Arch. Math., Vol. 65, 534–539 (1995)

0003-889X/95/6506-0534 $ 2.70/0

Simple algebras of hermitian operators

By

X. R. SHEN and J. D. H. SMITH

1. Introduction. A *comtrans algebra* E over a commutative ring R with unit is a unital R-module E equipped with two trilinear operations, a *commutator* $[x, y, z]$ and a *translator* $\langle x, y, z\rangle$, such that the commutator is *left alternative*:

(1.1) $$[x, x, z] = 0,$$

the translator satisfies the *Jacobi identity*:

(1.2) $$\langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle = 0,$$

and together the commutator and translator satisfy the *comtrans identity*:

(1.3) $$[x, y, z] = \langle x, y, x\rangle .$$

Comtrans algebras were introduced [6] as part of the algebraic structure in the tangent bundle corresponding to the coordinate n-ary loop of an $(n + 1)$-web. Thus their relationship with smooth n-ary loops is analogous to the relationship of Lie algebras with Lie groups. In fact the theory of comtrans algebras is modelled on (and to some extent may subsume) the theory of Lie algebras. The present paper is part of a continuing programme (cf. [3], [4]) of classifying simple comtrans algebras. Previously known simple comtrans algebras have arisen from rectangular matrices [3], simple Lie algebras [3] and spaces equipped with a bilinear form having trivial radical [4]. Theorem 4.2 below, the main result of the current paper, shows that the vector space H_n of n-dimensional Hermitian matrices furnishes a simple comtrans algebra. Comtrans algebra structure on spaces of Hermitian operators was introduced in [7], where applications to quantum mechanics were discussed.

Section 2 covers the necessary algebraic fundamentals of comtrans algebras: ideals, simplicity, and the enveloping algebra. Section 3 gives the definition of the comtrans algebra on spaces of Hermitian operators, and introduces notation for the special elements that feature in the proof of simplicity. Section 4 derives the main simplicity result from the somewhat technical Slimming Lemma (Proposition 4.1) whose proof comprises the final Section 5.

2. Ideals and enveloping algebras. For elements x, y of a comtrans algebra E over the ring R, there are R-module endomorphisms of E defined by

(2.1) $$K(x, y) : E \to E;\ z \mapsto [z, x, y],$$

(2.2) $R(x, y): E \to E;\ z \mapsto \langle z, x, y\rangle,$

(2.3) $L(x, y): E \to E;\ z \mapsto \langle y, x, z\rangle$

The *enveloping algebra* $M(E)$ of the comtrans algebra E is the subalgebra of the R-module endomorphism algebra $\mathrm{End}_R E$ generated by $\{K(x, y), R(x, y), L(x, y) | x, y \in E\}$ [5]. An *ideal* J of the comtrans algebra E is a submodule of the $M(E)$-module E. The ideals of E are precisely the kernels of comtrans algebra homomorphisms with domain E [3, Prop. 3.1]. A comtrans algebra is *abelian* if the commutators and translators are all zero. A non-abelian comtrans algebra E is *simple* if it is irreducible as an $M(E)$-module, i.e. if it has no proper non-trivial ideals. For example, a Lie algebra L furnishes a comtrans algebra $CT(L)$ whose commutator and translator are both equal to the repeated commutator $[[z, x], y]$ in the Lie algebra. Thus $K(x, y) = \mathrm{Ad}(x)\mathrm{Ad}(y)$ and $R(x, y) = \mathrm{Ad}(x)\mathrm{Ad}(y)$. Then simplicity of the Lie algebra L is equivalent to simplicity of the comtrans algebra $CT(L)$ [3, Th. 3.2].

3. Hermitian operators. Let K be a field of odd or zero characteristic, for example the field $\mathbb{R}$ of reals. Let $L = K(q)$ be an extension of K by an algebraic element q with minimal polynomial $x^2 + 1$, for example the field $\mathbb{C}$ of complex numbers with $q = i$. In general, elements of K in L are called *real*, while elements of qK in L are called *imaginary*. Define an involutory automorphism $^-: L \to L; q \mapsto -q$ of L, having K as the field of fixed points. Fix an invertible diagonal $n \times n$ matrix T over K. Let H_n be the K-space of $n \times n$ *Hermitian* matrices over L, i.e. $n \times n$ matrices A over L satisfying $T^{-1}A^t T = \bar{A}$. Then H_n is closed under the Lie product

(3.1) $[x, y] = q(xy - yx)$

and the Jordan product

(3.2) $x \circ y = \frac{1}{2}(xy + yx).$

These products are connected by the identities

(3.3) $\frac{1}{4}[y, [x, z]] = x \circ (y \circ z) - (x \circ y) \circ z$

and

(3.4) $[x \circ y, z] = x \circ [y, z] + y \circ [x, z]$

(cf. [2, § 2.2]). Comtrans algebra structure is defined on H_n by taking

(3.5) $[x, y, z] = 2x \circ (y \circ z) - 2y \circ (z \circ x) - [x, y] \circ z$

as the commutator and

(3.6) $\langle x, y, z\rangle = 2z \circ (x \circ y) - 2y \circ (z \circ x) + [x, y \circ z]$

as the translator. The identities (1.1)–(1.3) are readily verified.

It is useful to have some notation available for the proof of the simplicity of the comtrans algebra H_n of $n \times n$ Hermitian matrices. Let E^{ij} denote the $n \times n$ matrix with zeroes everywhere except for a one located at the intersection of row i and column j. Let B^{ij} denote the symmetric matrix $E^{ij} + E^{ji}$. Let C^{ij} denote the skew-symmetric matrix $q(E^{ji} - E^{ij})$. Let D^{ij} denote the traceless diagonal matrix $E^{ii} - E^{jj}$. Let I denote the identity matrix $\sum_{i=1}^{n} E^{ii}$. Note that the matrices I, B^{ij}, C^{ij}, D^{ij} are all Hermitian, and that H_n has the n^2 elements $B^{ij}(1 \leqq i \leqq j \leqq n)$ and $C^{ij}(1 \leqq i < j \leqq n)$ as a basis (cf. (4.1), (4.2) below).

Remark 3.1. In the case $K = \mathbb{R}$, $L = \mathbb{C}$, $q = i$, $T = I$, the matrices B^{12}, C^{12} and D^{12} in H_2 are observables representing the three components σ_x, σ_y and σ_z of electron spin [1, § 37].

The following formulae record the Lie and Jordan products in H_n for pairs of matrices B^{ij}, C^{ij}, D^{ij}:

$$(3.7)\quad \begin{cases} [B^{ij}, B^{kl}] = -\{\delta_{jl} C^{ij} + \delta_{jk} C^{il} + \delta_{il} C^{jk} + \delta_{jk} C^{jl}\} \\ [B^{ij}, C^{kl}] = -\{\delta_{jl} B^{ik} - \delta_{jk} B^{il} + \delta_{il} B^{jk} - \delta_{ik} B^{jl}\} \\ [B^{ij}, D^{kl}] = -\{\delta_{jl} C^{ik} - \delta_{jl} C^{il} + \delta_{ik} C^{jk} - \delta_{il} C^{jl}\} \\ [C^{ij}, C^{kl}] = -\{\delta_{jl} C^{ik} - \delta_{jk} C^{il} - \delta_{il} C^{jk} + \delta_{ik} C^{jl}\} \\ [C^{ij}, D^{kl}] = -\{-\delta_{jk} B^{ik} + \delta_{jl} B^{il} + \delta_{ik} B^{jk} - \delta_{il} B^{jl}\} \\ [D^{ij}, D^{kl}] = -\{\delta_{ik} C^{ik} - \delta_{il} C^{il} - \delta_{jk} C^{jk} + \delta_{jl} C^{jl}\} \end{cases};$$

$$(3.8)\quad \begin{cases} B^{ij} \circ B^{kl} = \frac{1}{2}\{\delta_{jl} B^{ik} + \delta_{jk} B^{il} + \delta_{il} B^{jk} + \delta_{jk} B^{jl}\} \\ B^{ij} \circ C^{kl} = \frac{1}{2}\{-\delta_{jl} C^{ik} + \delta_{jk} C^{il} - \delta_{il} C^{jk} + \delta_{ik} C^{jl}\} \\ B^{ij} \circ D^{kl} = \frac{1}{2}\{\delta_{jl} B^{ik} - \delta_{jl} B^{il} + \delta_{ik} B^{jk} - \delta_{il} B^{jl}\} \\ C^{ij} \circ C^{kl} = \frac{1}{2}\{\delta_{jl} B^{ik} - \delta_{jk} B^{il} - \delta_{il} B^{jk} + \delta_{ik} B^{jl}\} \\ C^{ij} \circ D^{kl} = \frac{1}{2}\{\delta_{jk} C^{ik} - \delta_{jl} C^{il} - \delta_{ik} C^{jk} + \delta_{il} C^{jl}\} \\ D^{ij} \circ D^{kl} = \frac{1}{2}\{\delta_{ik} B^{ik} - \delta_{il} B^{il} - \delta_{jk} B^{jk} + \delta_{jl} B^{jl}\} \end{cases}$$

A typical application of (3.7) in the subsequent sections is to determine $AK(A', I) = [A, A']$ for various Hermitian matrices A, A'.

4. Simplicity and the Slimming Lemma. The sets

$$(4.1)\qquad \{B^{ij} \mid 1 \leqq i \leqq j \leqq n\}$$

of *real basis elements* and

$$(4.2)\qquad \{C^{st} \mid 1 \leqq s < t \leqq n\}$$

of *imaging basis elements* together comprise a basis of the K-space H_n. Consider an element

$$(4.3)\qquad A = \sum_{1 \leqq i \leqq j \leqq n} b_{ij} B^{ij} + \sum_{1 \leqq s < t \leqq n} c_{st} C^{st}$$

of H_n. It is said to have *real weight* $|\{b_{ij} \mid b_{ij} \neq 0\}|$ and *imaginary weight* $|\{c_{st} \mid c_{st} \neq 0\}|$. Its (*total*) *weight* is the sum of its real and imaginary weights. The following result, whose proof is relegated to the final section, is known as the *Slimming Lemma*.

Proposition 4.1. *An Hermitian operator* (4.3) *of weight bigger than one may be reduced to an operator of strictly smaller positive weight by the action of the enveloping algebra.*

The Slimming Lemma is the key result yielding the simplicity of the algebras H_n.

Theorem 4.2. *For* $n > 1$, *the comtrans algebra* H_n *of Hermitian operators is simple.*

Proof. Let J be a non-zero ideal of H_n. Recall that J is invariant under the action of the enveloping algebra. Let A be a non-zero element of J. By successive application of the Slimming Lemma (and possibly a scalar multiplication), it follows that a real or imaginary basis element is an image of A under the action of the enveloping algebra, and thus an element of J. In fact, since $\frac{1}{2}C^{st}K(B^{tt}, I) = B^{st}$, this image may be taken to be B^{ij} for some $i \leqq j$. For each of an exhaustive set of three cases, it will be shown that $B^{ij} \in J$ entails containment of all the basis elements within J, so that J is improper and H_n is simple.

Case 1. $i = j$. For $k \neq i = j$, one has

$$B^{ij}K(C^{kj}, -I) = B^{ik} \tag{4.4}$$

in J. Then $B^{kk} = B^{ii} + B^{ik}K(C^{ik}, I) \in J$. Using (4.4) again, all the real basis elements lie in J. Finally, $\frac{1}{2}B^{st}K(D^{st}, I) = C^{st} \in J$ for $s < t$.

Case 2. $i \neq j$, $n > 2$. For $i \neq k \neq j$, (4.4) holds, so $B^{ik} \in J$. Similarly, one obtains $B^{st} \in J$ for $s \neq t$. Then $-\frac{1}{2}B^{st}K(C^{st}, I) = D^{st} \in J$ and $\frac{1}{2}B^{st}K(D^{st}, I) = C^{st} \in J$. Also $\frac{1}{2}D^{st} - \frac{1}{4}D^{st}K(B^{st}, C^{st}) = E^{ss} \in J$, so $B^{ss} \in J$ for $1 \leqq s \leqq n$.

Case 3. $n = 2$, $i = 1$, $j = 2$. First, $-\frac{1}{2}B^{12}K(C^{12}, I) = D^{12} \in J$ and $\frac{1}{2}B^{12}K(D^{12}, I) = C^{12} \in J$. Then $-\frac{1}{2}D^{12}K(B^{12}, C^{12}) = I \in J$. □

5. Proof of the Slimming Lemma. In this section, the Slimming Lemma (Prop. 4.1) is proved. The proof is divided into three lemmas corresponding to an exhaustive set of distinct cases.

Lemma 5.1. *The Slimming Lemma holds for Hermitian operators*

$$A = bB^{ij} + cC^{st} \tag{5.1}$$

of real and imaginary weight one.

Proof. If the index sets $\{i, j\}$ and $\{s, t\}$ are distinct, or if $i \neq s < t = j$, or if $j > s < t = i$, then

$$AK(B^{ss}, I) = -2cB^{st} \tag{5.2}$$

has a positive weight strictly smaller than that of A. If $i = s < t = j$, consider

$$AK(B^{ij}, I)^2 = c(B^{ii} - B^{jj})K(B^{ij}, I) = -4cC^{ij}. \tag{5.3}$$

Otherwise, i.e. if $i = s < t \neq j$ or $j = s < t > i$,

$$AK(B^{tt}, I) = 2cB^{st}. \quad \square \tag{5.4}$$

Lemma 5.2. *The Slimming Lemma holds for Hermitian operators* (4.3) *of imaginary weight bigger than one.*

Proof. If (4.3) comprises non-zero coefficients c_{st} and $c_{s't'}$ with distinct index sets $\{s, t\}$ and $\{s', t'\}$, or with $t = t'$, $s \neq s'$, or with $s < t = s' < t'$, then it may be written in the form $A = c_{st} C^{st} + c_{s't'} C^{s't'} + \sum_{s<q\neq t} c_{sq} C^{sq} + \sum_{p<s<t} c_{ps} C^{ps} + \sum_{\substack{s\neq p<q\neq s \\ (p,q)\neq(s',t')}} c_{pq} C^{pq} + b_{ss} B^{ss} + \sum_{s<q} b_{sq} B^{sq} + \sum_{p<s} b_{ps} B^{ps} + \sum_{s\neq p\leqq q\neq s} b_{pq} B^{pq}$. For an operator of positive weight strictly smaller than that of A, one may then take $AK(B^{ss}, I) = -2 c_{st} C^{st} - 2 \sum_{s<q\neq t} c_{sq} B^{sq} + 2 \sum_{p<s<t} c_{ps} B^{ps} + 2 \sum_{s<q} b_{bq} C^{sq} - 2 \sum_{p<s} b_{ps} C^{ps}$. Otherwise, i.e. if $s = s' < t' \neq t$, (4.3) may be written in the form $A = c_{st} C^{st} + c_{s't'} C^{s't'} + \sum_{s\neq p<t} c_{pt} C^{pt} + \sum_{s<t<q} c_{tq} C^{tq} + \sum_{\substack{t\neq p<q\neq t \\ (p,q)\neq(s',t')}} c_{pq} C^{pq} + b_{tt} B^{tt} + \sum_{p<t} b_{pt} B^{pt} + \sum_{t<q} b_{tq} B^{tq} + \sum_{t\neq p\leqq q\neq t} b_{pq} B^{pq}$. For an operator of positive weight strictly smaller than that of A, one may then take $AK(B^{tt}, I) = 2 c_{st} B^{st} + 2 \sum_{s\neq p<t} c_{pt} B^{pt} - 2 \sum_{s<t<q} c_{tq} B^{tq} - 2 \sum_{p<t} b_{pt} C^{pt} + 2 \sum_{t<q} b_{tq} C^{tq}$. □

Lemma 5.3. *The Slimming Lemma holds for Hermitian operators* (4.3) *of real weight bigger than one and imaginary weight one or zero.*

Proof. The Hermitian operator (4.3) may be written in the form

$$A = c C^{st} + \sum_{1\leqq i\leqq j\leqq n} b_{ij} B^{ij} \tag{5.5}$$

with $s < t$. The proof breaks up into two distinct cases.

Case 1. $\exists (i, j) \neq (i', j')$. $b_{ij} \neq 0 \neq b_{i'j'}$, $\{i, j\} \cap \{i', j'\} \neq \emptyset$. Without loss of generality, $i < j$. If $j \in \{i', j'\}$, then A may be written in the form $A = b_{ij} B^{ij} + b_{i'j'} B^{i'j'} + \sum_{i\leqq q\neq j} b_{iq} B^{iq} + \sum_{p\leqq i} b_{pi} B^{pi} + \sum_{i\neq p\leqq q\neq i} b_{pq} B^{pq} + c C^{st}$. For an operator of positive weight strictly smaller than that of A, one may take $AK(B^{ii}, I) = 2 b_{ij} C^{ij} + 2 \sum_{i<q\neq j} b_{iq} C^{iq} - 2 \sum_{p<i} b_{pi} C^{pi} + 2 c \delta_{it} B^{is} - 2 c \delta_{is} B^{it}$. If $i \in \{i', j'\}$, then A may be written in the form $A = b_{ij} B^{ij} + b_{i'j'} B^{i'j'} + \sum_{i\neq p\leqq j} b_{pj} B^{pj} + \sum_{j\leqq q} b_{jq} B^{jq} + \sum_{j\neq p\leqq q\neq j} b_{pq} B^{pq} + c C^{st}$. For an operator of positive weight strictly smaller than that of A, one may take $AK(B^{jj}, I) = -2 b_{ij} C^{ij} - 2 \sum_{i\neq p<j} b_{pj} C^{pj} + 2 \sum_{j<q} b_{jq} C^{jq} + 2 c \delta_{jt} B^{js} - 2 c \delta_{js} B^{jt}$.

Case 2. $\forall (i, j) \neq (i', j')$, $b_{ij} \neq 0 \neq b_{i'j'} \Rightarrow \{i, j\} \cap \{i', j'\} = \emptyset$. If there is a non-zero b_{ij} with $i \neq j$, then A may be written in the form $A = b_{ij} B^{ij} + \sum_{\{i,j\}\not\ni p\leqq q\notin\{i,j\}} b_{pq} B^{pq} + c C^{st}$. For an operator of positive weight strictly smaller than that of A, one may take $AK(B^{ii}, I) = 2 b_{ij} C^{ij} + 2 c \delta_{it} B^{is} - 2 c \delta_{is} B^{it}$. Otherwise, the real basis elements having non-zero coefficients in A are all diagonal. If a diagonal real basis element B^{jj} has zero coefficient in A, and $b_{ii} \neq 0$, write $A = b_{ii} B^{ii} + \sum_{i\neq k\neq j} b_{kk} B^{kk} + c C^{st}$. Then $AK(B^{ij}, I) = -2 b_{ii} C^{ij}$

$+ c\delta_{jt}B^{is} - c\delta_{js}B^{it} + c\delta_{it}B^{js} - c\delta_{is}B^{jt}$. This has positive weight strictly smaller than that of A except in the case where $c \neq 0$, $\{i, j\} = \{s, t\}$, and A has real weight 2, i.e. where A can be written in one of the forms $A = b_{ss}B^{ss} + b_{kk}B^{kk} + cC^{st}$ with $0 \notin \{b_{ss}, b_{kk}, c\}$ and $s \neq k \neq t$ or $A = b_{tt}B^{tt} + b_{kk}B^{kk} + cC^{st}$ with $0 \notin \{b_{tt}, b_{kk}, c\}$ and $s \neq k \neq t$. For A having the first of these forms, (5.2) holds. For A having the second of these forms, (5.4) holds.

The remaining possibility within Case 2 is where A has the form

$$A = cC^{st} + \sum_{i=1}^{n} b_{ii}B^{ii} \tag{5.6}$$

with all b_{ii} non-zero. If $A = cC^{st} + bI$ with $c \neq 0$, then (5.2) holds. If A is a non-zero scalar multiple of the identity matrix, then $A(K(C^{12}, D^{12}) + R(C^{12}, D^{12}) + L(C^{12}, D^{12})) = b[C^{12}, D^{12}, I] = bC^{12}K(D^{12}, I) = -2bB^{12}$ has smaller positive weight. Otherwise, $b_{ii} \neq b_{jj}$ for some i, j. Then $AK(B^{ij}, I) = 2(b_{jj} - b_{ii})C^{ij} + c\delta_{jt}B^{is} - c\delta_{js}B^{it} + c\delta_{it}B^{js} - c\delta_{is}B^{jt}$. This has positive weight smaller than that of A unless $c \neq 0$, $n = 2$, and $A = b_{ss}B^{ss} + b_{tt}B^{tt} + cC^{st}$ with $s = 1$, $t = 2$. In this case (5.2) holds. □

References

[1] P. A. M. Dirac, The Principles of Quantum Mechanics. Oxford 1967.

[2] G. A. Saizew, Algebraic Problems of Mathematical and Theoretical Physics (Russian). Moscow 1974. German translation: Berlin 1979.

[3] X. R. Shen and J. D. H. Smith, Simple multilinear algebras, rectangular matrices and Lie algebras. J. Algebra **160**, 424–433 (1993).

[4] X. R. Shen and J. D. H. Smith, Comtrans algebras and bilinear forms. Arch. Math. **59**, 327–333 (1992).

[5] X. R. Shen and J. D. H. Smith, Representation theory of comtrans algebras. J. Pure Appl. Algebra **80**, 177–195 (1992).

[6] J. D. H. Smith, Multilinear algebras and Lie's Theorem for formal n-loops. Arch. Math. **51**, 169–177 (1988).

[7] J. D. H. Smith, Comtrans algebras and their physical applications. Banach Center Publ. **28**, 319–326 (1993).

Eingegangen am 23. 6. 1994*)

Anschriften der Autoren:

J. D. H. Smith
Department of Mathematics
Iowa State University
Ames, Iowa 50011
USA

X. R. Shen
Department of Mathematics
and Computer Science
Merrimack College
North Andover
Massachusetts, 01845
USA

*) Eine Neufassung ging am 26. 4. 1995 ein.