.

2. Semilattice Dualities

The variety $\underline{\underline{\mathrm{Sl}}}$ of *semilattices* is the variety of commutative, idempotent semigroups. The semigroup multiplication will usually be denoted by a period or by juxtaposition of the arguments. A semilattice H or $(H, .)$ may be considered as a poset $(H, \leq)$ or $(H, \leq .)$ – a *meet semilattice* – under the relation

$$x \leq y \Leftrightarrow x \cdot y = x. \tag{2.1}$$

In turn, the poset $(H, \leq)$ may be considered as a small category H with set H of objects. The morphism set $H(x, y)$ is the singleton $x \to y$ if $x \leq y$, and otherwise $H(x, y)$ is empty. Another way of looking at a semilattice H is as a topological space with topology

$$\Omega(H) = \{G \subseteq H | x \leq y \in G \Rightarrow x \in G\}, \tag{2.2}$$

the *Alexandrov topology* $\Upsilon(H, \geq)$ of the dual poset $(H, \geq)$ [13, II. 1.8]. The poset $(H, \leq)$ may be identified with the subposet $\{\downarrow h | h \in H\}$ of $(\Omega(H), \subseteq)$ consisting of principal subordinate subsets. In this guise, H forms a basis for the topology $\Omega(H)$. At various times throughout the current work, it becomes convenient to regard semilattices as semigroups, posets, categories, topological spaces, or as bases of topologies.

Beyond the basic variety $\underline{\underline{\mathrm{Sl}}}$ of semilattices, various varieties of semilattices with constants are needed. A *bounded semilattice* is a semilattice H with two constants forming lower and upper bounds in the poset $(H, \leq)$. The variety $\underline{\underline{\mathrm{Sl}}}_\mu$ is the variety of semilattices with a single constant (i.e. the tensor product [31, 232] of $\underline{\underline{\mathrm{Sl}}}$ with the variety of pointed sets). The variety $\underline{\underline{\mathrm{Sl}}}_0$ is the variety of commutative idempotent monoids. (The suffix 0 here pertains to the additive notation for semilattices used in [25]. Despite the current use of multiplicative notation, retention of this designation for the variety of monoids facilitates use of [25] and related references.) Finally, the variety $\underline{\underline{\mathrm{Sl}}}_1$ is the variety of semilattices with an absorbing constant (i.e. an initial object in the corresponding category). Along with the notation for semilattice varieties, which are categories having all homomorphisms as morphisms, notations for the dual categories of representation spaces are needed. Recall that a *Stone* topological space is one having a compact Hausdorff zero-dimensional topology, or characterized by the equivalent conditions of [13, Theorem II. 4.2]. Then $\mathfrak{B}$ denotes the category of bounded Stone topological semilattices and continuous homomorphisms between them. Following [10], $\underline{Z}$ denotes the category of Stone topological $\underline{\underline{\mathrm{Sl}}}_0$-algebras (the mnemonic Z for "zero" is helpful here). As posets, $\mathfrak{B}$-spaces are complete (indeed algebraic) lattices [10, p. 39]. A subset X of a $\mathfrak{B}$-space G is a *cover* of an element g of G iff $g \leq \sup X$. An element c of G is *compact* if it is not the lower bound, and if each cover of c contains a finite subcover. Then $\underline{U}$ denotes the full subcategory of $\mathfrak{B}$ consisting of $\mathfrak{B}$-spaces with compact upper bound. Finally, $\underline{M}$ is the category of $\mathfrak{B}$-spaces with a compact constant and $\mathfrak{B}$-morphisms respecting these constants.

The dualities for semilattice varieties may be listed as follows:

$$C : \underline{\underline{\mathrm{Sl}}} \rightleftarrows \mathfrak{B} : F; \tag{2.3}$$

$$C_0 : \underline{\underline{\mathrm{Sl}}}_0 \rightleftarrows \underline{Z} : F_0; \tag{2.4}$$

$$C_1 : \underline{\underline{\mathrm{Sl}}}_1 \rightleftarrows \underline{U} : F_1; \tag{2.5}$$

$$C_\mu : \underline{\underline{\mathrm{Sl}}}_\mu \rightleftarrows \underline{M} : F_\mu. \tag{2.6}$$

Duality (2.4) is covered very thoroughly in [10], and briefly in [13, VI. 3.6]. Duality (2.3) is given explicitly in [6, p. 158], [5, p. 28]. Both these dualities arise from the role of the 2-element semilattice $2 = \{0 < 1\}$ as a "schizophrenic object", i.e. as an object (with appropriate structure) of each of the categories appearing in the duality. Thus $C = \underline{\underline{\mathrm{Sl}}}(\ \ ,2)$, $F = \mathfrak{B}(\ \ ,2)$, $C_0 = \underline{\underline{\mathrm{Sl}}}_0(\ \ ,2)$, and $F_0 = \underline{Z}(\ \ ,2)$. The dualities (2.5) and (2.6) do not seem to have been treated in the literature (and (2.6) does not appear to arise from a schizophrenic object). They are discussed below on the basis of the duality (2.3).

As an object $\underset{\sim}{2}$ of $\mathfrak{B}$, the two-element semilattice has the discrete topology, lower bound 0, and upper bound 1. For a semilattice H, the $\mathfrak{B}$-space HC is defined to be the closed subspace $\underline{\underline{\mathrm{Sl}}}(H,2)$ of the product space $\underset{\sim}{2}^H$. Elements of HC are called *characters* of H. The characteristic function of a subset Θ of a semilattice $(H,.)$ is a character of $(H,.)$ iff the subset Θ is a *wall* of $(H,.)$, i.e. iff

$$\forall h,k \in H,\ (h \cdot k \in \Theta) \Leftrightarrow (h \in \Theta, k \in \Theta) \tag{2.7}$$

[33, Prop. 2.2]. In meet semilattices, walls are often described as "filters" (cf. [10, Defn. 2.1]), while in join semilattices they are often described as "ideals" (cf. [32, Defn. 4.1]). Let HW denote the set of walls of H.

PROPOSITION 2.1. *Under intersection, HW forms a subsemilattice of the power set of H. Moreover, there is a natural $\mathfrak{B}$-isomorphism*

$$HC \to HW;\ \chi \mapsto \chi^{-1}\{1\}. \tag{2.8}$$

PROOF. Cf. [10, Prop. II. 2.4(ii)]. The least element of HW is the empty wall. ■

PROPOSITION 2.2. *In a $\mathfrak{B}$-space G, the set GK of compact elements forms a join semilattice (i.e. a subcategory with coproducts of pairs of objects).*

PROOF. See [3, Th. VIII. 8], [10, Prop. II. 1.10 and Th. II. 3.3] or [31, 64]. ■

For an element h of a semilattice H, the wall [h] is the intersection of all the walls of H containing h. Such a wall is called *principal.*

PROPOSITION 2.3. *For a semilattice H, an element Θ of HW is compact iff it is principal. There is a natural isomorphism*

$$H \to HWK;\ h \mapsto [h]. \tag{2.9}$$

PROOF. See [10, Prop. II. 3.8], [31, 344 (i)] or [32, Th. 5.1]. The latter two references apply on considering the variety of stammered semilattices [31, 327]. ■

In the other direction, one has the following

PROPOSITION 2.4. *For a $\mathfrak{B}$-space G, there is a natural isomorphism*

$$\gamma_G : G \to GKW;\ g \mapsto GK \cap \downarrow g. \tag{2.10}$$

PROOF. Cf. [10, Prop. II. 3.9]. The bottom element of G maps to the empty wall: each compact element of G is strictly above the bottom. ■

PROPOSITION 2.5. *There is a contravariant functor $K : \mathfrak{B} \to \underline{\underline{\mathrm{Sl}}}$ given by Proposition 2.2. For a $\mathfrak{B}$-space G, define*

$$\kappa_G : GF \to GK;\ \theta \mapsto \inf \theta^{-1}\{1\}. \tag{2.11}$$

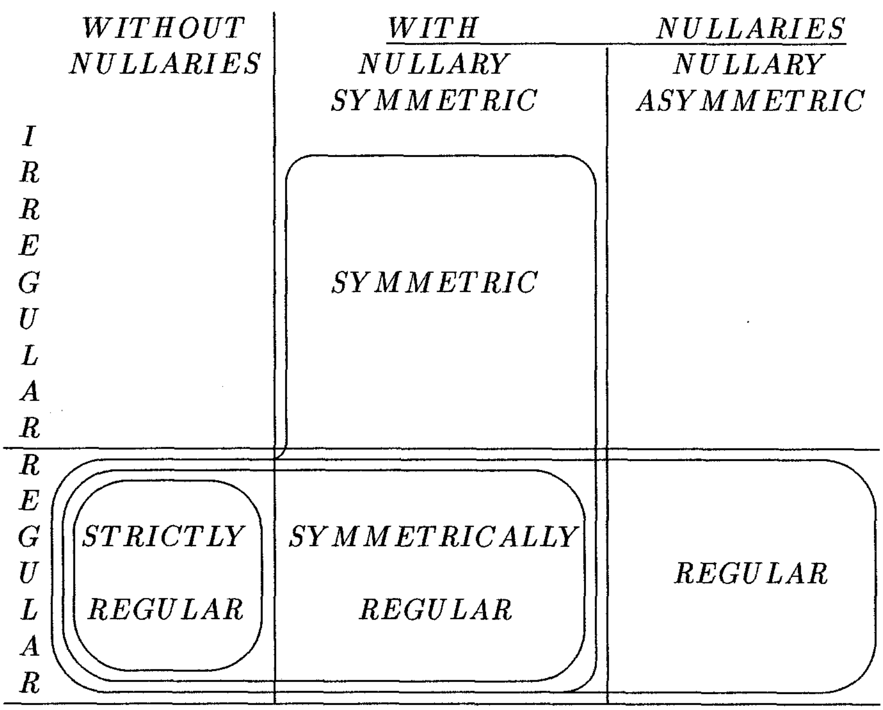
Then $\kappa : F \dot{\to} K$ is a natural isomorphism.

PROOF. See [10, Th. II.3.7 and Prop. II. 3.20]. ■

With these details of the basic semilattice duality (2.3) established, it becomes easy to treat the dualities (2.5) and (2.6). For an $\underline{\underline{\mathrm{Sl}}}_\mu$-object $(H, ., \mu)$, the corresponding $\underline{M}$-object HC_μ may be defined with the aid of Propositions 2.1 and 2.3. Namely HC_μ is the $\mathfrak{B}$-object HW with the principal wall $[\mu]$ selected as the compact constant. In the other direction, suppose given an $\underline{M}$-object G with compact constant m. Then GF_μ is the semilattice GF with constant $m\kappa_G^{-1}$, defined using the natural isomorphism of (2.11). The functors C_1 and F_1 of (2.5) are the respective restrictions of C_μ and F_μ to the full subcategories $\underline{\underline{\mathrm{Sl}}}_1$ of $\underline{\underline{\mathrm{Sl}}}_\mu$ and $\underline{U}$ of $\underline{M}$.

3. Identities and Płonka Sums

The varieties leading to semilattice-based dualities are determined by properties of their defining identities. One thus needs a fairly detailed classification of different kinds of identity. An identity is *regular* if it involves exactly the same set of arguments on each side [31, 13]. It is *strictly regular* if it is regular, and does not involve nullary operations on either side. It is *nullary symmetric* if it does involve nullary operations on each side. It is *nullary asymmetric* if it involves nullary operations on one side only. It is *symmetric* if it is strictly regular or nullary symmetric. Finally, it is *symmetrically regular* if it is regular, but not nullary asymmetric. This classification may be displayed by the following Venn diagram.



There is a nice correspondence between the classes of identities outlined in the diagram and the varieties of semilattices listed on the left hand side of (2.3) – (2.6). Let $\tau : B \to \mathbb{N}$ be a *plural* type, i.e. with $\tau^{-1}\{n \in \mathbb{N} | n > 1\}$ non-empty. Set $B^+ = \tau^{-1}\mathbb{Z}^+$ and $B_0 = \tau^{-1}\{0\}$. A semilattice $(H, .)$ may be realized as a B^+-algebra, a B^+*-semilattice*, on setting

(3.1) $$h_1 \dots h_{\omega\tau}\omega = h_1 \cdot \ldots \cdot h_{\omega\tau}$$

for ω in B^+. A pointed semilattice $(H, ., \mu)$ from $\underline{\underline{\mathrm{Sl}}}_\mu$ or its subvarieties $\underline{\underline{\mathrm{Sl}}}_0, \underline{\underline{\mathrm{Sl}}}_1$ may be realized as a B-algebra for non-empty B_0 via (3.1) and

$$\omega = \mu \tag{3.2}$$

for ω in B_0. Such a B-algebra is called a *B-semilattice.* The classification of identities in the type τ carries over to varieties of B-algebras. A variety $\mathfrak{V}$ of B-algebras is said to be symmetrically regular, for example, if it is precisely the class of all B-algebras satisfying a set of symmetrically regular identities.

PROPOSITION 3.1. *Let $\tau : B \to \mathbb{N}$ be a plural type, and let $\mathfrak{V}$ be a variety of B-algebras.*

(a) *Suppose that B_0 is empty. Then $\mathfrak{V}$ contains the variety $\underline{\underline{\mathrm{Sl}}}$ of B-semilattices iff $\mathfrak{V}$ is* (*strictly*) *regular. In particular, $\underline{\underline{\mathrm{Sl}}}$ is the variety of B-algebras satisfying all* (*strictly*) *regular identities in the type τ.*

(b) *Suppose that B_0 is non-empty. Then $\mathfrak{V}$ contains the varieties* (i) $\underline{\underline{\mathrm{Sl}}}_0$, (ii) $\underline{\underline{\mathrm{Sl}}}_1$, (iii) $\underline{\underline{\mathrm{Sl}}}_\mu$ *of B-semilattices iff $\mathfrak{V}$ is respectively* (i) *regular,* (ii) *symmetric,* (iii) *symmetrically regular. In particular, the varieties* (i) $\underline{\underline{\mathrm{Sl}}}_0$, (ii) $\underline{\underline{\mathrm{Sl}}}_1$, (iii) $\underline{\underline{\mathrm{Sl}}}_\mu$ *are the varieties of B-algebras satisfying respectively all* (i) *regular,* (ii) *symmetric,* (iii) *symmetrically regular identities in the type τ.*

PROOF. (a) Cf. [20], [21], [25, Prop 2.1], [31, 235]. (b) Cf. [25, Prop. 9.1]. For (i), cf. [12, Cor. 3.8], [22]. For (ii), cf. [23]. For (iii), cf. [24]. ∎

DEFINITION 3.2. Let $\tau : B \to \mathbb{N}$ be a plural type, and let $\mathfrak{V}$ be a variety of B-algebras.

(a) Suppose that B_0 is empty. Then the (*strict*) *regularization* $\widetilde{\mathfrak{V}}$ of $\mathfrak{V}$ is the variety of all B-algebras satisfying all the (strictly) regular identities satisfied in $\mathfrak{V}$.

(b) Suppose that B_0 is non-empty. Then the (i) *regularization* $\widetilde{\mathfrak{V}}_0$, (ii) *symmetrization* $\widetilde{\mathfrak{V}}_1$, (iii) *symmetric regularization* $\widetilde{\mathfrak{V}}_\mu$ of $\mathfrak{V}$ are the varieties of all B-algebras satisfying respectively all the (i) regular, (ii) symmetric, (iii) symmetrically regular identities satisfied in $\mathfrak{V}$.

(c) The (*non-constant*) *reduction* $\mathfrak{V}_+$ of $\mathfrak{V}$ is the variety of B^+-algebras satisfying each of the identities satisfied by all of the B^+-reducts of algebras from $\mathfrak{V}$.

Models for the various varieties introduced in Definition 3.2 (a) (b) may be obtained by a construction known as the Płonka sum from representations of semilattices or sheaves over Alexandrov spaces (2.2). Let H be a semilattice, considered as a small category. A *set representation* of H is a contravariant functor

$$R : H \to \underline{\underline{\mathrm{Set}}} \tag{3.3}$$

from H to the category of sets, i.e. an object in the functor category $\widehat{H} = \underline{\underline{\mathrm{Set}}}^{H^{op}}$ of presheaves on H. Since the elements of H, as principal subordinate subsets, are join-irreducible in the poset $(\Omega(H), \subseteq)$, the condition [17, II. 1(9)] for a presheaf on H to be a sheaf is trivially satisfied. By the Comparison Lemma for Grothendieck topoi [17, Th. II.1.3 and App., Cor. 3(a)], the category $\widehat{H}$ is equivalent to the category $\underline{\underline{\mathrm{Sh}}}(H)$ of sheaves (of sets) over the space H under the Alexandrov topology (2.2). By [17, Cor. II. 6.3], the category $\underline{\underline{\mathrm{Sh}}}(H)$ is in turn equivalent to the category $\underline{\underline{\mathrm{Etale}}}H$ of étale bundles $\pi : E \to H$ over the space H. Given a representation (presheaf) $R : H \to \underline{\underline{\mathrm{Set}}}$, the corresponding bundle $\pi : E \to H$ (or more loosely just the total space E) is the bundle $R\Lambda$ of germs of the sheaf $R : \Omega(H) \to \underline{\underline{\mathrm{Set}}}$. An alternative, purely algebraic description of the equivalence between semilattice representations and étale bundles may be given. The variety $\underline{\underline{\mathrm{Lz}}}$ of *left trivial* or *left zero bands* is the variety of semigroups satisfying

$$x * y = x \tag{3.4}$$

[11, 119], [31, 225]. The category $\underline{\underline{\mathrm{Lz}}}$ is isomorphic to the category $\underline{\underline{\mathrm{Set}}}$ of sets. In one direction, forget the multiplication (3.4). In the other direction, the multiplication on any set is just the projection $(x, y) \mapsto x$ from the direct square. The (strict) regularization $\widetilde{\underline{\underline{\mathrm{Lz}}}}$ is the variety of *left normal bands*, the variety of idempotent semigroups (bands) satisfying

$$x * y * z = x * z * y \tag{3.5}$$

[11, 119] [31, 223]. The *bundle* $\pi : E \to H$ of a left normal band $(E, *)$ is its projection onto its semilattice replica [31, 17]. One obtains a corresponding representation

$$E\Gamma : H \to \underline{\underline{\mathrm{Set}}};\ h \mapsto \pi^{-1}\{h\}, \tag{3.6}$$

defined on the morphism level by

$$(h \to k)E\Gamma : \pi^{-1}\{k\} \to \pi^{-1}\{h\};\ x \mapsto x * y \tag{3.7}$$

for any y in $\pi^{-1}\{h\}$. In the other direction, a presheaf or representation (3.3) gives a contravariant functor

$$R : H \to (*) \tag{3.8}$$

to the category $(*)$ of groupoids or magmas (sets with a binary multiplication) and homomorphisms. Here (3.8) is obtained from (3.3) on interpreting each set hR as the left zero band $(hR, *)$. Defining $\pi = \dot{\bigcup\limits_{h \in H}} (hR \to \{h\})$ and

$$x * y = x(x^\pi * y^\pi \to x^\pi)^R \tag{3.9}$$

for x, y in $E = \dot{\bigcup\limits_{h \in H}} hR$ gives the bundle

$$R\Lambda : E \to H \tag{3.10}$$

of a left normal band $(E, *)$. The constructions Γ of (3.6) and Λ of (3.10) extend to functors providing an equivalence

$$\Lambda : \widehat{H} \rightleftarrows (\widetilde{\underline{\underline{\mathrm{Lz}}}}, H) : \Gamma \tag{3.11}$$

between the presheaf category $\widehat{H}$ for a fixed semilattice H and the comma category $(\widetilde{\underline{\underline{\mathrm{Lz}}}}, H)$ of left normal bands over H (cf. [16], §II. 6). The equivalence (3.11) is an algebraic analogue of the equivalence

$$\Lambda : \underline{\underline{\mathrm{Sh}}}H \rightleftarrows \underline{\underline{\mathrm{Etale}}}\ H : \Gamma \tag{3.12}$$

of sheaf theory [13, Cor. V. 1.5(i)], [17, Cor. II. 63].

The algebraic equivalence (3.11) may be extended. Let $\underline{\underline{C}}$ be a category whose objects are small categories and whose morphisms are functors. Let $\underline{\underline{D}}$ be a category. Then one may define a new category $(\underline{\underline{C}}; \underline{\underline{D}})$, called a *semicolon category*, as follows. Its objects are covariant functors $R : C \to \underline{\underline{D}}$ from an object C of $\underline{\underline{C}}$ to $\underline{\underline{D}}$. Given two such objects $R : C \to \underline{\underline{D}}$ and $R' : C' \to \underline{\underline{D}}$, a morphism $(\sigma, f) : R \to R'$ is a pair consisting of a $\underline{\underline{C}}$-morphism $f : C \to C'$ and a natural transformation $\sigma : R \dot{\to} fR'$. The composition of morphisms in $(\underline{\underline{C}}; \underline{\underline{D}})$ is defined by

$$(\sigma, f)(\tau, g) = (\sigma(f\tau), fg) \tag{3.13}$$

(cf. [10, §0.1] [16, Ex. V. 2.5 (b)]. (Note that Mac Lane used the name "supercomma"and the symbol $\downarrow$ in place of the semicolon.) For a concrete category $\underline{\underline{D}}$, let $(\underline{\underline{\mathrm{Sl}}}; \underline{\underline{D}}^{op})'$ denote the full subcategory of $(\underline{\underline{\mathrm{Sl}}}; \underline{\underline{D}}^{op})$ comprising the functor $\varnothing \to \underline{\underline{D}}^{op}$ and functors from non-empty semilattices to the full subcategory of $\underline{\underline{D}}^{op}$ consisting of non-empty $\underline{\underline{D}}$-objects. Then the equivalence (3.11) may be extended to the equivalence

$$\Lambda : (\underline{\underline{\mathrm{Sl}}}; \underline{\underline{\mathrm{Set}}}^{op})' \rightleftarrows \widetilde{\underline{\underline{\mathrm{Lz}}}} : \Gamma. \tag{3.14}$$

Consider a left normal band morphism $F : E \to E'$, with semilattice replica $f : H \to H'$. Define $R = E\Gamma : H \to \underline{\underline{\mathrm{Set}}}$ and $R' = E'\Gamma : H' \to \underline{\underline{\mathrm{Set}}}$. A natural transformation $\varphi : R \dot{\to} fR'$ is defined by its components

$$\varphi_h : hR \to hfR'; \ x \mapsto xF \tag{3.15}$$

at objects h of H. The $(\underline{\underline{\mathrm{Sl}}}; \underline{\underline{\mathrm{Set}}}^{op})'$-morphism $F\Gamma : E\Gamma \to E'\Gamma$ is then defined as the pair (φ, f). Conversely, given such a pair as an $(\underline{\underline{\mathrm{Sl}}}; \underline{\underline{\mathrm{Set}}}^{op})'$-morphism, a left normal band morphism $f = (\varphi, f)\Gamma : R\Gamma \to R'\Gamma$ is defined as the disjoint union of the components (3.15).

The general equivalence (3.14) may be specialized in various ways. Of particular interest in the current context is the case of a strongly irregular variety $\mathfrak{V}$ of B-algebras, considered as a category with homomorphisms as morphisms. Such a variety $\mathfrak{V}$ is said to be *strongly irregular* if there is a left zero band operation $*$ derived from the basic B^+-operations in $\mathfrak{V}$. For example, the variety of groups (as usually presented with multiplication, inversion and unit) is strongly irregular by virtue of

$$x * y = (xy)y^{-1}. \tag{3.16}$$

Then Płonka's Theorem describing regularizations of strongly irregular varieties may be formulated as follows:

THEOREM 3.3. *Let $\tau : B \to \mathbb{N}$ be a plural type, and let $\mathfrak{V}$ be a strongly irregular variety of B-algebras.*

(a) *Suppose that B_0 is empty. Then the equivalence* (3.14) *specializes to an equivalence*

$$\Lambda : (\underline{\underline{\mathrm{Sl}}}; \mathfrak{V}^{op})' \rightleftarrows \widetilde{\mathfrak{V}} : \Gamma. \tag{3.17}$$

(b) *Suppose that B_0 is non-empty. Then the equivalence* (3.14) *specializes to an equivalence*

$$\Lambda_0 : (\underline{\underline{\mathrm{Sl}}}_0; \mathfrak{V}^{op}) \rightleftarrows \widetilde{\mathfrak{V}}_0 : \Gamma_0. \tag{3.18}$$

PROOF. (a) Cf. [20], [21], [25, 7.1], [31, 239]. (b) Cf. [22], [25, 11.1(b)]. Consider a semilattice representation $R : H \to \mathfrak{V}$. For an operation ω in B^+, the operation ω on the bundle $E = R\Lambda$ is defined by

$$\ldots x_i \ldots \omega = \ldots x_i(\ldots x_i^\pi \ldots \omega \to x_i^\pi)^R \ldots \omega. \tag{3.19}$$

For ω in B_0, the constant ω in the bundle $E = R\Lambda = R\Lambda_0$ is defined to be the constant ω in the B-algebra representing the unit of the monoid H. ■

The functors Λ of (3.17) and Λ_0 of (3.18) are known as *Płonka sums*, Λ_0 "*with constants*".

Płonka's Theorem 3.3 describes algebras from $\widetilde{\mathfrak{V}}$ and $\widetilde{\mathfrak{V}}_0$ as Płonka sums. For non-empty B_0, there are analogous descriptions of algebras from $\widetilde{\mathfrak{V}}_1$ and $\widetilde{\mathfrak{V}}_\mu$. However, the representations of the semilattice replicas are not in the category $\mathfrak{V}$, but in a more complicated category ${}^2\mathfrak{V}$ obtained from $\mathfrak{V}$. In fact, this more complicated category is itself a sort of Płonka sum of categories over the two-element semilattice. First note the forgetful functor

$$U : \mathfrak{V} \to \mathfrak{V}_+ \tag{3.20}$$

obtained by taking B^+-reducts. The class of objects of ${}^2\mathfrak{V}$ is the disjoint union

$$Ob({}^2\mathfrak{V}) = Ob(\mathfrak{V}_+) \,\dot{\cup}\, Ob(\mathfrak{V}). \tag{3.21}$$

The morphisms of ${}^2\mathfrak{V}$ are either morphisms of $\mathfrak{V}$ or $\mathfrak{V}_+$, or else of the form

$$f : X \to Y \tag{3.22}$$

with $X \in Ob(\mathfrak{V}_+)$, $Y \in Ob(\mathfrak{V})$, $f \in \mathfrak{V}_+(X, YU)$. Then for $g \in \mathfrak{V}(Y, Z)$, one has $fg \in {}^2\mathfrak{V}(X, Z)$ with $fg^U \in \mathfrak{V}_+(X, ZU)$. For $h \in \mathfrak{V}_+(W, X)$, one has $hf \in {}^2\mathfrak{V}(W, Y)$ with $hf \in \mathfrak{V}_+(W, YU)$. Define

$$(\underline{\underline{\mathrm{Sl}}}_\mu;\ {}^2\mathfrak{V}^{op})_\mu \tag{3.23}$$

to be the full subcategory of $(\underline{\underline{\mathrm{Sl}}}_\mu;\ {}^2\mathfrak{V}^{op})$ consisting of functors $R : H \to {}^2\mathfrak{V}^{op}$ with restrictions $R : (\downarrow \mu) \to \mathfrak{V}$ and $R : (H - \downarrow \mu) \to \mathfrak{V}_+$. Objects in the image of the latter restriction are required to be non-empty. Define $(\underline{\underline{\mathrm{Sl}}}_1,\ {}^2\mathfrak{V}^{op})_1$ as the full subcategory of $(\underline{\underline{\mathrm{Sl}}}_\mu;\ {}^2\mathfrak{V}^{op})_\mu$ consisting of functors whose domain semilattices have initial constant objects. Then Płonka's Theorem describing symmetrizations and symmetric regularizations of strongly irregular varieties may be formulated as follows:

THEOREM 3.4. *Let $\tau : B \to \mathbb{N}$ be a plural type with $\tau^{-1}\{0\}$ non-empty. Let $\mathfrak{V}$ be a strongly irregular variety of B-algebras. Then the equivalence* (3.14) *specializes to equivalences*

$$\Lambda_1 : (\underline{\underline{\mathrm{Sl}}}_1;\ {}^2\mathfrak{V}^{op})_1 \rightleftarrows \widetilde{\mathfrak{V}}_1 : \Gamma_1 \tag{3.24}$$

and

$$\Lambda_\mu : (\underline{\underline{\mathrm{Sl}}}_\mu;\ {}^2\mathfrak{V}^{op})_\mu \rightleftarrows \widetilde{\mathfrak{V}}_\mu : \Gamma_\mu. \tag{3.25}$$

PROOF. For (3.24), cf. [23], [25, 11.1(c)]. For (3.25), cf. [24], [25, 11.1(a)]. ∎

The functors Λ_1 of (3.24) and Λ_μ of (3.25) are known as *Plonka sums with constants.*

4. Semilattice-based Dualities

A *duality* (of concrete categories) is written as

$$D : \mathfrak{A} \rightleftarrows \mathfrak{X} : E \tag{4.1}$$

and understood as follows. Firstly, $\mathfrak{A}$ is a concrete category of objects called *algebras* (e.g. a variety of algebras of type $\tau : B \to \mathbb{N}$), while $\mathfrak{X}$ is a concrete category of objects called *representation spaces for $\mathfrak{A}$-algebras.* There are (covariant) functors $D : \mathfrak{A} \to \mathfrak{X}^{op}$ and $E : \mathfrak{X} \to \mathfrak{A}^{op}$ furnishing an (*adjoint*) *equivalence* between $\mathfrak{A}$ and $\mathfrak{X}^{op}$ (as in [16, Th. IV. 4.1]). Thus there are natural isomorphisms $\varepsilon : 1_{\mathfrak{A}} \dot{\to} DE$ and $\eta : 1_{\mathfrak{X}} \dot{\to} ED$. For example, in the duality (2.3) for semilattices, these natural isomorphisms are given by (2.9) and (2.10) respectively.

Suppose given a duality (of concrete categories)

$$D : \mathfrak{V} \rightleftarrows \mathfrak{Y} : E \tag{4.2}$$

for a variety $\mathfrak{V}$ of B-algebras with B_0 non-empty. Let the natural isomorphisms for the duality (4.2) be $\varepsilon : 1_{\mathfrak{V}} \dot{\to} DE$ and $\eta : 1_{\mathfrak{Y}} \dot{\to} ED$. Further, suppose given a duality (of concrete categories)

$$D_+ : \mathfrak{V}_+ \rightleftarrows \mathfrak{Y}_+ : E_+ \tag{4.3}$$

for the reduction $\mathfrak{V}_+$ of $\mathfrak{V}$, with natural isomorphisms $\varepsilon^+ : 1_{\mathfrak{V}_+} \dot{\to} D_+E_+$ and $\eta^+ : 1_{\mathfrak{Y}_+} \dot{\to} E_+D_+$. Recall the forgetful functor $U : \mathfrak{V} \to \mathfrak{V}_+$ of (3.20). Then the dualities (4.2) and (4.3) are said to be *compatible* if there is a functor (preserving underlying sets)

$$Z : \mathfrak{Y} \to \mathfrak{Y}_+ \tag{4.4}$$

such that the following *compatibility conditions* are satisfied:

$$\begin{cases} (a) & DZ = UD_+; \\ (b) & EU = ZE_+; \\ (c) \text{ for } A \in Ob(\mathfrak{V}), & \varepsilon_A U = \varepsilon^+_{AU}; \\ (d) \text{ for } Y \in Ob(\mathfrak{Y}), & \eta_Y Z = \eta^+_{YZ}. \end{cases} \tag{4.5}$$

(Thus the pair (U, Z) is a "map of adjunctions" in the sense of [16, §IV. 7].) Define a dual Płonka sum $\mathfrak{Y}^2$ of $\mathfrak{Y}$ and $\mathfrak{Y}_+$ by

$$Ob(\mathfrak{Y}^2) = Ob(\mathfrak{Y}) \dot{\cup} Ob(\mathfrak{Y}_+), \tag{4.6}$$

such that morphisms of $\mathfrak{Y}^2$ are either morphisms of $\mathfrak{Y}$ or $\mathfrak{Y}_+$, or else of the form

$$f : Y \to Y^+ \tag{4.7}$$

with $Y \in Ob(\mathfrak{Y}), Y^+ \in Ob(\mathfrak{Y}_+), f \in \mathfrak{Y}_+(YZ, Y^+)$ (cf. (3.21) and (3.22)).

PROPOSITION 4.1.

(a) *The category* ${}^2\mathfrak{V}$ *is complete, and has directed colimits.*

(b) *The category* $\mathfrak{Y}^2$ *is cocomplete, and has directed limits.*

PROOF. It will be shown that ${}^2\mathfrak{V}$ is complete. Cocompleteness of $\mathfrak{Y}^2$ is proved dually. First, consider a pair (f, g) of parallel morphisms in ${}^2\mathfrak{V}$. If the pair lies in $\mathfrak{V}$, then its equalizer in ${}^2\mathfrak{V}$ is its equalizer in $\mathfrak{V}$. If the pair lies in $\mathfrak{V}_+$, then its equalizer in ${}^2\mathfrak{V}$ is its equalizer in $\mathfrak{V}_+$. Suppose the pair has domain A^+ in $\mathfrak{V}_+$ and codomain A in $\mathfrak{V}$. Thus there is a parallel pair in $\mathfrak{V}_+(A^+, AU)$. The $\mathfrak{V}_+$-equalizer of this pair is then the ${}^2\mathfrak{V}$-equalizer of (f, g). Thus ${}^2\mathfrak{V}$ has equalizers.

Next, consider a set $(A_i | i \in I)$ of ${}^2\mathfrak{V}$-objects, partitioned so that $I = I_0 \dot{\cup} I_1$ with set $(A_i | i \in I_0)$ of $\mathfrak{V}_+$-objects and set $(A_i | i \in I_1)$ of $\mathfrak{V}$-objects. Form the $\mathfrak{V}_+$-product $P = (\prod_{i \in I_0} A_i) \times (\prod_{i \in I_1} A_i U)$ with projections $p_i : P \to A_i$ for i in I_0 and $p_i : P \to A_i U$ for i in I_1. Then P becomes the product of $(A_i | i \in I)$ in ${}^2\mathfrak{V}$, equipped with projections $p_i : P \to A_i$ in ${}^2\mathfrak{V}$ corresponding to $p_i : P \to A_i U$ in $\mathfrak{V}_+$ for i in I_1. Since ${}^2\mathfrak{V}$ has equalizers and products, it follows [16, §V. 2] that ${}^2\mathfrak{V}$ is complete.

Now it will be shown that $\mathfrak{Y}^2$ has directed limits. The proof that ${}^2\mathfrak{V}$ has directed colimits is dual. Consider an (upwardly-)directed poset J, and a contravariant functor $R : J \to \mathfrak{Y}^2$. There are two cases to consider.

$$(a) \quad \exists k \in J. \quad kR \in Ob(\mathfrak{Y}).$$

In this case $\varprojlim(R : J \to \mathfrak{Y}^2) = \varprojlim(R : \uparrow k \to \mathfrak{Y})$. Indeed, any cone including a morphism to kR in $Ob(\mathfrak{Y})$ must have its vertex in $\mathfrak{Y}$, and then the universality property for $\varprojlim(R : \uparrow k \to \mathfrak{Y})$ in $\mathfrak{Y}$ yields the universality property for $\varprojlim(R : J \to \mathfrak{Y}^2)$ in $\mathfrak{Y}^2$.

$$(b) \quad \forall k \in J, \ kR \in Ob(\mathfrak{Y}_+).$$

In this case $\varprojlim(R : J \to \mathfrak{Y}^2) = \varprojlim(R : J \to \mathfrak{Y}_+)$. For a cone with vertex in $\mathfrak{Y}_+$, the universality property for $\varprojlim(R : J \to \mathfrak{Y}_+)$ in $\mathfrak{Y}_+$ yields the requisite universality property for $\varprojlim(R : J \to \mathfrak{Y}^2)$ in $\mathfrak{Y}^2$. For a cone with vertex Y in $\mathfrak{Y}$, the $\mathfrak{Y}^2$-morphisms $Y \to kR$ to the limiting cone correspond to $\mathfrak{Y}_+$-morphisms $YZ \to kR$. The universality property for $\varprojlim(R : J \to \mathfrak{Y}_+)$ in $\mathfrak{Y}_+$ yields a unique $\mathfrak{Y}_+$-morphism $YZ \to \varprojlim R$, giving the requisite unique $\mathfrak{Y}^2$-morphism $Y \to \varprojlim R$. ∎

THEOREM 4.2. *Given compatible dualities* (4.2) *and* (4.3), *there is a duality* (*of concrete categories*)

$$^2D : \ ^2\mathfrak{V} \rightleftarrows \mathfrak{Y}^2 : E^2. \tag{4.8}$$

PROOF. The dual of the non-trivial $^2\mathfrak{V}$-morphism $f : A^+ \to A$ corresponding to the $\mathfrak{V}_+$-morphism $f : A^+ \to AU$ is the non-trivial $\mathfrak{Y}^2$-morphism $f^2D : AD \to A^+D_+$ corresponding to the $\mathfrak{Y}_+$-morphism $fD_+ : AUD_+ \to A^+D_+$. Compatibility condition (4.5) (a) guarantees that this $\mathfrak{Y}_+$-morphism has the correct domain. The functor ^2D acts as D on Mor $(\mathfrak{V})$ and as D_+ on Mor $(\mathfrak{V}_+)$. The functor E^2 is defined dually. In particular, it sends a non-trivial $\mathfrak{Y}^2$-morphism $f : Y \to Y^+$ to the non-trivial $^2\mathfrak{V}$-morphism $fE^2 : Y^+E_+ \to YE$ corresponding to the $\mathfrak{V}_+$-morphism $fE_+ : Y^+E_+ \to YZE_+$. Compatibility condition (4.5) (b) guarantees that this $\mathfrak{V}_+$-morphism has the correct codomain. The natural transformation $^2\varepsilon : 1_{2_\mathfrak{V}} \dot{\to} {}^2DE^2$ is defined by $^2\varepsilon_A = \varepsilon_A$ for $A \in Ob(\mathfrak{V})$ and $^2\varepsilon_A = \varepsilon^+_A$ for $A \in Ob(\mathfrak{V}_+)$. The natural transformation $\eta^2 : 1_{\mathfrak{Y}_2} \dot{\to} E^2\ {}^2D$ is defined similarly by $\eta^2_Y = \eta_Y$ for $y \in Ob(\mathfrak{Y})$ and $\eta^2_y = \eta^+_y$ for $Y \in Ob(\mathfrak{Y}_+)$. The naturality of $^2\varepsilon$ on morphisms entirely within $\mathfrak{V}$ or $\mathfrak{V}_+$ is immediate. Consider a non-trivial $^2\mathfrak{V}$-morphism $f : A^+ \to A$ corresponding to the $\mathfrak{V}_+$-morphism $f : A^+ \to AU$. Then $^2\varepsilon$ is natural at this $^2\mathfrak{V}$-morphism f since $f^2\varepsilon_A = f\varepsilon_A = f\varepsilon^U_A = f\varepsilon^+_{AU} = \varepsilon^+_{A+} f^{D_+E_+} =^2 \varepsilon_{A+} f^{2DE^2}$. The first and last equalities here follow by definition of $^2\varepsilon$, ^2D, and E^2. The second equality follows by composition in $^2\mathfrak{V}$, the third by compatibility condition (4.5)(c), and the fourth by naturality of ε^+ at the $\mathfrak{V}_+$-morphism $f : A^+ \to AU$. The naturality of η^2 is obtained dually. ∎

EXAMPLE 4.3. The semilattice dualities (2.6) and (2.3) are compatible. The functor $U : \underline{\underline{\mathrm{Sl}}}_\mu \to \underline{\underline{\mathrm{Sl}}}$ forgets the constant, while the functor $Z : \underline{M} \to \mathfrak{B}$ forgets the compact constant.

Now suppose that a duality of the form (4.1) is given, with complete $\mathfrak{A}$ having directed colimits, so that $\mathfrak{X}$ is cocomplete and has directed limits. Such a duality may come from (4.2) with B_0 empty, or from (4.8) of Theorem 4.2. Set $\widetilde{\mathfrak{A}} = (\underline{\underline{\mathrm{Sl}}};\ \mathfrak{A}^{op})$. For a $\mathfrak{B}$-space G, a representation (contravariant functor) $R : G \to \mathfrak{X}$ is said to be *$\mathfrak{B}$-continuous* if

$$gR = \varprojlim(R : GK \cap \downarrow g \to \mathfrak{X}) \tag{4.9}$$

for each element g of G. (The limit on the right hand side of (4.9) is the limit of the restriction of R to the upwardly directed ordered subset of G consisting of compact elements below g.) The category $\widetilde{\mathfrak{X}}$ is defined to be the full subcategory of $(\mathfrak{B};\ \mathfrak{X}^{op})$ consisting of $\mathfrak{B}$-continuous representations of $\mathfrak{B}$-spaces in $\mathfrak{X}$. By virtue of (3.14), representations R in $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{X}}$ may be identified with the (total spaces of the) corresponding bundles $R\Lambda$. Define a contravariant functor $\widetilde{D} : \widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{X}}$ by

$$(R : H \to \mathfrak{A})\widetilde{D} = (HC \to \mathfrak{X};\ \theta \mapsto [\varinjlim(R : \theta^{-1}\{1\} \to \mathfrak{A})]D). \tag{4.10}$$

To see that $R\widetilde{D}$ is $\mathfrak{B}$-continuous, it is convenient to identify characters θ of H with the walls $\theta^{-1}\{1\}$ that they determine, according to (2.8). Under the identification, HCK is the set of principal walls (Proposition 2.3). For an element h of H, with corresponding principal wall [h], one has $[h]\ R^{\widetilde{D}} = \left[\varinjlim(R : [h] \to \mathfrak{A})\right] D = hRD$. Then for an arbitrary wall Θ of H, it follows that $\varprojlim(R^{\widetilde{D}} : HCK \cap \downarrow \Theta \to \mathfrak{X}) = \varprojlim(R^{\widetilde{D}} : \{[h] | h \in \Theta\} \to \mathfrak{X}) = \varprojlim(RD : \Theta \to \mathfrak{X}) = [\varinjlim(R : \Theta \to \mathfrak{A})]D = \Theta R^{\widetilde{D}}$, as required. The penultimate equality holds since the covariant functor $D : \mathfrak{A} \to \mathfrak{X}^{op}$, having $E : \mathfrak{X}^{op} \to \mathfrak{A}$ as a right adjoint, preserves colimits [16, §V. 5]. The definition of $\widetilde{E} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{A}}$ is quite direct. Recall the natural isomorphism $\kappa_G : GF \to GK$ between the meet semilattice GF dual to the $\mathfrak{B}$-space G and the join semilattice GK of compact elements of G (Proposition 2.5). The composite of κ_G with the order-preserving embedding $j : GK \to G$ of the poset GK in G gives a contravariant functor $\kappa_G j : GF \to G$. The definition of $\widetilde{E}$ is then given by

$$(R : G \to \mathfrak{X})\widetilde{E} = (\kappa_G\, jRE : GF \to \mathfrak{A}). \tag{4.11}$$

The right hand side of (4.11), as the composite of three contravariant functors, is contravariant. It thus forms a representation of the semilattice GF in $\mathfrak{A}$, determining an $\widetilde{\mathfrak{A}}$-algebra.

THEOREM 4.4. *Suppose given a duality* (4.1) *between a complete concrete category $\mathfrak{A}$ with directed colimits and a concrete category $\mathfrak{X}$ of representation* *spaces. Then the functors* $\widetilde{D}$ *of* (4.10) *and* $\widetilde{E}$ *of* (4.11) *yield a duality*

$$\widetilde{D} : \widetilde{\mathfrak{A}} \rightleftarrows \widetilde{\mathfrak{X}} : \widetilde{E} \tag{4.12}$$

between the category $\widetilde{\mathfrak{A}} = (\underline{\underline{\mathrm{Sl}}}; \mathfrak{A}^{op})$ *and the full subcategory* $\widetilde{\mathfrak{X}}$ *of* $(\mathfrak{B}; \mathfrak{X}^{op})$ *consisting of* $\mathfrak{B}$*-continuous representations.*

PROOF. Cf. [34, Th. 4.3]. Cocompleteness of $\mathfrak{A}$ was assumed there, but the existence of directed colimits suffices. ■

Theorem 4.4 uses the semilattice duality (2.3). An analogous result may be formulated and proved on the basis of the semilattice monoid duality (2.4).

THEOREM 4.5. *Under the assumptions of Theorem* 4.4*, there is a duality*

$$\widetilde{D}_0 : \widetilde{\mathfrak{A}}_0 \rightleftarrows \widetilde{\mathfrak{X}}_0 : \widetilde{E}_0 \tag{4.13}$$

between the category $\widetilde{\mathfrak{A}}_0 = (\underline{\underline{\mathrm{Sl}}}_0; \mathfrak{A}^{op})$ *and the full subcategory* $\widetilde{\mathfrak{X}}_0$ *of* $(\underline{Z}; \mathfrak{X}^{op})$ *consisting of continuous representations.*

Płonka's Theorems 3.3 and 3.4 may now be combined with Theorems 4.4 and 4.5 to yield dualities for the regularizations and symmetrization of a strongly irregular variety.

THEOREM 4.6. *Let* $\tau : B \to \mathbb{N}$ *be a plural type, and let* $\mathfrak{V}$ *be a strongly irregular variety of* B*-algebras, equipped with a duality of concrete categories*

$$D : \mathfrak{V} \rightleftarrows \mathfrak{Y} : E. \tag{4.14}$$

(a) *Suppose that* B_0 *is empty. Then there is a duality*

$$\widetilde{D} : \widetilde{\mathfrak{V}} \rightleftarrows \widetilde{\mathfrak{Y}} : \widetilde{E} \tag{4.15}$$

for the regularization $\widetilde{\mathfrak{V}}$ *of* $\mathfrak{V}$*.*

(b) *Suppose that* B_0 *is non-empty. Then there is a duality*

$$\widetilde{D}_0 : \widetilde{\mathfrak{V}}_0 \rightleftarrows \widetilde{\mathfrak{Y}}_0 : \widetilde{E}_0 \tag{4.16}$$

for the regularization $\widetilde{\mathfrak{V}}_0$ *of* $\mathfrak{V}$*. Suppose further that there is a duality*

$$D_+ : \mathfrak{V}_+ \rightleftarrows \mathfrak{Y}_+ : E_+ \tag{4.17}$$

for the reduction $\mathfrak{V}_+$ *of* $\mathfrak{V}$ *compatible with the duality* (4.14). *Then there are dualities*

$$(4.18) \qquad \widetilde{D}_1 : \widetilde{\mathfrak{V}}_1 \rightleftarrows \widetilde{\mathfrak{Y}}_1 : \widetilde{E}_1$$

for the symmetrization $\widetilde{\mathfrak{V}}_1$ *of* $\mathfrak{V}$ *and*

$$(4.19) \qquad \widetilde{D}_\mu : \widetilde{\mathfrak{V}}_\mu \rightleftarrows \widetilde{\mathfrak{Y}}_\mu : \widetilde{E}_\mu$$

for the symmetric regularization $\widetilde{\mathfrak{V}}_\mu$ *of* $\mathfrak{V}$.

PROOF. (a) Applying Theorem 4.4 with (4.14) in place of (4.1) yields a duality

$$(4.20) \qquad \widetilde{D} : (\underline{\underline{\mathrm{Sl}}}; \mathfrak{V}^{op}) \rightleftarrows \widetilde{\mathfrak{X}} : \widetilde{E}$$

between $(\underline{\underline{\mathrm{Sl}}}; \mathfrak{V}^{op})$ and the full subcategory $\widetilde{\mathfrak{X}}$ of $(\mathfrak{B}; \mathfrak{Y}^{op})$ consisting of $\mathfrak{B}$-continuous representations. This duality restricts to

$$(4.21) \qquad \widetilde{D} : (\underline{\underline{\mathrm{Sl}}}; \mathfrak{V}^{op})' \rightleftarrows \widetilde{\mathfrak{Y}} : \widetilde{E}$$

for a suitable subcategory $\widetilde{\mathfrak{Y}}$ of $\widetilde{\mathfrak{X}}$. By the equivalence (3.17) of Theorem 3.3, the category on the left hand side of (4.21) may be replaced with $\widetilde{\mathfrak{V}}$, yielding (4.15).
(b) Applying Theorem 4.5 with (4.14) in place of (4.1) yields (4.16). Applying Theorem 4.4 to the duality (4.8) of Theorem 4.2 yields a duality

$$(4.22) \qquad {}^2\widetilde{D} : (\underline{\underline{\mathrm{Sl}}};\ {}^2\mathfrak{V}^{op}) \rightleftarrows \widetilde{\mathfrak{X}} : \widetilde{E}^2$$

for the full subcategory $\widetilde{\mathfrak{X}}$ of $(\mathfrak{B};\ \mathfrak{Y}^{2\ op})$ consisting of continuous representations. By (3.25), the symmetric regularization $\widetilde{\mathfrak{V}}_\mu$ is equivalent to the category $(\underline{\underline{\mathrm{Sl}}}_\mu;\ {}^2\mathfrak{V}^{op})_\mu$ of (3.23). This category in turn has an equivalence

$$(4.23) \qquad L : (\underline{\underline{\mathrm{Sl}}}_\mu;\ {}^2\mathfrak{V}^{op})_\mu \rightleftarrows \mathfrak{S} : J$$

with a subcategory $\mathfrak{S}$ of $(\underline{\underline{\mathrm{Sl}}};\ {}^2\mathfrak{V}^{op})$. The functor L forgets ("Loses") the constant μ in the domain semilattice of a representation. The functor J recovers the constant in the domain of suitable representations (viz. those in $\mathfrak{S}$) as the least upper bound ("Join") of the semilattice elements represented by ${}^2\mathfrak{V}$-objects in $\mathfrak{V}$. Define a functor

$$(4.24) \qquad \widetilde{D}_\mu : \widetilde{\mathfrak{V}}^\mu \to \widetilde{\mathfrak{X}}$$

as the composite $\widetilde{D}_\mu = \Gamma_\mu L\ {}^2\widetilde{D}$ of the functors of (3.25), (4.23), and (4.22). Let $\widetilde{\mathfrak{Y}}_\mu$ be the image of $\widetilde{D}_\mu$ in $\widetilde{\mathfrak{X}}$. Let $\widetilde{E}_\mu$ be the restriction of $\widetilde{E}^2$ to $\widetilde{\mathfrak{Y}}_\mu$. This yields the duality (4.19). Finally, (4.18) is obtained from (4.19) by restriction. ∎

5. First applications

The current and subsequent sections discuss some typical applications of the general Theorem 4.6. In these applications, the category $\mathfrak{V}$ of (4.14) is a strongly irregular variety of finitary B-algebras with nullary operations. The category $\widetilde{\mathfrak{V}}_0$ is the regularization of $\mathfrak{V}$, the category $\widetilde{\mathfrak{V}}_1$ is the symmetrization of $\mathfrak{V}$, and finally $\widetilde{\mathfrak{V}}_\mu$ is the symmetric regularization of $\mathfrak{V}$. The duality (4.14) arises from a schizophrenic object T, appearing simultaneously as an object $\underline{T}$ of $\mathfrak{V}$ and as an object $\underset{\sim}{T}$ of $\mathfrak{Y}$, in such a way that there are natural isomorphisms

$$\lambda : D \dot{\to} \mathfrak{V}(\quad, \underline{T}) \tag{5.1}$$

and

$$\rho : E \dot{\to} \mathfrak{Y}(\quad, \underset{\sim}{T}). \tag{5.2}$$

Moreover, the underlying sets of $\underline{T}$ and $\underset{\sim}{T}$ coincide with T. The category $\mathfrak{V}$ is the closure $ISP\{\underline{T}\}$ of the singleton class $\{\underline{T}\}$ under the closure operations P of power, S of subalgebra, and I of isomorphic copy. The representation space $\underset{\sim}{T}$ is a compact Hausdorff structure with closed relations. These relations on $\underset{\sim}{T}$ are subalgebras of powers of T with respect to the algebra structure on T given by $\underline{T}$. Moreover, operations $\omega : T \to T$ of $\underline{T}$ are continuous with respect to the topology of $\underset{\sim}{T}$. The category $\mathfrak{Y}$ is the closure $ISP\{\underset{\sim}{T}\}$ under the closure operations P of power, S of (closed) substructure, and I of isomorphic (homeomorphic) copy. (Cf. [5].) By Theorem 4.6, there are dualities

$$\widetilde{D}_0 : \widetilde{\mathfrak{V}}_0 \rightleftarrows \widetilde{\mathfrak{Y}}_0 : \widetilde{E}_0, \tag{4.16}$$

$$\widetilde{D}_1 : \widetilde{\mathfrak{V}}_1 \rightleftarrows \widetilde{\mathfrak{Y}}_1 : \widetilde{E}_1, \tag{4.18}$$

$$\widetilde{D}_\mu : \widetilde{\mathfrak{V}}_\mu \rightleftarrows \widetilde{\mathfrak{Y}}_\mu : \widetilde{E}_\mu \tag{4.19}$$

for algebras from the regularization $\widetilde{\mathfrak{V}}_0$, the symmetrization $\widetilde{\mathfrak{V}}_1$, and the symmetric regularization $\widetilde{\mathfrak{V}}_\mu$, respectively. Similarly as in (4.14), the dualities (4.15) $\widetilde{D} : \widetilde{\mathfrak{V}} \rightleftarrows \widetilde{\mathfrak{Y}} : \widetilde{E}$ and (4.16) arise from schizophrenic objects. Given a schiophrenic object T for the duality (4.14), there is a representation J of the two-element semilattice 2 in $\mathfrak{V}$, with $1J = T$ and $0J = \infty$, the terminal object of $\mathfrak{V}$ or $\mathfrak{Y}$, that may be interpreted either as an object $\underline{J}$ or $\underline{T}^\infty$ representing $\underline{2}$ in $\mathfrak{V}_1$ or as an object $\underset{\sim}{Y}$ or $\underset{\sim}{T}^\infty$ representing $\underset{\sim}{2}$ in $\mathfrak{Y}$. There is a similar representation of the two-element semilattice monoid 2_0 in $\mathfrak{V}_0$, that may be interpreted either as an object $\underline{T}_0^\infty$ representing $\underline{2}_0$ in $\mathfrak{V}_0$, or as an object $\underset{\sim}{T}_0^\infty$ representing $\underset{\sim}{2}_0$ in $\mathfrak{Y}_0$.

Proposition 5.1. *If T is a schizophrenic object for the duality* (4.14), *then T^∞ is a schizophrenic object for the duality* (4.15) *and T_0^∞ is a schizophrenic object for the duality* (4.16).

Proof. For the first part, see [34, Th. 6.1]. The second part follows similarly. ∎

As in the case of the duality (2.6) for semilattices, the duality (4.19) does not seem to arise from a schizophrenic object.

Before discussing our first examples, let us note that if a basis for the identities holding in the variety of $\mathfrak{V}$ is known, then it is very easy to find a basis for each of the varieties $\widetilde{\mathfrak{V}}, \widetilde{\mathfrak{V}}_0, \widetilde{\mathfrak{V}}_1$, and $\widetilde{\mathfrak{V}}_\mu$. Since $\mathfrak{V}$ is a strongly irregular variety, there is a basis for the identities satisfied in $\mathfrak{V}$ consisting of a set of symmetrically regular identities and a unique irregular identity of the form

$$x * y = x \tag{5.3}$$

as discussed in Section 3. (See [18], [29].) The bases for the varieties $\widetilde{\mathfrak{V}}, \widetilde{\mathfrak{V}}_0, \widetilde{\mathfrak{V}}_1$ and $\widetilde{\mathfrak{V}}_\mu$ can be obtained by means of the following theorem.

Theorem 5.2. *Let $\tau : B \to \mathbb{N}$ be a plural type, and let $\mathfrak{V}$ be a strongly irregular variety of B-algebras defined by a set Σ of symmetrically regular identities and the identity* (5.3). *Then the* (*strict*) *regularization, symmetrization and symmetrical regularization of $\mathfrak{V}$ may be defined by the following sets of identities, respectively:*

(a) *If B_0 is empty, then $\widetilde{\mathfrak{V}}$ is defined by the identities Σ, the identities defining $*$ as a left normal band operation, and for each ω in B the identities*

$$x_1 \dots x_{\omega\tau}\omega * y = (x_1 * y) \dots (x_{\omega\tau} * y)\omega, \tag{5.4}$$

$$y * (x_1 \dots x_{\omega\tau}\omega) = y * x_1 * \dots * x_{\omega\tau}. \tag{5.5}$$

(b) *If B_0 is non-empty, then $\widetilde{\mathfrak{V}}_\mu$ is defined by the identities as for $\widetilde{\mathfrak{V}}$ and the identity*

$$\omega_0 * \omega_0^1 = \omega_0 \tag{5.6}$$

for all ω_0, ω_0^1 in B_0.

(c) *If B_0 is non-empty, then $\widetilde{\mathfrak{V}}_0$ is defined by the identities as for $\widetilde{\mathfrak{V}}_\mu$ and the identity*

$$x * \omega_0 = x \tag{5.7}$$

for each ω_0 in B_0, and $\widetilde{\mathfrak{V}}_1$ is defined by the identities as for $\widetilde{\mathfrak{V}}_\mu$ and the identity

$$\omega_0 * x = \omega_0 \tag{5.8}$$

for each ω_0 in B_0.

PROOF. See [20], [21], [29], [30], [22], [23], [24], [25]. ∎

EXAMPLE 5.3. (REGULARIZED PRIESTLEY DUALITY) The variety $\underline{\underline{\mathrm{Dl}}}$ of distributive lattices may be defined by the set of symmetrically regular (indeed regular) identities axiomatizing $+$ and $\cdot$ as semilattice operations and connecting them by the distributive laws, and moreover one strongly irregular identity

$$x * y := x + xy = x \tag{5.9}$$

of absorption. As shown in [19], the set of identities from Theorem 5.2 defining the variety $\underline{\underline{\mathrm{Dd}}}$ of doubly distributive dissemilattices (cf. [31, 109], [19], [25]), that is the regularization of the variety $\underline{\underline{\mathrm{Dl}}}$, may be reduced to the regular ones among those defining $\underline{\underline{\mathrm{Dl}}}$. The algebras in $\underline{\underline{\mathrm{Dd}}}$ are also called distributive quasilattices or bisemilattices. (Cf. [19], [32]). The variety $\underline{\underline{\mathrm{Dlz}}}$ of distributive lattices with a lower bound 0 is axiomatized by the identities defining $\underline{\underline{\mathrm{Dl}}}$ together with the symmetrically regular identity

$$x + 0 = x + x0. \tag{5.10}$$

Finally, the variety $\underline{\underline{\mathrm{Dlb}}}$ of bounded distributive lattices with a lower bound 0 and an upper bound 1 is axiomatized by the identities defining $\underline{\underline{\mathrm{Dlz}}}$ and the symmetrically regular identity

$$x \cdot 1 = x(x + 1). \tag{5.11}$$

By Theorem 5.2, it is easy to see that the symmetrically regular identities from those defining $\underline{\underline{\mathrm{Dlz}}}$ describe the regularization $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_0$ of $\underline{\underline{\mathrm{Dlz}}}$. And similarly, the symmetrically regular identities from those defining $\underline{\underline{\mathrm{Dlb}}}$ describe the regularization $\widetilde{\underline{\underline{\mathrm{Dlb}}}}_0$ of $\underline{\underline{\mathrm{Dlb}}}$. Note that the dissemilattices in $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_0$ are those that have recently found applications in the theory of program semantics [15], [28]. They are called "snack algebras" in [15], and are used there to describe the structure of certain types of so called "edible" power domains.

For each of the three varieties of distributive lattices $\underline{\underline{\mathrm{Dl}}}$, $\underline{\underline{\mathrm{Dlz}}}$ and $\underline{\underline{\mathrm{Dlb}}}$ considered above, there is a version of Priestley duality. Each one arises from a schizophrenic object that is the two-element lattice 2 without or with suitable constants. The functor $\widetilde{D}$ or $\widetilde{D}_0$ of (4.15) or (4.16) is naturally isomorphic to the functor given by homming to the schizophrenic object considered as a lattice. Similarly, the functor $\widetilde{E}$ or $\widetilde{E}_0$ of (4.15) or (4.16) is naturally isomorphic to the functor given by homming to the same schizophrenic object considered as a corresponding ordered space. In particular, one has the following dualities

(5.12) $$\underline{\underline{\mathrm{Dl}}}(\ ,\underline{2}) : \underline{\underline{\mathrm{Dl}}} \rightleftarrows \underline{\underline{\mathrm{OSb}}} : \underline{\underline{\mathrm{OSb}}}(\ ,\underset{\sim}{2}_{0,1}),$$

(5.13) $$\underline{\underline{\mathrm{Dlz}}}(\ ,\underline{2}_0) : \underline{\underline{\mathrm{Dlz}}} \rightleftarrows \underline{\underline{\mathrm{OSz}}} : \underline{\underline{\mathrm{OSz}}}(\ ,\underset{\sim}{2}_0),$$

(5.14) $$\underline{\underline{\mathrm{Dlb}}}(\ ,\underline{2}_{0,1}) : \underline{\underline{\mathrm{Dlb}}} \rightleftarrows \underline{\underline{\mathrm{OS}}} : \underline{\underline{\mathrm{OS}}}(\ ,\underset{\sim}{2})$$

between distributive lattices and bounded ordered Stone (or bounded Priestley) spaces, between distributive lattices with a lower bound and ordered Stone (or Priestley) spaces with lower bound, and finally between bounded distributive lattices and ordered Stone (or Priestley) spaces. (Cf. [6]) The schizophrenic object of (5.12) is the two-element lattice $\underline{2} = (\{0,1\},+,\cdot)$ or the order space $\underset{\sim}{2}_{0,1} = (\{0,1\},\leq,0,1,\mathcal{T})$. The schizophrenic object of (5.13) is the two element lattice $\underline{2}_0 = (\{0,1\},+,\cdot,0)$ or the ordered space $\underset{\sim}{2}_0 = (\{0,1\},\leq,0,\mathcal{T})$. Finally, the schizophrenic object of (5.14) is the lattice $\underline{2}_{0,1} = (\{0,1\},+,\cdot,0,1)$ or the ordered space $\underset{\sim}{2} = (\{0,1\},\leq,\mathcal{T})$.

The dualities (5.12), (5.13) and (5.14) may be regularized using Theorem 4.6. The duality for the variety $\underline{\underline{\mathrm{Dd}}} = \widetilde{\underline{\underline{\mathrm{Dl}}}}$ of (non-empty) distributive dissemilattices is thoroughly treated in [7]. It has the form

(5.15) $$\underline{\underline{\mathrm{Dd}}}(\ ,\underline{3}) : \underline{\underline{\mathrm{Dd}}} \rightleftarrows \underline{\underline{\mathrm{OSNB}}} : \underline{\underline{\mathrm{OSNB}}}(\ ,\underset{\sim}{3}),$$

where the category $\underline{\underline{\mathrm{OSNB}}}$ is axiomatized as the category of ordered Stone topological left normal bands with three constants c_0, c_1, c_α satisfying

(5.16) $$\begin{cases} x * c_\alpha = c_\alpha * x = c_\alpha, \\ x * c_0 = x * c_1 = x, \\ c_0 \sqsubseteq x \leq c_1,\ c_\alpha \leq x \sqsubseteq c_2, \\ c_0 * x = c_1 * x \Rightarrow x = c_\alpha, \end{cases}$$

where $\sqsubseteq$ is the ordering relation defined by

$$x \sqsubseteq y \Leftrightarrow x * y \leq y \text{ and } y * x = y. \tag{5.17}$$

The schizophrenic object $3 = 2^\infty$ is interpreted either as the dissemilattice $\underline{3} = (\{0,1,\alpha\},+,\cdot)$, the Płonka sum of a two-element lattice and a one-element lattice, or as the ordered Stone topological left normal band $\underset{\sim}{3}_{0,1} = (\{0,1,\alpha\},\leq,*,0,1,\alpha,\mathcal{T})$, the Płonka sum of a two-element ordered left zero band and a one-element left zero band, with all three elements serving as the constants. The constantless reducts of the schizophrenic objects of the regularizations of the dualities (5.13) and (5.14) are the same as the constantless reducts of $\underline{3}$ and $\underset{\sim}{3}_{0,1}$. Moreover $\underline{3}_0 = \underline{2}_0^\infty$ and $\underset{\sim}{3}_0 = \underset{\sim}{2}_0^\infty$ have one constant operation selecting the lower bound of the two-element lattice component. The object $\underline{3}_{0,1} = \underline{2}_{0,1}^\infty$ has two constant operations, selecting the upper and lower bounds of the two-element lattice component respectively, and the object $\underset{\sim}{3} = \underset{\sim}{2}^\infty$ has the set of these bounds as a (closed) unary relation, say τ. The category $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_0 = ISP(\underline{2}_0^\infty)$, the regularization of $\underline{\underline{\mathrm{Dlz}}}$, is equivalent to the category $(\underline{\underline{\mathrm{Sl}}}_0;\underline{\underline{\mathrm{Dlz}}}^{op})$ (cf. (3.18).) The category $\widetilde{\underline{\underline{\mathrm{OSz}}}}_0 = ISP(\underset{\sim}{2}_0^\infty)$ is the full subcategory of $(\underline{Z};\underline{\underline{\mathrm{OSz}}}^{op})$ consisting of continuous representations of $\underline{Z}$-semilattices in the category $\underline{\underline{\mathrm{OSz}}}$. It may be axiomatized as the category of ordered Stone topological left normal bands with one constant c_0 satisfying the conditions

$$x * c_0 = x \text{ and } c_0 * x \leq x. \tag{5.18}$$

The category $\widetilde{\underline{\underline{\mathrm{Dlb}}}}_0 = ISP(\underline{2}_{0,1}^\infty)$, the regularization of $\underline{\underline{\mathrm{Dlb}}}$, is equivalent to the category $(\underline{\underline{\mathrm{Sl}}}_0;\underline{\underline{\mathrm{Dlb}}}^{op})$ (cf. (3.18)). Let $\underline{\underline{\mathrm{Dlb}}}^*$ be the subcategory of $\underline{\underline{\mathrm{Dlb}}}$ consisting of non-trivial algebras. The category $(\underline{\underline{\mathrm{Sl}}}_0;\underline{\underline{\mathrm{Dlb}}}^{op})$ has a full subcategory $(\underline{\underline{\mathrm{Sl}}}_0;\underline{\underline{\mathrm{Dlb}}}^{*op})$ with $\widetilde{\underline{\underline{\mathrm{Dlb}}}}_0^* = (\underline{\underline{\mathrm{Sl}}}_0;\underline{\underline{\mathrm{Dlb}}}^{*op})\Lambda_0$.

The elements of $\widetilde{\underline{\underline{\mathrm{Dlb}}}}_0^*$ may be described as Płonka sums (with constant) of non-trivial bounded distributive lattices. The category $\widetilde{\underline{\underline{\mathrm{OS}}}}_0$ is the full subcategory of $(\underline{Z};\underline{\underline{\mathrm{OS}}}^{op})$ consisting of continuous representations of $\underline{Z}$-semilattices in the category $\underline{\underline{\mathrm{OS}}}$. This category cannot be axiomatized similarly as the category $\widetilde{\underline{\underline{\mathrm{OSz}}}}_0$. The reason is that $\widetilde{\underline{\underline{\mathrm{OS}}}}_0$ contains representations $R : G \to \underline{\underline{\mathrm{OS}}}$ with some gR empty. Note that if for g in G, $gR = \varnothing$ and $g \leq h$, then $hR = \varnothing$, too. In particular, the dual of each representation $R : H \to \underline{\underline{\mathrm{Dlb}}}$ with $hR = \infty$ for all h in H is the representation $R^{\widetilde{D}_0} : HW \to \underline{\underline{\mathrm{OS}}}$ with $\Theta R^{\widetilde{D}_0} = \varnothing$ for all Θ in HW. More generally, if there is a subordinate set $D \subseteq H$ such that for all h in $D, hR = \infty$, then $\uparrow hR^{\widetilde{D}_0} = \varnothing$ for all h in D. Let us call such representations *trivial.* Consider the subcategory $\underline{\underline{\widetilde{\mathrm{OSz}}}}_0^*$ of $\underline{\underline{\widetilde{\mathrm{OSz}}}}_0$ consisting of non-trivial representations in $\underline{\underline{\widetilde{\mathrm{OSz}}}}_0$. Then the duality

$$(5.18) \qquad \widetilde{D}_0 : \underline{\underline{\widetilde{\mathrm{Dlb}}}}_0 \rightleftarrows \underline{\underline{\widetilde{\mathrm{OS}}}}_0 : \widetilde{E}_0$$

restricts to the duality

$$(5.19) \qquad \widetilde{D}_0^* : \underline{\underline{\widetilde{\mathrm{Dlb}}}}_0^* \rightleftarrows \underline{\underline{\widetilde{\mathrm{OS}}}}_0^* : \widetilde{E}_0^*$$

between the category of Płonka sums of nontrivial bounded lattices and the category of non-trivial representations of $\underline{Z}$-semilattices in $\underline{\underline{\mathrm{OS}}}$. The category $\underline{\underline{\widetilde{\mathrm{OS}}}}_0^* = ISP(\underset{\sim}{2}^\infty)$ may now be axiomatized as the category of ordered Stone topological left normal bands having a closed unary relation τ, satisfying

$$(5.20) \qquad \forall x \in \tau,\ \forall y \in \tau,\ x = x * y \text{ and } y = y * x$$

and

$$(5.21) \qquad \forall x, \exists y. \exists z \in \tau.\ x = x * y,\ y = y * x \text{ and } y = z * y.$$

Axiom (5.20) says that τ is a class of the semilattice replica, while axiom (5.21) says that τ is the unique maximal class under the (meet-) semilattice ordering of the replica. These axioms guarantee that semilattice replicas of $\underline{\underline{\widetilde{\mathrm{OS}}}}_0^*$-morphisms are genuine $\underline{Z}$-morphisms.

Example 5.4. (Regularized Stone duality) Stone duality may be construed as the duality $\underline{\underline{\mathrm{Bool}}} \rightleftarrows \underline{\underline{\mathrm{Stone}}}$ between the variety of Boolean algebras and the category of Stone spaces. The category $\underline{\underline{\mathrm{Stone}}}$ is isomorphic with the category $\underline{\underline{\mathrm{SLz}}}$ of Stone topological left zero bands. The schizophrenic object for the duality

$$(5.22) \qquad D : \underline{\underline{\mathrm{Bool}}} \rightleftarrows \underline{\underline{\mathrm{SLz}}} : E$$

has two elements, and is interpreted as the two-element Boolean algebra $\underline{2}_b = (\{0,1\}, +, \cdot, ', 0, 1)$ or the two-element (topological) left-zero band $\underset{\sim}{2}_b = (\{0,1\}, *, \mathcal{T})$. So the schizophrenic object 2_b^∞ for the regularization of the duality (5.22) appears in $\underline{\underline{\widetilde{\mathrm{Bool}}}}_0$ as the three-element Płonka sum of the singleton ∞ and the Boolean algebra $\underline{2}_b$. As $\underset{\sim}{2}_b^\infty$, the schizophrenic object is a Stone topological normal band $(\{0,1,\alpha\}, *, \mathcal{T})$, the Płonka sum of the singleton ∞ and the left-zero band $\underset{\sim}{2}_b$, together with a constant operation c_α selecting ∞ and a unary relation τ consisting of the subset $\{0,1\}$. The variety $\underline{\underline{\widetilde{\mathrm{Bool}}}}_0$, the regularization of the variety $\underline{\underline{\mathrm{Bool}}}$, can be axiomatized by the identities defining the variety $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_0$ together with the following:

$$
(5.23) \qquad \begin{cases} (x+y)' = x'y', \\ x'' = x, \\ x.0 = xx', \\ 0' = 1. \end{cases}
$$

(See [22] and Theorem 5.2.) The category $\widetilde{\underline{\underline{\mathrm{Bool}}}}_0$ contains a full subcategory $\widetilde{\underline{\underline{\mathrm{Bool}}}}_0^*$ of Płonka sums of non-trivial Boolean algebras. The category $\widetilde{\underline{\underline{\mathrm{SLz}}}}_0$ is the full subcategory of $(\underline{Z}; \underline{\underline{\mathrm{SLz}}}^{op})$ consisting of continuous representations of $\underline{Z}$-semilattices in the category $\underline{\underline{\mathrm{SLz}}}$. Similarly as in the case of $\widetilde{\underline{\underline{\mathrm{OS}}}}_0$, the category $\widetilde{\underline{\underline{\mathrm{SLz}}}}_0$ also contains trivial representations. However, again similarly as in that case, the duality

$$
(5.24) \qquad \tilde{D}_0 : \widetilde{\underline{\underline{\mathrm{Bool}}}}_0 \rightleftarrows \widetilde{\underline{\underline{\mathrm{SLz}}}}_0 : \tilde{E}_0
$$

restricts to a duality

$$
(5.25) \qquad \tilde{D}_0^* : \widetilde{\underline{\underline{\mathrm{Bool}}}}_0^* \rightleftarrows \widetilde{\underline{\underline{\mathrm{SLz}}}}_0^* : \tilde{E}_0^*
$$

between the category of Płonka sums of non-trivial Boolean algebras and the category of non-trivial continuous representations of $\underline{Z}$-semilattices in $\underline{\underline{\mathrm{SLz}}}$. The category $\widetilde{\underline{\underline{\mathrm{SLz}}}}_0^* = ISP(\underset{\sim}{2}_0^{\infty})$ is axiomatized as the category of Stone topological left-normal bands with one constant operation c_α acting as zero and with one closed unary relation τ satisfying conditions (5.20) and (5.21).

Example 5.5. (Regularized Pontryagin duality) (See [34].) This is the regularization

$$
(5.26) \qquad \widetilde{\underline{\underline{\mathrm{Ab}}}}_0(\ \ ,\underline{T}^{\infty}) : \widetilde{\underline{\underline{\mathrm{Ab}}}}_0 \rightleftarrows \widetilde{\underline{\underline{\mathrm{CHAb}}}} : \widetilde{\underline{\underline{\mathrm{CHAb}}}}(\ \ ,\underset{\sim}{T}^{\infty})
$$

of the Pontryagin duality

$$
(5.27) \qquad \underline{\underline{\mathrm{Ab}}}(\ \ ,\underline{T}) : \underline{\underline{\mathrm{Ab}}} \rightleftarrows \underline{\underline{\mathrm{CHAb}}} : \underline{\underline{\mathrm{CHAb}}}(\ \ ,\underset{\sim}{T})
$$

between the variety $\underline{\underline{\mathrm{Ab}}}$ of abelian groups and the category of compact Hausdorff abelian groups. The circle group $T = \mathbb{R}/\mathbb{Z}$ is a schizophrenic object for the duality (5.27). The variety $\widetilde{\underline{\underline{\mathrm{Ab}}}}_0$ is the variety of commutative inverse semigroups with identity 0 satisfying

$$
(5.28) \qquad \begin{cases} -(-x) = x, \\ -(x+y) = (-x)+(-y), \\ x - x + x = x \\ x + 0 = x. \end{cases}
$$

It is the class of Płonka sums with constants of abelian groups [22], [25], and is equivalent to the category $(\underline{\underline{\mathrm{Sl}}}_0; \underline{\underline{\mathrm{Ab}}}^{op})$. The class $\underline{\underline{\widetilde{\mathrm{CHAb}}}}$ is axiomatized as the category of compact Hausdorff $\underline{\underline{\widetilde{\mathrm{Ab}}}}_0$-algebras. For an $\underline{\underline{\widetilde{\mathrm{Ab}}}}_0$-monoid G, the dual $\underline{\underline{\widetilde{\mathrm{Ab}}}}_0(G, \underline{T}^\infty)$ is the set $\widehat{G}^*$ of semicharacters of G in the notation of [9, Defn. 5.3].

In much the same way, one can find regularizations of dualities for the varieties $\underline{\underline{\mathrm{A}}}_m$ of abelian groups of exponent m and varieties $\underline{\underline{\mathrm{V}}}_q$ of vector spaces over the finite field $GF(q)$ (see [6], [5]).

6. Symmetrically regularized Priestley duality

In this section, the category $\mathfrak{V}$ of (4.14) is the variety $\underline{\underline{\mathrm{Dlz}}}$ of distributive lattices with a lower bound 0, and the category $\mathfrak{Y}$ of (4.14) is the category $\underline{\underline{\mathrm{OSz}}}$ of ordered Stone spaces with a lower bound 0. The category $\mathfrak{V}_+$ is the variety $\underline{\underline{\mathrm{Dlz}}}_+ = \underline{\underline{\mathrm{Dl}}}$ of distributive lattices, and the category $\mathfrak{Y}_+$ is the category $\underline{\underline{\mathrm{OSz}}}_+ = \underline{\underline{\mathrm{OSb}}}$ of bounded ordered Stone spaces. There is a duality

$$(6.1) \qquad D_+ = \underline{\underline{\mathrm{Dl}}}(\quad, \underline{2}) : \underline{\underline{\mathrm{Dl}}} \rightleftarrows \underline{\underline{\mathrm{OSb}}} : \underline{\underline{\mathrm{OSb}}}(\quad, \underset{\sim}{2}_{0,1}) = E_+,$$

(cf. 5.12) with natural isomorphisms $\varepsilon^+ : 1_{\underline{\underline{\mathrm{Dl}}}} \dot{\rightarrow} D_+E_+$ and $\mathfrak{Y}^+ : 1_{\underline{\underline{\mathrm{DSb}}}} \dot{\rightarrow} E_+D_+$. In order to achieve compatibility with (6.1), the duality (5.13) has to be modified slightly. The categories $\mathfrak{V}$ and $\mathfrak{Y}$ are as defined above. Instead of the functors $\underline{\underline{\mathrm{Dlz}}}(\quad, \underline{2}_0)$ and $\underline{\underline{\mathrm{OSz}}}(\quad, \underset{\sim}{2}_0)$, two other functors D and E are defined. To this end, let the "creation operator"

$$(6.2) \qquad Z_+ := Z : \underline{\underline{\mathrm{OSz}}} \to \underline{\underline{\mathrm{OSb}}}$$

be the functor that assigns, to each space X in $\underline{\underline{\mathrm{OSz}}}$, the space $X \dot{\bigcup} \{1\}$, where 1 is not in X, and for each x in $X, x < 1$. Morphisms between Z-images of members of $\underline{\underline{\mathrm{OSz}}}$ extend $\underline{\underline{\mathrm{OSz}}}$-morphisms by preserving 1. The category $\underline{\underline{\mathrm{OSz}}}$ becomes concrete by virtue of the composite $Z_+G : \underline{\underline{\mathrm{OSz}}} \to \underline{\underline{\mathrm{Set}}}$ of the creation operator $Z_+ : \underline{\underline{\mathrm{OSz}}} \to \underline{\underline{\mathrm{OSb}}}$ with the standard forgetful functor $G : \underline{\underline{\mathrm{OSb}}} \to \underline{\underline{\mathrm{Set}}}$. Now let the "destruction operator"

$$(6.3) \qquad Z_- : \underline{\underline{\mathrm{OSz}}}Z \to \underline{\underline{\mathrm{OSz}}}$$

be the functor removing the element 1 from each space Y in the image $\underline{\underline{\mathrm{OSz}}}Z$ of $\underline{\underline{\mathrm{OSz}}}$ under Z. Note that $Z_-Z_+ = 1_{\underline{\underline{\mathrm{OSz}}}Z}$ and $Z_+Z_- = 1_{\underline{\underline{\mathrm{OSz}}}}$. Now recall the forgetful functor $U : \mathfrak{V} \to \mathfrak{V}_+$ of (3.20) and accordingly define

$$(6.4) \qquad U_- := U : \underline{\underline{\mathrm{Dlz}}} \to \underline{\underline{\mathrm{Dl}}}, \text{ and}$$

$$(6.5) \qquad U_+ : \underline{\underline{\mathrm{Dlz}}}U_- \to \underline{\underline{\mathrm{Dlz}}}.$$

The first functor U_- forgets the constant 0, and the second one recalls it. As before, $U_-U_+ = 1_{\underline{\underline{\mathrm{Dlz}}}}$ and $U_+U_- = 1_{\underline{\underline{\mathrm{Dlz}}}U}$. The functors D and E are defined as follows:

$$(6.6) \qquad D := U_-D_+Z_- \text{ and } E := Z_+E_+U_+.$$

PROPOSITION 6.1. *Let L denote an object of $\underline{\underline{\mathrm{Dlz}}}$. There is a natural isomorphism $\varphi : \underline{\underline{\mathrm{Dlz}}}(\quad, \underline{2}_0) \dot{\to} D = U_-D_+Z_-$ between the functors $\underline{\underline{\mathrm{Dlz}}}(\quad, \underline{2}_0)$ of* (5.13) *and D of* (6.6) *given by*

$$(6.7) \quad \varphi_L : \underline{\underline{\mathrm{Dlz}}}(L, \underline{2}_0) \to LU_-D_+Z_-; ((L,+,\cdot,0) \to \underline{2}_0) \mapsto ((L,+,\cdot) \to \underline{2}).$$

Let Y denote an object of $\underline{\underline{\mathrm{OSz}}}$. There is a natural isomorphism $\psi : \underline{\underline{\mathrm{OSz}}}(\quad, \underset{\sim}{2}_0) \dot{\to} E = Z_+E_+U_+$ between the functors $\underline{\underline{\mathrm{OSz}}}(\quad, \underset{\sim}{2}_0)$ of (5.13) *and E of* (6.6) *given by*

$$(6.8) \quad \psi_Y : \underline{\underline{\mathrm{OSz}}}(Y, \underset{\sim}{2}_0) \to YX_+E_+U_+; \\ ((Y, \leq, 0, \mathcal{T}) \to \underset{\sim}{2}_0) \mapsto ((Y \dot{\cup} \{1\}, \leq, 0, 1, \mathcal{T}) \to \underset{\sim}{2}_{0,1}).$$

PROOF. Both functions φ_L and ψ_Y are obviously bijective. The upper bound of LU_-D_+ is the function $f : L \to \underline{2};\ a \mapsto 1$ with $Lf = \{1\}$. ■

Since $\varphi : \underline{\underline{\mathrm{Dlz}}}(\quad, \underline{2}_0) \dot{\to} D$ and $\psi : \underline{\underline{\mathrm{OSz}}}(\quad, \underset{\sim}{2}_0) \dot{\to} E$ are natural isomorphisms, one obtains a duality

$$(6.9) \qquad D : \underline{\underline{\mathrm{Dlz}}} \rightleftarrows \underline{\underline{\mathrm{OSz}}} : E.$$

The natural isomorphisms for (6.9), $\varepsilon : 1_{\underline{\underline{\mathrm{Dlz}}}} \dot{\to} DE = U_-D_+Z_-Z_+E_+U_+ = U_-D_+E_+U_+$ and $\eta = 1_{\underline{\underline{\mathrm{OSz}}}} \dot{\to} ED = Z_+E_+U_+U_-D_+Z_- = Z_+E_+D_+Z_-$, are given by

$$(6.10) \qquad \varepsilon_L = \varepsilon^+_{LU}U_+ \text{ and } \eta_Y = \eta^+_{YZ}Z_-.$$

PROPOSITION 6.2. *The dualities* (6.9) $D : \underline{\underline{\mathrm{Dlz}}} \rightleftarrows \underline{\underline{\mathrm{OSz}}} : E$ *and* (6.1) $D_+ : \underline{\underline{\mathrm{Dl}}} \rightleftarrows \underline{\underline{\mathrm{OSb}}} : E_+$ *are compatible.*

PROOF. One has to check the conditions (a)-(d) of (4.5). First note that $DZ = U_-D_+Z_-Z_+ = U_-D_+ = UD_+$ and $ZE_+ = Z_+E_+ = Z_+E_+U_+U_- = EU_- = EU$. This proves (a) and (b). Now, for each lattice L in $\underline{\underline{\mathrm{Dlz}}}$ and space Y in $\underline{\underline{\mathrm{OSz}}}$, the conditions (6.10) give that $\varepsilon_LU = \varepsilon_LU_- = \varepsilon^+_{LU}U_-U_+ =$

ε^+_{LU} and similarly $\eta_Y Z = \eta_Y Z_+ = \eta^+_{YZ} Z_- Z_+ = \eta^+_{YZ}$. This proves (c) and (d). ■

Since the dualities (6.9) and (6.1) are compatible, Proposition 4.1 and Theorem 4.2 yield the duality

$$ {}^2D : {}^2\underline{\underline{\mathrm{Dlz}}} \rightleftarrows \underline{\underline{\mathrm{OSz}}}^2 : E^2. \tag{6.11}$$

Note that the category $\underline{\underline{\mathrm{OSz}}}^2$ is concrete by virtue of the faithful functor into $\underline{\underline{\mathrm{Set}}}$ that acts as the standard forgetful functor G on the subcategory $\underline{\underline{\mathrm{OSz}}}_+$, as Z_+G on the subcategory $\underline{\underline{\mathrm{OSz}}}$, and that assigns to each $\underline{\underline{\mathrm{OSz}}}^2$-morphism of the form $f : Y \to X$, where Y is in $Ob(\underline{\underline{\mathrm{OSz}}})$ and X is in $Ob(\underline{\underline{\mathrm{OSz}}}_+)$, the $\underline{\underline{\mathrm{Set}}}$-morphism $f : YZ \to X$. The following diagram helps to summarize the dualities and functors discussed in this section:

$$\begin{tikzcd}
\underline{\underline{\mathrm{Set}}} \arrow[dd, equal] & \underline{\underline{\mathrm{Dlz}}} \arrow[l, "G"'] \arrow[r, shift left, "\mathrm{D}"] \arrow[d, shift right, "\mathrm{U}_-=\mathrm{U}"'] & \underline{\underline{\mathrm{OSz}}} \arrow[l, shift left, "\mathrm{E}"] \arrow[r, "Z_+G"] \arrow[d, shift right, "\mathrm{Z}=\mathrm{Z}_+"'] & \underline{\underline{\mathrm{Set}}} \arrow[dd, equal] \\
 & \underline{\underline{\mathrm{Dlz}}}U \arrow[u, shift right, "\mathrm{U}_+"'] \arrow[d] & \underline{\underline{\mathrm{OSz}}}Z \arrow[u, shift right, "\mathrm{Z}_-"'] \arrow[d] & \\
\underline{\underline{\mathrm{Set}}} & \underline{\underline{\mathrm{Dlz}}}_+ \arrow[l, "G"'] \arrow[r, shift left, "\mathrm{D}_+"] & \underline{\underline{\mathrm{OSz}}}_+ \arrow[l, shift left, "\mathrm{E}_+"] \arrow[r, "G"'] & \underline{\underline{\mathrm{Set}}}
\end{tikzcd}$$

Theorem 4.6 now yields the dualities

$$\widetilde{D}_1 : \widetilde{\underline{\underline{\mathrm{Dlz}}}}_1 \rightleftarrows \widetilde{\underline{\underline{\mathrm{OSz}}}}_1 : \widetilde{E}_1 \tag{6.12}$$

for the symmetrization $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_1$ of $\underline{\underline{\mathrm{Dlz}}}$ and

$$\widetilde{D}_\mu : \widetilde{\underline{\underline{\mathrm{Dlz}}}}_\mu \rightleftarrows \widetilde{\underline{\underline{\mathrm{OSz}}}}_\mu : E_\mu \tag{6.13}$$

for the symmetric regularization $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_\mu$ of $\underline{\underline{\mathrm{Dlz}}}$. An equational axiomatization of the varieties $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_1$ and $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_\mu$ can be deduced from Theorem 5.2. The variety $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_1$ is defined by the symmetrically regular identities defining $\underline{\underline{\mathrm{Dlz}}}$ (cf. Section 5) together with the identity

$$0 + 0x = 0, \tag{6.14}$$

and the variety $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_\mu$ is defined by the symmetrically regular identities from those defining $\underline{\underline{\mathrm{Dlz}}}$.

Note that the dualities for $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_\mu$ and $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_1$ obtained in this section do not seem to arise from schizophrenic objects. However, it is likely that a duality with a schizophrenic object exists at least for $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_1$. It should be based on a duality for $\underline{\underline{\mathrm{Sl}}}_1$ obtained by means of the two-element semilattice with an absorbing element selected as a constant nullary operation serving as a schizophrenic object. We did not pursue this line in the current work. Similarly, we did not consider another possible duality for $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_0$ that can be obtained from our duality for $\widetilde{\underline{\underline{\mathrm{Dlz}}}}_\mu$ by taking $\mu = 0$.

7. Lindenbaum-Tarski duality

Here the basic variety $\mathfrak{V}$ is the variety $\underline{\underline{\mathrm{Lz}}}$ of left zero bands, discussed in Section 3, isomorphic to the category $\underline{\underline{\mathrm{Set}}}$ of sets. The categorical version of the well-known Lindenbaum-Tarski duality [1], [6], [13, VI. 4.6(b)], [34] may be written as

$$(7.1) \qquad \underline{\underline{\mathrm{Lz}}}(\quad, \underline{2}) : \underline{\underline{\mathrm{Lz}}} \rightleftarrows \underline{\underline{\mathrm{St\ B}}} : \underline{\underline{\mathrm{St\ B}}}(\quad, \tilde{2}),$$

where $\underline{\underline{\mathrm{St\ B}}}$ is the category of Stone topological Boolean algebras, isomorphic to the category of complete, atomic Boolean algebras. For a left zero band E, the space $\underline{\underline{\mathrm{Lz}}}(E, \underline{2})$ consists of characteristic functions $\varphi_T : E \to \underline{2}$ of subsets T of E. One can then identify the space $\underline{\underline{\mathrm{Lz}}}(E, \underline{2})$ with the (topological) Boolean algebra of subsets of the set E. The natural isomorphisms ε^{LT} and η^{LT} for the duality (7.1) are given, as usual, by

$$(7.2) \qquad \varepsilon^{LT}_E : E \to \underline{\underline{\mathrm{St\ B}}}(\underline{\underline{\mathrm{Lz}}}(E, \underline{2}), \underset{\sim}{2}); \ a \mapsto (\varphi \mapsto a\varphi)$$

and

$$(7.3) \qquad \eta^{LT}_X ;\ X \to \underline{\underline{\mathrm{Lz}}}(\underline{\underline{\mathrm{St\ B}}}(X, \underset{\sim}{2}), \underline{2}); \ x \mapsto (h \mapsto xh).$$

The schizophrenic two-element object 2 appears in $\underline{\underline{\mathrm{Lz}}}$ as the band $\underline{2} = (\{0,1\}, *)$, and in $\underline{\underline{\mathrm{St\ B}}}$ as the Boolean algebra $\underset{\sim}{2} = (\{0,1\}, +, \cdot, ', 0, 1, \mathcal{T})$ with discrete topology $\mathcal{T}$. A duality for the variety $\underline{\underline{\mathrm{Ln}}}$ of left normal bands, the (strict) regularization of the variety $\underline{\underline{\mathrm{Lz}}}$, was obtained in [34]. It has the form

$$(7.4) \qquad \underline{\underline{\mathrm{Ln}}}(\quad, 3) : \underline{\underline{\mathrm{Ln}}} \rightleftarrows \widetilde{\underline{\underline{\mathrm{St\ B}}}} : \widetilde{\underline{\underline{\mathrm{St\ B}}}}(\quad, \underset{\sim}{3}).$$

The algebraic part of the category $\widetilde{\underline{\underline{\mathrm{St\ B}}}}$ is axiomatized as the regularization $\widetilde{\underline{\underline{\mathrm{Bool}}}}$ of the variety $\underline{\underline{\mathrm{Bool}}}$ of Boolean algebras. Here the variety $\underline{\underline{\mathrm{Bool}}}$

may be axiomatized by the identities defining the variety $\underline{\underline{\mathrm{Dd}}}$ of distributive dissemilattices together with (regular) identities

$$(7.5) \qquad \begin{cases} (x+y)' = x'y', \\ xx' + yy' = xx'yy', \\ x'' = x \end{cases}$$

and one irregular identity

$$(7.6) \qquad x + xy = x$$

of absorption. The regularization $\underline{\underline{\widetilde{\mathrm{Bool}}}}$ is axiomatized by the axioms for $\underline{\underline{\mathrm{Dd}}}$ and the identities (7.5). The category $\underline{\underline{\widetilde{\mathrm{St\ B}}}}$ is described as the class of Stone topological $\underline{\underline{\widetilde{\mathrm{Bool}}}}$-algebras with three nullary operations c_α, c_0, c_1 satisfying

$$(7.7) \qquad \begin{cases} x + xc_0 = x, \\ c_0 + x = x = xc_1, \\ c_2 + x = c_\alpha = xc_\alpha, \\ c_0 + c_0x = c_1 + x \Rightarrow x = c_\alpha. \end{cases}$$

The schizophrenic object 3 appears in $\underline{\underline{\mathrm{Ln}}}$ as the Płonka sum $\underline{3}$ of the two-element left zero band $\underline{2}$ and a one-element left zero band, and in $\underline{\underline{\widetilde{\mathrm{St\ B}}}}$ as a 3-element regularized Boolean algebra $\underset{\sim}{3}$, with discrete topology, and with its three elements $\alpha, 0$ and 1 selected as constants by nullary operations c_α, c_0 and c_1 respectively. The algebra $\underset{\sim}{3}$ is well known under various names such as the "weak extension of Boolean logic" [8], [14] and the "Bochvar system of logic" [2], [4].

The duality (7.1) can be modified to obtain duality for pointed sets. Here the basic category $\mathfrak{V}$ of (4.14) is the category $\underline{\underline{\mathrm{Lz}}}^P$ of left-zero bands with a (non-empty) set P of nullary operations, isomorphic to the category $\underline{\underline{\mathrm{Set}}}^P$ of P-pointed sets. One easily obtains the duality

$$(7.8) \qquad D : \underline{\underline{\mathrm{Lz}}}^P \rightleftarrows \underline{\underline{\mathrm{St\ B}}}^P : E$$

between the category $\underline{\underline{\mathrm{Lz}}}^P$ isomorphic to $\underline{\underline{\mathrm{Set}}}^P$ and the category of Stone topological Boolean algebras with a set of pointed atoms selected by nullary operations in P. The natural isomorphisms $\varepsilon : 1_{\underline{\underline{\mathrm{Set}}}_P} \dot{\to} DE$ and $\eta : 1_{\underline{\underline{\mathrm{St\ B}}}_P} \dot{\to} ED$ for the duality (7.8) are given by (7.2) and (7.3) with $\varepsilon = \varepsilon^{LT}$ and $\eta = \eta^{LT}$, but one requires additionally that for any pointed set (S, P) and each p in P, the element of SD selected as a constant by the nullary operation p is the atom $\{p\}$ containing the element p in the Boolean algebra of subsets of S. Similarly, for a pointed Stone Boolean algebra X with a pointed atom p

in P, the element of XE selected as a constant by the nullary operation p is the morphism h with $(\uparrow p)h = \{1\}$ and $(X - \uparrow p)h = \{0\}$. The natural isomorphisms ε and η act on constant elements p in P accordingly:

$$p \mapsto (p\varepsilon_S : SD \to \underset{\sim}{2};\ \varphi_T \mapsto p\varphi_T), \tag{7.9}$$

$$p \mapsto (p\eta_X : XE \to \underline{2};\ h \mapsto ph). \tag{7.10}$$

Note that the schizophrenic object 2 for the duality (7.1) is no longer a schizophrenic object for the duality (7.8). Homomorphisms of Boolean algebras do not necessarily respect nullary operations in P. Nevertheless the duality (7.8) satisfies the assumptions of Theorem 4.6, whence there is a duality

$$\widetilde{D}_0 : \widetilde{\underline{\underline{\mathrm{Lz}}}_0^P} \rightleftarrows \widetilde{\underline{\underline{\mathrm{St\ B}}}_0^P} : \widetilde{E}_0 \tag{7.11}$$

for the regularization $\widetilde{\underline{\underline{\mathrm{Lz}}}_0^P}$ of $\underline{\underline{\mathrm{Lz}}}^P$. The variety $\widetilde{\underline{\underline{\mathrm{Lz}}}_0^P}$ is axiomatized by the identities defining the variety $\underline{\underline{\mathrm{Ln}}}$ and, for each p in P, the identity $x * p = x$. The category $\widetilde{\underline{\underline{\mathrm{St\ B}}}_0^P}$ is axiomatized as the category of Stone topological $\widetilde{\underline{\underline{\mathrm{Bool}}}}$-algebras with the set $P \cup \{c_\alpha, c_1, c_0\}$ of nullary operations. For a representation $R : (H, \cdot, 1) \to \underline{\underline{\mathrm{Lz}}}$, all P-constants lie in the fibre $1R$.

To find dualities for the symmetrization and symmetrical regularization of the variety $\underline{\underline{\mathrm{Lz}}}^P$, one needs to specify the categories $\mathfrak{V}_+$ and $\mathfrak{Y}_+$ of (4.3). For $\mathfrak{V}_+$ one takes the variety $\underline{\underline{\mathrm{Lz}}}$, for $\mathfrak{Y}_+$ the category $\underline{\underline{\mathrm{St\ B}}}$, and for (4.3) the duality (7.1). One has the following diagram of dualities and functors similar to that for $\underline{\underline{\mathrm{Dlz}}}$ and $\underline{\underline{\mathrm{Dlz}}}_+$.

$$\begin{tikzcd}
\underline{\underline{\mathrm{Set}}} \arrow[d, equal] & \underline{\underline{\mathrm{Lz}}}^P \arrow[l, "G"'] \arrow[r, shift left, "\mathrm{D}"] \arrow[d, shift left] & \underline{\underline{\mathrm{St\ B}}}^P \arrow[l, shift left, "\mathrm{E}"] \arrow[r, "G"] \arrow[d, shift left] & \underline{\underline{\mathrm{Set}}} \arrow[d, equal] \\
\underline{\underline{\mathrm{Set}}} & \underline{\underline{\mathrm{Lz}}} \arrow[l, "G"] \arrow[u, shift left, "\mathrm{U}"'] \arrow[r, shift left, "\mathrm{D}_+"] & \underline{\underline{\mathrm{St\ B}}} \arrow[l, shift left, "\mathrm{E}_+"] \arrow[r, "G"'] \arrow[u, shift left, "\mathrm{Z}"'] & \underline{\underline{\mathrm{Set}}}
\end{tikzcd}$$

Here both functors $U : \underline{\underline{\mathrm{Lz}}}^P \to \underline{\underline{\mathrm{Lz}}}$ and $Z : \underline{\underline{\mathrm{St\ B}}}^P \to \underline{\underline{\mathrm{St\ B}}}$ forget the nullary operations in P. The functors D_+ and E_+ in the bottom line are the $\underline{\underline{\mathrm{Lz}}}(\quad, \underline{2})$ and $\underline{\underline{\mathrm{St\ B}}}(\quad, \underset{\sim}{2})$ of (7.1) respectively.

LEMMA 7.1. *The dualities* (7.8) $D : \underline{\underline{\mathrm{Lz}}}^P \rightleftarrows \underline{\underline{\mathrm{St\ B}}}^P : E$ *and* (7.1) $D_+ : \underline{\underline{\mathrm{Lz}}} \rightleftarrows \underline{\underline{\mathrm{St\ B}}} : E$ *are compatible.*

PROOF. The conditions (a) and (b) of (4.5) follow directly from the definitions of the functors in the diagram above. The conditions (c) and (d) are also obvious. ■

Now since the assumptions of Theorem 4.6 are obviously satisfied, it follows that there are dualities

$$\widetilde{D}_1 : \widetilde{\underline{\underline{\mathrm{Lz}}}}{}_1^P \rightleftarrows \widetilde{\underline{\underline{\mathrm{St\ B}}}}{}_1^P : \widetilde{E}_1 \tag{7.10}$$

for the symmetrization $\widetilde{\underline{\underline{\mathrm{Lz}}}}{}_1^P$ of $\underline{\underline{\mathrm{Lz}}}^P$ and

$$\widetilde{D}_\mu : \widetilde{\underline{\underline{\mathrm{Lz}}}}{}_\mu^P \rightleftarrows \widetilde{\underline{\underline{\mathrm{St\ B}}}}{}_\mu^P : \widetilde{E}_\mu \tag{7.11}$$

for the symmetric regularization $\widetilde{\underline{\underline{\mathrm{St\ B}}}}{}_\mu^P$ of $\underline{\underline{\mathrm{St\ B}}}^P$. The variety $\widetilde{\underline{\underline{\mathrm{Lz}}}}{}_1^P$ is axiomatized by the identities defining the variety $\underline{\underline{\mathrm{Ln}}}$ and, for each p in P, the identity $p * x = p$, whereas the variety $\widetilde{\underline{\underline{\mathrm{Lz}}}}{}_\mu^P$ is axiomatized by the identities defining $\underline{\underline{\mathrm{Ln}}}$ and all the identities $p * q = p$ for p, q in P. For a representation $R : (H, \cdot, 0) \to \underline{\underline{\mathrm{Lz}}}^P$, all P-constants are located in the fibre 0R, and for a representation $R : (H, \cdot, \mu) \to \underline{\underline{\mathrm{Lz}}}^P$, all P-constants are located in the fibre μR. The non-constant reducts of $\underline{\underline{\mathrm{St\ B}}}{}_1^P$- and $\widetilde{\underline{\underline{\mathrm{St\ B}}}}{}_\mu^P$-spaces are defined as $\widetilde{\underline{\underline{\mathrm{St\ B}}}}$-spaces. Each $\widetilde{\underline{\underline{\mathrm{St\ B}}}}{}_1^P$-space has the set P of nullary operations. For a representation $R : (G, \cdot, 1) \to \underline{\underline{\mathrm{St\ B}}}^P$, all P-constants are located in the fibre $1R$. For a representation $R : (G, \cdot, m) \to \underline{\underline{\mathrm{St\ B}}}^P$, there is the set P of nullary operations in each fibre gR for $g \geq m$. Denote by P_g the set of constants in gR. Then the space corresponding to the representation R has the set $\dot{\bigcup}_{g \geq m} P_g$ of constants.

8. Definable constants

For this final section, suppose that the category $\mathfrak{V}$ of (4.14) is a strongly irregular variety of B-algebras with definable constant operations. (Recall that a constant is *definable* if the nullary operation selecting it is identically equal to an operation derived from the basic non-nullary operations.) Examples are furnished by Boolean algebras, where $0 = x \cdot x'$ and $1 = x + x'$, and by abelian groups, where $0 = x - x$. It follows that the functor $U : \mathfrak{V} \to \mathfrak{V}_+$ of (3.20) is an isomorphism between $\mathfrak{V}$ and $\mathfrak{V}U$. Moreover $Ob(\mathfrak{V}_+) = Ob(\mathfrak{V}U) \,\dot\cup\, \{\varnothing\}$. Given a duality (4.2) for such a variety, there may be no compatible duality (4.3) for $\mathfrak{V}_+$ enabling one to apply Theorem 4.6 to obtain dualities for $\widetilde{\mathfrak{V}}_\mu$

and $\widetilde{\mathfrak{V}}_1$. What is always possible, however, as illustrated in this section by Pontryagin and Stone dualities, is that one may replace the category $\mathfrak{V}_+$ in Section 4 by $\mathfrak{V}'_+ := \mathfrak{V}U$. The category $\mathfrak{Y}_+$ is then replaced by a category $\mathfrak{Y}'_+$ that is a disjoint copy $\mathfrak{Y}Z$ of the category $\mathfrak{Y}$ via an isomorphism Z. The duality (4.2) then yields a compatible duality

$$D'_+ : \mathfrak{V}'_+ \rightleftarrows \mathfrak{Y}'_+ : E'_+ \tag{8.1}$$

with $D_+ = U^{-1}DZ, D'_+ = Z^{-1}EU$, etc. Thus (8.1) may replace (4.3), and more generally the work of Section 4 applies once symbols with "+" acquire primes. Modified in this way, Theorem 4.6 gives dualities (4.19) and (4.18) for $\widetilde{\mathfrak{V}}_\mu$ and $\widetilde{\mathfrak{V}}_1$.

EXAMPLE 8.1. (SYMMETRICALLY REGULARIZED PONTRYAGIN DUALITY) The duality (4.14) here is the Pontryagin duality

$$D = \underline{\underline{\mathrm{Ab}}}(\quad, \underline{T}) : \underline{\underline{\mathrm{Ab}}} \rightleftarrows \underline{\underline{\mathrm{CHAb}}} : \underline{\underline{\mathrm{CHAb}}}(\quad, \underset{\sim}{T}) = E \tag{8.2}$$

(cf. (5.27) discussed in Example 5.5). The category $\underline{\underline{\mathrm{Ab}}}'_+$ is the category of non-empty abelian groups defined as algebras $(G, +, -)$ (without 0 as a basic nullary operation) satisfying the identities

$$\begin{cases} (x+y)+z = x+(y+z) \\ x+y = y+x \end{cases} \tag{8.3}$$

$$x + y - y = x \tag{8.4}$$

(cf. [21]). Note that abelian groups as algebras $(G, +, -, 0)$ may be defined by the same set of axioms together with the identity

$$0 + 0 = 0. \tag{8.5}$$

Among all these identities only (8.4) is irregular. The identities defining the symmetric regularization $\underline{\underline{\widetilde{\mathrm{Ab}}}}_\mu$ are (8.3), (8.5) and

$$\begin{cases} -(-x) = x \\ -(x+y) = (-x) - (y) \\ x + x - x = x. \end{cases} \tag{8.6}$$

Non-constant reducts of $\underline{\underline{\widetilde{\mathrm{Ab}}}}_\mu$-semigroups are commutative inverse semigroups, Płonka sums of $\underline{\underline{\mathrm{Ab}}}'_+$-groups. The category $\underline{\underline{\widetilde{\mathrm{Ab}}}}_\mu$ is equivalent to a category of representations of $\underline{\underline{\mathrm{Sl}}}_\mu$-semilattices in ${}^2\underline{\underline{\mathrm{Ab}}}$. For such a representation

$R : (H, \cdot, \mu) \to {}^2\underline{\underline{\mathrm{Ab}}}$, the constant 0 is selected from the fibre μR. For $\mu = 1$, one obtains the symmetrization $\underline{\underline{\widetilde{\mathrm{Ab}}}}_1$ of $\underline{\underline{\mathrm{Ab}}}$. It is defined by the axioms for $\underline{\underline{\widetilde{\mathrm{Ab}}}}_\mu$ and the identity

$$(8.7) \qquad 0 + x - x = 0.$$

The category $\underline{\underline{\mathrm{CHAb}}}'_+$ is the category of non-empty compact Hausdorff abelian groups $(G, +, -, \mathcal{T})$. The category $\underline{\underline{\widetilde{\mathrm{CHAb}}}}_\mu$ is equivalent to a category of continuous representations of $\underline{M}$-semilattices in the category $\underline{\underline{\mathrm{CHAb}}}^2$. Non-constant algebraic reducts of $\underline{\underline{\widetilde{\mathrm{CHAb}}}}_\mu$-semigroups are commutative inverse semigroups, Płonka sums of $\underline{\underline{\mathrm{Ab}}}'_0$-groups. For such a representation $R : (G, \cdot, m) \to \underline{\underline{CH\mathrm{Ab}}}^2$, there is one nullary operation 0_g in each fibre gR for $g \geq m$. The space corresponding to the representation R has the set $\{0_g | g \in G \text{ and } g \geq m\}$ of constants.

Example 8.2. Symmetrically regularized Stone duality The basic duality (4.14) is again the duality

$$(8.8) \qquad D : \underline{\underline{\mathrm{Bool}}} \rightleftarrows \underline{\underline{\mathrm{Stone}}} \simeq \underline{\underline{\mathrm{SLz}}} : E$$

between the variety $\underline{\underline{\mathrm{Bool}}}$ of Boolean algebras and the category $\underline{\underline{\mathrm{Stone}}}$ of Stone spaces or $\underline{\underline{\mathrm{SLz}}}$ of Stone topological left zero bands (cf. Example 5.4). The variety $\underline{\underline{\mathrm{Bool}}}$ is defined by the axioms for the variety $\underline{\underline{\mathrm{Dd}}}$ and the identities (7.5) and (7.6) of Section 7 together with the symmetrically regular identities (5.10) and (5.11) of Section 5. The category $\underline{\underline{\mathrm{Bool}}}'_+$ is the category of non-empty Boolean algebras defined as algebras without basic nullary operations by the same axioms as $\underline{\underline{\mathrm{Bool}}}$ but without (5.10) and (5.11). The symmetric regularization $\underline{\underline{\widetilde{\mathrm{Bool}}}}_\mu$ of $\underline{\underline{\mathrm{Bool}}}$ is defined by the symmetrical identities defining $\underline{\underline{\mathrm{Bool}}}$ and the identities

$$(8.9) \qquad 0 + 0.1 = 0 \text{ and } 1 + 1 \cdot 0 = 1.$$

If $\mu = 1$, one should also add the identities

$$(8.10) \qquad 0 + 0.x = 0 \text{ and } 1 + 1 \cdot x = 1.$$

The category $\underline{\underline{\mathrm{Bool}}}_\mu$ is equivalent to a category of representations of $\underline{\underline{\mathrm{Sl}}}_\mu$-semilattices in the category ${}^2\underline{\underline{\mathrm{Bool}}}$. Just as for $\underline{\underline{\widetilde{\mathrm{Bool}}}}_0$, the category $\underline{\underline{\widetilde{\mathrm{Bool}}}}_\mu$ contains representations $R : (H, \cdot, \mu) \to {}^2\underline{\underline{\mathrm{Bool}}}$ with trivial fibres hR. The full subcategory $\underline{\underline{\widetilde{\mathrm{Bool}}}}^*_\mu$ of $\underline{\underline{\widetilde{\mathrm{Bool}}}}_\mu$ consists of Płonka sums of non-trivial $\underline{\underline{\mathrm{Bool}}}'_+$-algebras, with constants 0 and 1 selected by the two nullary operations of the fibre μR. The categories $\underline{\underline{\mathrm{SLz}}}'_+$ and $\underline{\underline{\mathrm{SLz}}}$ coincide. The category $\widetilde{\underline{\underline{\mathrm{SLz}}}}_\mu$ is equivalent to a category of continuous representations of $\underline{M}$-semilattices in the category $\underline{\underline{\mathrm{SLz}}}^2$. It may contain trivial representations with empty fibres. The full subcategory $\widetilde{\underline{\underline{\mathrm{SLz}}}}^*_\mu$ of $\widetilde{\underline{\underline{\mathrm{SLz}}}}_\mu$ consists of non-trivial continuous representations of $\underline{M}$-semilattices in $\underline{\underline{\mathrm{SLz}}}^2$. Non-constant algebraic reducts of $\widetilde{\underline{\underline{\mathrm{SLz}}}}^*_\mu$-spaces are normal bands, Płonka sums of left zero bands. The duality

$$\widetilde{D}_\mu : \widetilde{\underline{\underline{\mathrm{Bool}}}}_\mu \rightleftarrows \widetilde{\underline{\underline{\mathrm{SLz}}}} : \widetilde{E}_\mu \tag{8.11}$$

restricts to a duality

$$\widetilde{D}^*_\mu : \widetilde{\underline{\underline{\mathrm{Bool}}}}^*_\mu \rightleftarrows \widetilde{\underline{\underline{\mathrm{SLz}}}}^* : \widetilde{E}^*_\mu. \tag{8.12}$$

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