DUALITY FOR QUASILATTICES AND GALOIS CONNECTIONS

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Abstract. The primary goal of the paper is to establish a duality for quasilattices. The main ingredients are duality for semilattices and their representations, the structural analysis of quasilattices as Płonka sums of lattices, and the duality for lattices developed by Hartonas and Dunn. Lattice duality treats the identity function on a lattice as a Galois connection between its meet and join semilattice reducts, and then invokes a duality between Galois connections and polarities. A second goal of the paper is a further examination of this latter duality, using the concept of a pairing to provide an algebraic equivalent to the relational structure of a polarity.

1. Preliminaries

1.1. **Introduction.** Quasilattices are algebras $(X, +, \times)$ with two semilattice structures $(X, +)$ and $(X, \times)$, satisfying the identities

$$[(x + y) \times z] + [y \times z] = (x + y) \times z \text{ and}$$
$$[(x \times y) + z] \times [y + z] = (x \times y) + z$$

(Definition 6.1). They were first introduced by Płonka [1], under the additional assumption that each semilattice operation distributes over the other, and then studied in full generality by Padmanabhan [2]. In turn, quasilattices form a special class of *Birkhoff systems*, algebras with two semilattice operations connected by the *Birkhoff identity*

$$x \times (x + y) = x + (x \times y)$$

— compare [3, 4].

A duality for the distributive quasilattices studied by Płonka was developed by Gierz and the first author [5]. The duality relied on three ingredients: Priestley duality for distributive lattices, duality for semilattices and their representations, and Płonka's structural analysis of a distributive quasilattice, as a union of distributive lattice fibers over

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the semilattice replica of the distributive quasilattice. Such structural behavior, in terms of a union of fibers over a semilattice, is now known as a *Płonka sum* in general algebra, or more narrowly as a *semilattice sum* in the context of semigroup theory. Geometrically, Płonka sums form the total spaces or bundles of sheaves over semilattices with the Alexandrov topology.

The current paper establishes a duality for general quasilattices. Two of the three ingredients in the new duality are analogues of those used for distributive quasilattices: duality for semilattices and their representations, and Padmanabhan's structural analysis of general quasilattices as Płonka sums of general lattices [2]. The third ingredient is the duality for lattices developed by Hartonas and Dunn [6]. Predecessors of this duality were Urquhart's lattice representations [7], and the duality for surjective lattice homomorphisms established by Hartung [8] on the basis of Wille's concept analysis [9].

The lattice duality of Hartonas and Dunn interprets the identity function on a lattice as a Galois connection between the meet and join semilattice reducts of the lattice, and then invokes a duality between semilattice Galois connections and polarities. Thus a second feature of our paper is a further examination of this latter duality, since it is of considerable interest in its own right. In particular, the topological Boolean algebras that were central to much of Rasiowa's work make an appearance here (Remark 3.9). In our approach to the duality for semilattice Galois connections, we introduce the concept of a *pairing* to provide an algebraic equivalent to the relational structure of a polarity. Pairings often allow us to give a more immediate and precise treatment of various features of the duality.

1.2. **Plan of the paper.** Section 2 summarizes the well-studied duality between the category **SL** of semilattices (understood as idempotent, commutative semigroups) and the category $\mathfrak{B}$ of bounded compact Hausdorff zero-dimensional topological semilattices, based on the so-called "Pontryagin duality" for idempotent, commutative monoids [10]. Objects of $\mathfrak{B}$ are described simply as $\mathfrak{B}$-*spaces*. Semilattice duality is fundamental to all the other dualities considered in the paper, starting with the duality for Galois connections between semilattices, due to Hartonas and Dunn, that is described in Section 3. Hartonas and Dunn presented the dual objects as *polarities*, i.e. relations between sets. Although there are certainly times when the relational language of polarities is appropriate, there are other times when an equivalent but more algebraic concept of a *pairing* is to be preferred. In particular, we are able to give a more direct characterization of the duals of semilattice Galois connections as *semilattice pairings* (Definition 3.17), compared with the semilattice polarities that were used originally.

Section 4 outlines the Hartonas-Dunn duality for lattices, based on the duality between semilattice Galois connections and semilattice polarities or pairings presented in the preceding section. The spaces dual to lattices are the so-called *lattice polarities* of Definition 4.2 or the equivalent *lattice pairings* of Definition 4.3 (the "L-frames" or "lattice frames" of Hartonas and Dunn). A lattice pairing is a pairing between $\mathfrak{B}$-spaces that satisfies the condition (4.6). Thus Hartonas-Dunn duality is interpreted as a duality between the category $\mathbf{L}$ of lattices and the category $\mathbf{LP}$ of lattice pairings.

Section 5 summarizes the duality for semilattice representations developed by the authors [11, 12] as a generalization of a technique used for distributive quasilattices in [5]. A duality (5.13) between complete and cocomplete categories $\mathbf{A}$ and $\mathbf{X}$ lifts to a duality (5.15) between the category $\widetilde{\mathbf{A}}$ of semilattice representations in $\mathbf{A}$ and the category $\widetilde{\mathbf{X}}$ of $\mathfrak{B}$-continuous representations of $\mathfrak{B}$-spaces in $\mathbf{X}$.

The duality for quasilattices is finally presented as Theorem 6.5. The language of semilattice representations and Płonka sums allows one to identify the category of quasilattices as the category $\widetilde{\mathbf{L}}$. Then the Hartonas-Dunn duality for lattices lifts to a duality between $\widetilde{\mathbf{L}}$ and the category $\widetilde{\mathbf{LP}}$ of $\mathfrak{B}$-continuous representations of $\mathfrak{B}$-spaces in $\mathbf{LP}$. Thus the objects of $\widetilde{\mathbf{LP}}$, the spaces dual to quasilattices, are the total spaces or bundles of sheaves of lattice pairings over $\mathfrak{B}$-spaces. A small example is worked in §6.3.

1.3. **Some notational conventions.** In general, notation that is not otherwise explicitly identified here will follow the conventions of [13]. In particular, in primal situations, we normally use algebraic notation placing functors and functions to the right of, or as superfixes to, their arguments. "Left-handed" Eulerian notation, with functions composed in backwards order, is then reserved for dual situations. In an ordered set $(X, \leq)$, we write $x^{\leq}$ for the "up-set" $\{y \in X \mid x \leq y\}$ of an element x, and $x^{\geq}$ for the "down-set" or subordinate subset $\{y \in X \mid x \geq y\}$.

An adjunction

$$\mathbf{A}^{\mathsf{op}} \underset{E'}{\overset{D'}{\rightleftarrows}} \mathbf{X}$$

that involves covariant functors D' and E' between a category $\mathbf{X}$ and the dual $\mathbf{A}^{\mathsf{op}}$ of a category $\mathbf{A}$ will be written as a *dual adjunction*

$$\mathbf{A} \underset{E}{\overset{D}{\rightleftarrows}} \mathbf{X} \tag{1.1}$$

involving contravariant functors D and E, with natural isomorphisms

$$\mathbf{A}(A, XE) \cong \mathbf{A}^{\mathsf{op}}(XE, A) \cong \mathbf{X}(X, AD) \tag{1.2}$$

for objects A of $\mathbf{A}$ and X of $\mathbf{X}$. The component

$$\eta_X \colon X \to XED$$

of the unit at an object X of $\mathbf{X}$ is the image of the identity 1_{XE} under the isomorphism (1.2). The component

$$\varepsilon_A \colon A \to ADE$$

of the counit at an object A of $\mathbf{A}$ is the image of the identity 1_{AD} under the isomorphism (1.2). The dual adjunction (1.1) becomes a *dual equivalence* if the unit and counit are natural isomorphisms.

2. Duality for semilattices

2.1. **Semilattices.** We begin by recalling various well-known aspects of semilattices, mainly to establish the terminology and conventions that will be used throughout.

2.1.1. *Semilattices as algebras.* The variety $\mathbf{SL}$ of *semilattices* is the variety of commutative, idempotent semigroups. (Note that in [10], members of $\mathbf{SL}$ were described as "protosemilattices".) The symbol $\mathbf{SL}$ will also be used to denote the category with object class $\mathbf{SL}$, where the morphisms are semigroup homomorphisms.

2.1.2. *Semilattices as posets.* Consider a semilattice $(H, \cdot)$. It becomes a *meet semilattice* $(H, \cdot, \leq)$ when equipped with the order relation

$$x \leq y \ \Leftrightarrow \ x = x \cdot y\,, \tag{2.1}$$

and a *join semilattice* $(H, \cdot, \leq)$ when equipped with the order relation $x \leq y \ \Leftrightarrow \ x \cdot y = y$. Order-theoretically, a meet semilattice is a poset in which each subset $\{x, y\}$ has a greatest lower bound $x \cdot y$, while a join semilattice is a poset in which each subset $\{x, y\}$ has a least upper bound $x \cdot y$.

2.1.3. *Semilattices as categories.* A small category $\mathbf{C}$ becomes a *poset category* if for each pair x, y of objects of $\mathbf{C}$, the condition $|\mathbf{C}(x,y) \cup \mathbf{C}(y,x)| \in \{0,1\}$ holds. The relation $x \leq y \;\Leftrightarrow\; |\mathbf{C}(x,y)| = 1$ then creates a poset $(\mathbf{C}_0, \leq)$ on the object set $\mathbf{C}_0$ of $\mathbf{C}$. Note that in a poset category $\mathbf{C}$, the only isomorphisms are the identity morphisms at objects of $\mathbf{C}$.

A meet semilattice $(H, \cdot, \leq)$ may be construed as a poset category, where $x \cdot y$ is the product of x and y. Dually, a join semilattice $(H, \cdot, \leq)$ may be construed as a poset category where $x \cdot y$ is the coproduct of x and y.

2.2. **Semilattice duality.** Duality for semilattices is treated explicitly in [14, §4.5] [15, p.158], [16, p.28], and also summarized in [11, 12]. It is given by a pair

$$\mathbf{SL} \underset{F}{\overset{C}{\rightleftarrows}} \mathfrak{B} \tag{2.2}$$

of contravariant functors. Objects of the category $\mathfrak{B}$ (i.e., $\mathfrak{B}$-*spaces*) are compact Hausdorff zero-dimensional ("Stone") topological meet semilattices, with both a least element 0 and a greatest element 1, selected by nullary operations. The morphisms of the category are the continuous homomorphisms.

The two-element semilattice $\mathbf{2} = \{0 < 1\}$ is a dualizing object for the duality (2.2). As an object of $\mathbf{SL}$, the dualizing object is the two-element meet semilattice. As an object of $\mathfrak{B}$, the dualizing object is additionally equipped with the discrete topology, along with nullary operations selecting 0 and 1. For a semilattice H, the $\mathfrak{B}$-space HC is defined as the closed subspace $\mathbf{SL}(H, \mathbf{2})$ of the compact product space $\mathbf{2}^H$. Elements of HC are called *characters* of H. For a semilattice homomorphism $f\colon H_1 \to H_2$, one obtains the dual $\mathfrak{B}$-morphism

$$f^C\colon H_2C \to H_1C; \theta \mapsto f\theta\,. \tag{2.3}$$

For a $\mathfrak{B}$-space G, the semilattice GF is the subsemilattice $\mathfrak{B}(G, \mathbf{2})$ of the semilattice reduct of the product $\mathbf{2}^G$. Then the dual semilattice homomorphism

$$f^F\colon G_2F \to G_1F; \theta \mapsto f\theta$$

of a $\mathfrak{B}$-morphism $f\colon G_1 \to G_2$ is obtained in similar fashion to (2.3).

2.3. **The set theory of semilattice duality.** The functors of the duality (2.2) will now be described in set-theoretical terms.

2.3.1. *Characters and walls.* Recall that a subset Θ of a semilattice $(H, \cdot)$ is a *wall* if and only if

$$\forall\, x, y \in H\,,\ (x \cdot y \in \Theta) \ \Leftrightarrow\ (x \in \Theta \text{ and } y \in \Theta)$$

[13, Defn. IV.1.2.2(b)]. In meet semilattices, walls are filters; in join semilattices, they are ideals.

Lemma 2.1. *Suppose that Θ is a subset of a semilattice $(H, \cdot)$. Then the characteristic function of Θ is a character of $(H, \cdot)$ if and only if Θ is a wall of $(H, \cdot)$.*

Let HW be the set of walls of a semilattice $(H, \cdot)$. Then under intersection, HW forms a subsemilattice of the power set of H. Let $f\colon H_1 \to H_2$ be a morphism of $\mathbf{SL}$. There is a well-defined function

$$f^W\colon H_2W \to H_1W; \Theta \mapsto f^{-1}(\Theta)\,.$$

Proposition 2.2. *The specifications given above yield a contravariant functor $W\colon \mathbf{SL} \to \mathfrak{B}$. There is a natural isomorphism $\gamma\colon C \to W$ with component*

$$\gamma_H\colon (HC, \cdot, \leq, 0, 1) \to (HW, \cap, \subseteq, \emptyset, H); \theta \mapsto \theta^{-1}(1)$$

at a semilattice H.

Proof. Compare [10, Prop. II.2.4(ii)], recalling that the least element of HW is the empty wall. □

2.3.2. *Compact elements.* As posets, $\mathfrak{B}$-spaces are complete lattices [10, p.39]. Let G denote a $\mathfrak{B}$-space. A subset X of G is a *cover* of an element g of G if and only if $g \leq \sup X$.

Definition 2.3. An element c of a $\mathfrak{B}$-space is *compact* if it is nonzero, and if each cover of c contains a finite cover.

Recall that a lattice is *algebraic* if it is complete, and if each element is the supremum of the set of compact elements that it dominates. For the following, compare [10, Cor. II.3.6].

Proposition 2.4. *$\mathfrak{B}$-spaces are algebraic lattices.*

If G is a $\mathfrak{B}$-space, let GK denote the set of compact elements of G. Let $f\colon G_1 \to G_2$ be a morphism of $\mathfrak{B}$. Then there is a well-defined function

$$f^K\colon G_2K \to G_1K; c \mapsto \inf f^{-1}(c^{\leq})\,.$$

Proposition 2.5. *The specifications given above yield a contravariant functor $K\colon \mathfrak{B} \to \mathbf{SL}$. Then there is a natural isomorphism $\kappa\colon F \to K$ with component*

$$\kappa_G\colon GF \to GK; \theta \mapsto \inf \theta^{-1}(1) \tag{2.4}$$

at a $\mathfrak{B}$-space G.

Proof. Compare [10, Th. II.3.7 and Prop. II.3.20]. □

2.3.3. *Equivalence using the set-theoretic functors.*

Definition 2.6. Let h be an element of a semilattice H. Then the *principal wall* $[h]$ is defined as the intersection of all the walls of H that contain h.

Making use of the set-theoretic functors defined above, the natural isomorphisms establishing the duality for semilattices may now be presented as follows.

Proposition 2.7. *Let H be a semilattice. An element Θ of HW is compact if and only if it is a principal wall. Then there is a natural isomorphism $\varepsilon\colon 1_{\mathbf{SL}} \to WK$ with component*

$$\varepsilon_H\colon (H,\leq) \to (HWK,\supseteq); h \mapsto [h]$$

at a semilattice H.

Proof. Compare [10, Prop. II.3.8]. In particular, note that

$$\begin{aligned} x \leq y &\Leftrightarrow\ \forall\ \theta \in HC\,,\ x\theta = 1 \Rightarrow y\theta = 1 \\ &\Leftrightarrow\ \forall\ \Theta \in HW\,,\ x \in \Theta \Rightarrow y \in \Theta \\ &\Leftrightarrow\ \bigcap_{x\in\Theta\in HW} \Theta \supseteq \bigcap_{y\in\Theta\in HW} \Theta \\ &\Leftrightarrow\ [x] \supseteq [y] \end{aligned}$$

for all x, y in a semilattice H. □

Corollary 2.8. *Let H be a semilattice. Then*

$$\widetilde{\varepsilon}_H\colon (H,\leq) \to \left(2^{HW},\subseteq\right); h \mapsto [h]^{\subseteq} = \{\Theta \in HW \mid h \in \Theta\} \tag{2.5}$$

is an order-preserving embedding.

Proof. The argument used for the proof of Proposition 2.7 may be modified to read as

$$\begin{aligned} x \leq y &\Leftrightarrow\ \forall\ \theta \in HC\,,\ x\theta = 1 \Rightarrow y\theta = 1 \\ &\Leftrightarrow\ \forall\ \Theta \in HW\,,\ x \in \Theta \Rightarrow y \in \Theta \\ &\Leftrightarrow\ \{\Theta \mid x \in \Theta\} \subseteq \{\Theta \mid y \in \Theta\} \end{aligned}$$

for elements x, y of a semilattice H. □

Proposition 2.9. *There is a natural isomorphism* $\eta\colon 1_{\mathfrak{B}} \to KW$ *with component*

$$\eta_G\colon G \to GKW; g \mapsto GK \cap g^{\geq} \tag{2.6}$$

at a $\mathfrak{B}$*-space* G.

Proof. Compare [10, Prop. II.3.9], noting that the zero element of G in (2.6) maps to the empty wall, since each compact element of G is strictly bigger than zero. □

2.3.4. *The topology of spaces dual to semilattices.* For the following result, which will be needed in the subsequent chapter, compare [17, §2.1], [10, Prop. II.2.4(i)]. The notation of (2.5) is used.

Proposition 2.10. *Let* H *be a semilattice. Then the set*

$$\{h\widetilde{\varepsilon}_H \mid h \in H\} \cup \{HW \smallsetminus h\widetilde{\varepsilon}_H \mid h \in H\}$$

forms a subbasis of clopen sets for the $\mathfrak{B}$*-topology on* HW.

3. The pairing/Galois connection duality

In their 1993 preprint [17], Hartonas and Dunn described respective dualities between polarities and two forms of Galois connections: dual adjunctions of posets, and dual adjunctions of semilattices. We will summarize the latter duality, since it underlies duality for lattices. The polarities that appear here are two-sorted relational structures. As such, they do not always fit well into our algebraic approach. Thus we will reinterpret them as dual pairings between semilattices, with values in the two-element semilattice.

3.1. **Polarities, pairings, and Galois connections.**

3.1.1. *Semilattice Galois connections.*

Definition 3.1. A (*semilattice*) *Galois connection* is a dual adjunction

$$(P, \leq) \underset{S}{\overset{R}{\rightleftarrows}} (Q, \leq) \tag{3.1}$$

between meet semilattices.

Remark 3.2. Let a be an element of a *relatively pseudo-complemented lattice* $(X, +, \times, \to)$ [18, §I.12], [19, §IV.1]. Define the maps

$$R(a)\colon X \to X; x \mapsto (a \times x) \quad \text{and} \quad S(a)\colon X \to X; x \mapsto (a \to x)\,.$$

Then

$$(X, \leq) \underset{S(a)}{\overset{R(a)}{\rightleftarrows}} (X, \leq)$$

is a semilattice Galois connection. In this case, the lattice X is distributive [13, p.267], [18, §I.12.1]. Galois connections between distributive lattices were studied by Orłowska and Rewitzky [20, §4].

The counit and unit of the dual adjunction (3.1) are the relationships

$$p \leq pRS \quad \text{and} \quad q \leq qSR \tag{3.2}$$

for elements p of P and q of Q. Then since R and S reverse orders, one has $pR \geq pRSR$ and $qS \geq qSRS$. On the other hand, setting $p = qS$ and $q = pR$ in (3.2) yields $qS \leq qSRS$ and $pR \leq pRSR$. Thus

$$pR = pRSR \quad \text{and} \quad qS = qSRS \tag{3.3}$$

for $p \in P$ and $q \in Q$.

The definition below follows Hartonas and Dunn [6, 17] in substituting for the standard "closure" terminology of Galois connections [13, p.261], since this standard terminology would clash with the topological notion of closure in the present context.

Definition 3.3. (a) Elements of the subset $QS = \{qS \mid q \in Q\}$ of P are described as *stable*.

(b) Elements of the subset $PR = \{pR \mid p \in P\}$ of Q are described as *stable*.

Proposition 3.4. *In a Galois connection* (3.1)*, the two contravariant functors R and S restrict to a pair*

$$(QS, \leq) \underset{S}{\overset{R}{\rightleftarrows}} (PR, \leq) \tag{3.4}$$

of mutually inverse order-reversing set isomorphisms.

Proof. The mutual inverse relation between the restrictions of R and S is established by (3.3). □

Definition 3.5. The relationship (3.4) is said to be the *Galois correspondence* furnished by the Galois connection (3.1).

3.1.2. *Polarities and pairings.* Polarities are usually defined as two-sorted relational structures (compare [13, III, Ex. 3.3.2(c)]).

Definition 3.6. Let X and Y be sets. Then a *polarity* (X, Y, α) consists of the ordered pair (X, Y), together with a subset α of $X \times Y$, a relation between X and Y. The set X is called the *domain* of the polarity, while the set Y is called the *codomain*.

Note that a polarity (X, Y, α) yields a bipartite graph. The vertex set of the graph is $X \cup Y$, while the edge set is $\{\{x, y\} \mid (x, y) \in \alpha\}$. On the other hand, the bipartite graph structure does not distinguish between the domain and codomain of the polarity. Polarities may be interpreted in terms of Wille's "concepts" [9] or Hardegree's "natural kinds" [21]. For a discussion, see [13, III, Exer. 3.3G].

While Definition 3.6 is standard, it is often convenient to have an equivalent notion that may fit better into the current algebraic context. We use the following.

Definition 3.7. Let X and Y be sets.

(a) A *pairing* $\langle X|Y\rangle$ is a function

$$\langle\ |\ \rangle \colon X \times Y \to \mathbf{2}; (x, y) \mapsto \langle x|y\rangle \tag{3.5}$$

from $X \times Y$ to the two-element lattice $(\mathbf{2}, \vee, \wedge)$ of truth values.

(b) The set X is called the *domain* of the pairing, while the set Y is called the *codomain*.

(c) Given a pairing (3.5), write

$$\langle x|\ \rangle \colon Y \to \mathbf{2}; y \mapsto \langle x|y\rangle$$

for each element x of X.

(d) Given a pairing (3.5), write

$$\langle\ |y\rangle \colon X \to \mathbf{2}; x \mapsto \langle x|y\rangle$$

for each element y of Y.

Pairings such as (3.5) correspond to the characteristic functions of the relations α in polarities (X, Y, α). In general, one often conflates subsets of a set with their characteristic functions, as is implicit in the notation 2^P for the power set of a set P. Nevertheless, if Θ is a wall of a meet semilattice $(P, \leq)$, we will continue to use θ as our notation for the corresponding character, namely the characteristic function of Θ.

3.1.3. *Galois connections from pairings.* With different conventions, in particular using polarities rather than pairings, the following result is standard. For a proof, compare [13, §III.3.3], [22, §§V.7–8]. Herrlich and Hušek refer to "Galois connections of the first kind" in this context [23, §1].

Proposition 3.8. *Let X and Y be sets. Then a pairing $\langle X|Y\rangle$ yields a semilattice Galois connection*

$$\big(\mathbf{Set}(X,\mathbf{2}),\leq\big) \underset{S}{\overset{R}{\rightleftarrows}} \big(\mathbf{Set}(Y,\mathbf{2}),\leq\big)\,, \tag{3.6}$$

with

$$R\colon \mathbf{Set}(X,\mathbf{2}) \to \mathbf{Set}(Y,\mathbf{2}); \theta \mapsto \bigwedge_{x\theta=1} \langle x|\ \ \rangle$$

and

$$S\colon \mathbf{Set}(Y,\mathbf{2}) \to \mathbf{Set}(X,\mathbf{2}); \varphi \mapsto \bigwedge_{y\varphi=1} \langle\ \ |y\rangle\,.$$

Here, the restricted maps R and S of the Galois correspondence

$$\big(\mathbf{Set}(Y,\mathbf{2})S,\leq\big) \underset{S}{\overset{R}{\rightleftarrows}} \big(\mathbf{Set}(X,\mathbf{2})R,\leq\big)$$

are anti-isomorphisms of complete lattices.

Remark 3.9. In the context of Proposition 3.8, the structures $\big(\mathbf{Set}(X,\mathbf{2}),RS\big)$ and $\big(\mathbf{Set}(Y,\mathbf{2}),SR\big)$ are closure systems. As such, they may each be construed as forming a *topological Boolean algebra*, a concept which was central to much of the work of Rasiowa [18, §III.1], [19, §VI.5].

3.1.4. *Pairings from Galois connections.*

Lemma 3.10. *Within a semilattice Galois connection* (3.1)*, consider characters θ of P and φ of Q. Then the following two conditions are equivalent:*

(a) $\exists\ p \in P\,.\ p\theta = 1$ *and* $pR\varphi = 1$;
(b) $\exists\ q \in Q\,.\ qS\theta = 1$ *and* $q\varphi = 1$.

Proof. Suppose that (a) holds. Take $q = pR$. Then $q\varphi = 1$. Also $qS = pRS \geq p$ by (3.2). Since $\theta\colon (P,\leq) \to (\mathbf{2},\leq)$ is a semilattice homomorphism, it follows that $qS\theta = pRS\theta \geq p\theta = 1$, so $qS = 1$ and (b) holds. The proof that (b) implies (a) is similar. □

Corollary 3.11. *For a semilattice Galois connection* (3.1)*, the equation*

$$\bigvee_{p\in P} p(\theta \wedge R\varphi) = \bigvee_{q\in Q} q(S\theta \wedge \varphi)$$

holds for all $\theta \in PC$ and $\varphi \in QC$.

Definition 3.12. Consider a semilattice Galois connection (3.1).

(a) Define

$$\langle\theta|\varphi\rangle = \bigvee_{p\in P} p(\theta \wedge R\varphi) = \bigvee_{q\in Q} q(S\theta \wedge \varphi) \tag{3.7}$$

for $\theta \in PC$ and $\varphi \in QC$.

(b) Write $\langle PC|QC\rangle_{R,S}$ for the pairing given by (3.7).

Set-theoretically, the pairing (3.7) corresponds to the polarity

$$(\Theta, \Phi) \in \alpha \ \Leftrightarrow \ \exists\, p \in \Theta\,.\ pR \in \Phi \tag{3.8}$$

with domain PW and codomain QW (compare [6, p.392], [17, p.11]). Consider the corresponding Galois connection

$$\big(2^{PW}, \subseteq\big) \underset{2^{SW}}{\overset{2^{RW}}{\rightleftarrows}} \big(2^{QW}, \subseteq\big)\,, \tag{3.9}$$

of the form (3.6) obtained from the polarity (PW, QW, α) according to Proposition 3.8. (Here, the functors are renamed to distinguish them from the original version as it appears in Proposition 3.15.) The following result (cf. [17, Lemma 3.2]) uses the notation of Corollary 2.8.

Proposition 3.13. *The Galois connection restricts to the pair*

$$\big(P\widetilde{\varepsilon}_P, \subseteq\big) \underset{SWK}{\overset{RWK}{\rightleftarrows}} \big(Q\widetilde{\varepsilon}_Q, \subseteq\big)\,.$$

of order-reversing functions, together constituting a semilattice Galois connection.

3.1.5. *The semilattice Galois connection category.* For the following, compare [17, p.12].

Definition 3.14. For semilattice Galois connections

$$(P, \leq) \underset{S}{\overset{R}{\rightleftarrows}} (Q, \leq) \quad \text{and} \quad (P', \leq) \underset{S'}{\overset{R'}{\rightleftarrows}} (Q', \leq)\,,$$

a *semilattice Galois connection morphism* is a pair

$$(f\colon P \to P', g\colon Q \to Q')$$

of semilattice homomorphisms such that the diagram

$$\begin{array}{ccc} (P, \leq) & \underset{S}{\overset{R}{\rightleftarrows}} & (Q, \leq) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (P', \leq) & \underset{S'}{\overset{R'}{\rightleftarrows}} & (Q', \leq)\,. \end{array}$$

commutes, in the sense that $Rg = fR'$ and $Sf = gS'$.

Semilattice Galois connection morphisms combine to constitute a category **SLGC** of semilattice Galois connections. For the following result, again compare [17, Lemma 3.2].

Proposition 3.15. *In the context of Proposition 3.13, the diagram*

$$\begin{array}{ccc} (P,\leq) & \underset{S}{\overset{R}{\rightleftarrows}} & (Q,\leq) \\ \widetilde{\varepsilon}_P \downarrow & & \downarrow \widetilde{\varepsilon}_Q \\ \big(P\widetilde{\varepsilon}_P,\subseteq\big) & \underset{SWK}{\overset{RWK}{\rightleftarrows}} & \big(Q\widetilde{\varepsilon}_Q,\subseteq\big)\,. \end{array} \tag{3.10}$$

represents an isomorphism of semilattice Galois connections.

3.2. Pairings dual to semilattice Galois connections.

3.2.1. *Semilattice polarities and pairings.* If $(X,\leq)$ is a $\mathfrak{B}$-space, set

$$X^* = \{k^{\leq} \mid k \in XK\}\,. \tag{3.11}$$

Note that the elements of X^* are clopen [10, Th. 3.3(3)]. Indeed, by Proposition 2.10 and semilattice duality, it follows that

$$\{k^{\leq} \mid k \in XK\} \cup \{X \smallsetminus k^{\leq} \mid k \in XK\}$$

is a subbasis for the topology on X, and that the set (3.11) is closed under intersection. (Compare the definition of "FSpaces" in [6] or "F_1Spaces" in [17].) The following, somewhat indirect definition recalls the "($\perp$-)frames" of [6].

Definition 3.16. A polarity (X,Y,α) is a *semilattice polarity* if X and Y are $\mathfrak{B}$-spaces, and if the induced semilattice Galois connection

$$\big(2^X,\subseteq\big) \underset{S}{\overset{R}{\rightleftarrows}} \big(2^Y,\subseteq\big) \tag{3.12}$$

restricts to a semilattice Galois connection

$$(X^*,\subseteq) \underset{S}{\overset{R}{\rightleftarrows}} (Y^*,\subseteq) \tag{3.13}$$

between the corresponding sets (3.11) of clopen subsets of X and Y.

For more direct conditions, Definition 3.16 may be rephrased in terms of pairings, making use of appropriately dualized versions of the formulas for R and S from Proposition 3.8.

Definition 3.17. A pairing $\langle X|Y\rangle$ between $\mathfrak{B}$-spaces X and Y is said to be a *semilattice pairing* if the conditions

$$\forall\ p \in \mathfrak{B}(X, \mathbf{2}), \quad \bigwedge_{p\chi=1} \langle \chi|\quad \rangle \in \mathfrak{B}(Y, \mathbf{2})$$

and

$$\forall\ q \in \mathfrak{B}(X, \mathbf{2}), \quad \bigwedge_{q\psi=1} \langle\quad |\psi\rangle \in \mathfrak{B}(X, \mathbf{2})$$

are satisfied.

3.2.2. *Categories of semilattice polarities and pairings.* The following definition recalls [17, pp.12–13].

Definition 3.18. Suppose that (X_1, Y_1, α_1) and (X_2, Y_2, α_2) are semilattice polarities. Then a *semilattice polarity morphism*

$$(f, g)\colon (X_1, Y_1, \alpha_1) \to (X_2, Y_2, \alpha_2) \tag{3.14}$$

is a pair of $\mathfrak{B}$-morphisms $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$ such that

$$\begin{array}{ccc} (X_2^*, \subseteq) & \underset{S_2}{\overset{R_2}{\rightleftarrows}} & (Y_2^*, \subseteq) \\ {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle g^{-1}} \\ (X_1^*, \subseteq) & \underset{S_1}{\overset{R_1}{\rightleftarrows}} & (Y_1^*, \subseteq)\,. \end{array}$$

is a semilattice Galois connection morphism between the corresponding analogues of (3.13) for the codomain and domain of (f, g).

Much as the somewhat indirect Definition 3.16 of a semilattice polarity was reformulated as the more direct Definition 3.17 of a semilattice pairing, so the indirect Definition 3.18 of a semilattice polarity morphism may be reformulated as follows.

Definition 3.19. Suppose that $\langle X_1|Y_1\rangle_1$ and $\langle X_2|Y_2\rangle_2$ are semilattice pairings. Then a *semilattice pairing morphism*

$$(f, g)\colon \langle X_1|Y_1\rangle_1 \to \langle X_2|Y_2\rangle_2$$

is a pair of $\mathfrak{B}$-morphisms

$$f\colon X_1 \to X_2; \chi_1 \mapsto f(\chi_1)$$

and

$$g\colon Y_1 \to Y_2; \psi_1 \mapsto g(\psi_1)$$

such that

$$\bigwedge_{p_2 f(\chi_1)=1} \langle \chi_1|\psi_1\rangle_1 = \bigwedge_{p_2\chi_2=1} \langle \chi_2|g(\psi_1)\rangle_2$$

for all $p_2 \in X_2F$ and $\psi_1 \in Y_1$, and dually

$$\bigwedge_{q_2 g(\psi_1)=1} \langle \chi_1 | \psi_1 \rangle_1 = \bigwedge_{q_2 \psi_2 = 1} \langle f(\chi_1) | \psi_2 \rangle_2$$

for all $q_2 \in Y_2F$ and $\chi_1 \in X_1$.

The semilattice polarity morphisms specified in Definition 3.18, or equivalently the semilattice pairing morphisms that are specified in Definition 3.19, combine to form a category **SLP** of semilattice polarities or semilattice pairings. One may choose between the two equivalent formulations, set-theoretic or algebraic, according to which is more appropriate for the task at hand. For example, within the proof of the duality of Theorem 3.20 given below, the counit is described in the set-theoretic language of semilattice polarities, while the unit is described in the algebraic language of semilattice pairings.

Definition 3.16 is set up to make the assignment

$$\Gamma\colon (X, Y, \alpha) \mapsto \Big[\ (X^*, \subseteq) \underset{S}{\overset{R}{\rightleftarrows}} (Y^*, \subseteq)\ \Big] \tag{3.15}$$

the object part of a functor $\Gamma\colon \mathbf{SLP} \to \mathbf{SLGC}$. The object part of a functor $\Pi\colon \mathbf{SLGC} \to \mathbf{SLP}$ is similarly given by the assignment

$$\Pi\colon \Big[\ (P, \leq) \underset{S}{\overset{R}{\rightleftarrows}} (Q, \leq)\ \Big] \mapsto \big((P, \leq)W, (Q, \leq)W, \alpha\big) \tag{3.16}$$

of the semilattice polarity of (3.8) to the semilattice Galois connection (3.1).

3.2.3. *Pairing/Galois connection duality.*

Theorem 3.20. *A dual equivalence*

$$\mathbf{SLGC} \underset{\Gamma}{\overset{\Pi}{\rightleftarrows}} \mathbf{SLP}$$

is provided by the functors Π *and* Γ.

Proof. The component of the counit at a semilattice Galois connection (3.1) is given by the isomorphism (3.10). For the unit, consider a semilattice pairing $\langle X|Y \rangle$. Application of the (pairing version of the) functor Γ yields a semilattice Galois connection

$$XF \underset{S}{\overset{R}{\rightleftarrows}} YF \tag{3.17}$$

with

$$R\colon p \mapsto \bigwedge_{p\chi=1} \langle \chi | \quad \rangle \quad \text{and} \quad S\colon q \mapsto \bigwedge_{q\psi=1} \langle \quad | \psi \rangle \,.$$

In turn, application of Π to (3.17) by means of Definition 3.12(a) yields a pairing $\langle\!\langle XFC | YFC \rangle\!\rangle$. Identifying XFC with X and YFC with Y by semilattice duality, one then has

$$\begin{aligned} \langle\!\langle \theta | \varphi \rangle\!\rangle = 1 \;\Leftrightarrow\; & \bigvee_{p \in XF} p(\theta \wedge R\varphi) = 1 \\ & \Leftrightarrow\; \exists\, p \in XF \,.\, p\theta = 1 \;\text{ and } pR\varphi = 1 \;\Leftrightarrow \\ & \exists\, p \in XF \,.\, p\theta = 1 \;\text{ and }\; \bigwedge_{p\chi=1} \langle \chi | \varphi \rangle = 1 \end{aligned} \tag{3.18}$$

for $\theta \in X$ and $\varphi \in Y$. Now the condition (3.18) certainly implies $1 = \bigwedge_{p\chi=1} \langle \chi | \varphi \rangle \leq \langle \theta | \varphi \rangle$, so $\langle \theta | \varphi \rangle = 1$. Conversely, suppose $\langle \theta | \varphi \rangle = 0$. It follows that $p\theta = 0$ or $\bigwedge_{p\chi=1} \langle \chi | \varphi \rangle = 0$ for all p in XF. Indeed, if $p\theta = 1$, then $\bigwedge_{p\chi=1} \langle \chi | \phi \rangle \leq \langle \theta | \varphi \rangle = 0$. Thus

$$\langle\!\langle \theta | \varphi \rangle\!\rangle = 1 \;\Leftrightarrow\; \langle \theta | \varphi \rangle = 1 \;:$$

The pairings $\langle\!\langle X | Y \rangle\!\rangle$ and $\langle X | Y \rangle$ coincide. □

4. Duality for lattices

4.1. **Lattices.**

4.1.1. *Semilattice reducts of lattices.* Order-theoretically, a lattice H is defined as a poset $(H, \leq)$ in which each subset $\{x, y\}$ has a greatest lower bound $x \times y$ and a least upper bound $x + y$. In terms of category theory, H is a poset category with binary products $x \times y$ and binary coproducts $x + y$. The poset $(H, \leq)$ is equal to the poset reduct of the meet semilattice $(H, \times, \leq_\times)$, and to the poset reduct of the join semilattice $(H, +, \leq_+)$. By taking the dual of the latter order, one may consider $(H, +, \geq_+)$ as a meet semilattice, since

$$x \geq_+ y \;\Leftrightarrow\; y \leq_+ x \;\Leftrightarrow\; x = x + y$$

— compare (2.1). The absorption property

$$x \leq_\times y \;\Leftrightarrow\; x \leq y \;\Leftrightarrow\; x \leq_+ y$$

of the lattice H may be formulated as

$$x \leq_\times y \;\Leftrightarrow\; y \geq_+ x$$

for all $x, y \in H$. This is the statement that the identity function 1_H on the set H furnishes order-reversing functions or contravariant functors that yield a semilattice Galois connection

$$(H, \leq_\times) \underset{1_H}{\overset{1_H}{\rightleftarrows}} (H, \geq_+) \tag{4.1}$$

from the meet semilattice reduct of H to the dual of its join semilattice reduct. When working with (4.1), it is sometimes helpful to write

$$x = x_\times = x_+ \tag{4.2}$$

for an element x of H, using $x_\times$ for x as a member of the meet semilattice $(H, \leq_\times)$, and x_+ for x as a member of the meet semilattice $(H, \geq_+)$.

4.1.2. *Polarities and pairings from lattices.* Given a lattice $(H, +, \times)$, application of the functor (3.16) to the semilattice Galois connection (4.1) yields a semilattice polarity

$$\big((H, \leq_\times)W, (H, \geq_+)W, \alpha\big) \tag{4.3}$$

with

$$(\Theta, \Phi) \in \alpha \ \Leftrightarrow \ \Theta \cap \Phi \neq \varnothing$$

[6, p.394], [17, p.19]. Equivalently, it yields the semilattice pairing $\big\langle (H, \leq_\times)C \big| (H, \geq_+)C \big\rangle$ with

$$\langle \theta | \phi \rangle = \bigvee_{h \in H} (h_\times \theta \wedge h_+ \varphi)$$

for characters θ of $(H, \leq_\times)C$ and φ of $(H, \geq_+)$, using the notation of (4.2). For the equivalence, note that

$$h \in \Theta \cap \Phi \ \Leftrightarrow \ h_\times \theta = 1 \ \text{ and } \ h_+ \varphi = 1$$

for each element h of H, with $\Theta = \theta^{-1}(1)$ and $\Phi = \varphi^{-1}(1)$ under the usual convention recalled at the end of §3.1.2.

To define one of the contravariant functors that constitute lattice duality, we will write

$$F \colon (H, +, \times) \mapsto \big((H, \leq_\times)W, (H, \geq_+)W, \alpha\big)$$

for the assignment of (4.3) or the equivalent semilattice pairing

$$\big\langle (H, \leq_\times)C \big| (H, \geq_+)C \big\rangle$$

to the lattice $(H, +, \times)$. Now if $f \colon (H_1, +, \times) \to (H_2, +, \times)$ is a lattice homomorphism, it follows that

(4.4)

$$\Big(\big(f^W \colon (H_2, \leq_\times)W \to (H_1, \leq_\times)W\big), \big(f^W \colon (H_2, \geq_+)W \to (H_1, \geq_+)W\big)\Big)$$

is a semilattice polarity homomorphism [6, Prop. 2.14].

4.2. **Representation and duality of lattices.**

4.2.1. *Representations of lattices.* Consider a lattice $(H,+,\times)$, with semilattice polarity (4.3). Use of the construction from §3.2.1 provides a set-theoretical representation of the lattice (compare [6, Th. 2.4], [17, Prop. 4.3, Th. 4.5]). Take X as the $\mathfrak{B}$-space $(H,\leq_\times)W$. Take Y as the $\mathfrak{B}$-space $(H,\geq_+)W$. Take R and S from (3.12).

Theorem 4.1. *With the closure operator RS on the complete Boolean algebra 2^X, the lattice H is isomorphic via*

$$\widetilde{\varepsilon}\colon h\mapsto \{\Theta\in(H,\leq_\times)W \mid h\in\Theta\}$$

to the lattice of clopen, stable subsets of X. One has

$$(x\times y)\widetilde{\varepsilon}=x\widetilde{\varepsilon}\cap y\widetilde{\varepsilon} \quad \text{and} \quad (x+y)\widetilde{\varepsilon}=\big(x\widetilde{\varepsilon}R\cap y\widetilde{\varepsilon}R\big)S$$

for elements x and y of H.

In anticipation of the second contravariant functor that constitutes lattice duality (see Theorem 4.9 below), we will write

$$\Delta\colon \big((H,\leq_\times)W,(H,\geq_+)W,\alpha\big)\mapsto(H,+,\times) \tag{4.5}$$

for the recovery of the lattice $(H,+,\times)$ from (4.3) or the equivalent semilattice pairing

$$\big\langle (H,\leq_\times)C\big|(H,\geq_+)C\big\rangle$$

by means of Theorem 4.1.

4.2.2. *Lattice polarities and pairings.* The following definition recalls the "L-frames" or "lattice frames" of [6, Defn. 2.3], [17, Defn. 4.6].

Definition 4.2. A semilattice polarity (X,Y,α) as in Definition 3.16 is a *lattice polarity* if the semilattice Galois connection (3.13) is a dual equivalence.

Since the only isomorphisms in poset categories are the identities, the condition in Definition 4.2 means that the order-reversing functions R and S of (3.13) are mutually inverse. Definition 4.2 may be rephrased in terms of pairings.

Definition 4.3. A semilattice pairing $\langle X|Y\rangle$ as in Definition 3.17 is a *lattice pairing* if

$$p=\bigwedge_{q\psi=1}\langle\ \ |\psi\rangle \ \Leftrightarrow\ q=\bigwedge_{p\chi=1}\langle\chi|\ \ \rangle \tag{4.6}$$

for all p in $\mathfrak{B}(X,\mathbf{2})$ and q in $\mathfrak{B}(Y,\mathbf{2})$.

In terms of the order-reversing functions R and S of (3.13), the condition in Definition 4.3 means that $p = qS$ if and only if $q = pR$. The condition may also be reformulated at an elementary level.

Lemma 4.4. *The condition*

$$\forall\ p \in \mathfrak{B}(X, \mathbf{2})\,,\ \forall\ q \in \mathfrak{B}(Y, \mathbf{2})\,,\ \forall\ \theta \in X\,,\ \forall\ \varphi \in Y\,,$$
$$p\theta = \bigwedge_{q\psi=1} \langle \theta | \psi \rangle \ \Leftrightarrow\ q\varphi = \bigwedge_{p\chi=1} \langle \chi | \varphi \rangle$$

is necessary and sufficient for a semilattice pairing $\langle X|Y \rangle$ *to be a lattice pairing.*

4.2.3. *Lattice duality.* Define $\mathbf{L}$ to be the category of lattices and lattice homomorphisms. Let $\mathbf{LP}$ be the full subcategory of $\mathbf{SLP}$ whose object class is the class of all lattice polarities or pairings.

Proposition 4.5. [6, Lemma 2.6] *If* $(H, +, \times)$ *is a lattice, then the semilattice polarity* $(H, +, \times)\digamma$ *is a lattice polarity.*

Defining $f\digamma$ by (4.4) for a lattice homomorphism $f\colon H_1 \to H_2$, one obtains a contravariant functor $\digamma\colon \mathbf{L} \to \mathbf{LP}$.

Definition 4.6. A lattice polarity (X, Y, α), and the equivalent lattice pairing $\langle X|Y \rangle$, are *canonical* if they are of the form $(H, +, \times)\digamma$ for a lattice $(H, +, \times)$.

In the terminology of [24, Def'n. 12.5], the following result states that the contravariant functor $\digamma\colon \mathbf{L} \to \mathbf{SLP}$, along with its corestriction $\digamma\colon \mathbf{L} \to \mathbf{LP}$, is dense.

Proposition 4.7. [6, Prop. 2.12] *In the category* $\mathbf{SLP}$*, and therefore also in the category* $\mathbf{LP}$*, each lattice polarity/pairing is isomorphic to a canonical lattice polarity/pairing.*

For the following, see [6, pp.399–401].

Proposition 4.8. *The contravariant functor* $\digamma\colon \mathbf{L} \to \mathbf{LP}$ *is full and faithful.*

By [25, Th. IV.4.1], Propositions 4.7 and 4.8 yield the following (compare [6, Th. 2.15]).

Theorem 4.9. *There is a dual equivalence*

$$\mathbf{L} \underset{\Delta}{\overset{\digamma}{\rightleftarrows}} \mathbf{LP} \tag{4.7}$$

provided by $\digamma$ *and a contravariant functor* Δ *with object part* (4.5).

5. Duality for semilattice representations

This chapter recalls some basic definitions and results from [11, 12].

5.1. **Płonka sums, sheaves, and bundles.**

5.1.1. *Set representations of semilattices.*

Definition 5.1. Suppose that $(H, \leq)$ is a meet semilattice.

(a) A (*set*) *representation* of $(H, \leq)$ is a contravariant functor

$$R\colon H \to \mathbf{Set} \tag{5.1}$$

from H (considered as a poset category according to §2.1.3) to the category of sets.

(b) A *morphism of set representations* of $(H, \leq)$ is a natural transformation between functors of the form (5.1).

(c) The functor category

$$\widehat{H} = \mathbf{Set}^{H^{\mathsf{op}}} \tag{5.2}$$

is the *category of set representations* of $(H, \leq)$.

5.1.2. *Semilattices as topological spaces.* Suppose that $(H, \leq)$ is a meet semilattice. One may consider H as a topological space under the *Alexandrov topology* $\Omega(H)$ of the dual poset $(H, \geq)$, where the open sets are the subordinate subsets of $(H, \leq)$ [26, II.1.8]. The poset $(H, \leq)$ is then represented as the subposet $\big(\{h^{\geq} \mid h \in H\}, \subseteq\big)$ of $\big(\Omega(H), \subseteq\big)$ consisting of the principal subordinate subsets. Thus the semilattice H is a basis for the Alexandrov topology $\Omega(H)$.

5.1.3. *Sheaves and bundles.* Using topological language, the functor category (5.2) is the category of presheaves on H. Now the elements of H, as principal lower sets in the poset $(\Omega, \subseteq)$, are join-irreducible. Thus the condition [27, II.1(9)] for a presheaf on H to be a sheaf is trivially satisfied. By the Comparison Lemma for Grothendieck topoi [28, Th. I.3.7], [27, Th. II.1.3 and App., Cor. 3(a)], the functor category $\widehat{H}$ is equivalent to the category $\mathrm{Sh}(H)$ of sheaves of sets over the space H under the Alexandrov topology. In turn, there is an equivalence

$$\mathrm{Sh}(H) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathrm{Etale}(H) \tag{5.3}$$

of the category $\mathrm{Sh}(H)$ of sheaves with the category $\mathrm{Etale}(H)$ of étale bundles $\pi\colon E \to H$ over the space H [26, Cor. V.1.5(i)], [27, Cor. II.6.3]. Specifically, for a representation $R\colon H \to \mathbf{Set}$, the bundle $\pi\colon E \to H$, or more loosely the total space E, is the bundle $R\Lambda$ of germs of the sheaf $R\colon \Omega(H) \to \mathbf{Set}$.

5.1.4. *The bundle of a left normal band.* In place of the topological description of the equivalence between set representations $R\colon H \to \mathbf{Set}$ and bundles $\pi\colon E \to H$ given in the preceding two paragraphs, one may give a purely algebraic description.

Definition 5.2. Let $(S, *)$ be a semigroup.

(a) If $(S, *)$ satisfies the identity $x * y = x$, then it belongs to the variety $\mathbf{Lz}$ of *left zero* or *left trivial bands* [29, p.119], [30, 225].
(b) If $(S, *)$ is idempotent and satisfies the identity $x*y*z = x*z*y$, then it belongs to the variety $\mathbf{Ln}$ of *left normal bands* [29, p.119], [30, 223].
(c) If $(E, *)$ is a left normal band, then its *bundle* $\pi\colon E \to H$ is its projection onto its *semilattice replica* $(H, *)$, its largest semilattice quotient [30, p.17].

An isomorphism between the category of sets and the category of left zero semigroups places the projection structure $x * y = x$ on a set. The inverse functor is the forgetful functor $\mathbf{Lz} \to \mathbf{Set}$. We thus identify $\mathbf{Set}$ with $\mathbf{Lz}$.

Suppose that $(E, *)$ is a left normal band. The bundle $\pi\colon E \to H$ yields a representation

$$E\Gamma\colon H \to \mathbf{Set}; h \mapsto \pi^{-1}\{h\}\,, \tag{5.4}$$

well-defined on morphisms by

$$(h \to k)E\Gamma\colon \pi^{-1}\{k\} \to \pi^{-1}\{h\}; x \mapsto x * y$$

for any element y of $\pi^{-1}\{h\}$.

5.1.5. *Płonka sums.* Let $(H, \leq)$ be a meet semilattice. Then a set representation $R\colon H \to \mathbf{Set}$ may be interpreted as a contravariant functor

$$R\colon H \to \mathbf{Lz} \tag{5.5}$$

taking the left projection structure on each set hR for $h \in H$. The functor (5.5) summarizes the data for a construction known as a "strong semilattice" in semigroup theory [29, p.90], and more generally as a *Płonka sum* [30, 236], [31], [32, §6]. Let $E = \sum_{h\in H} hR$ be the coproduct (disjoint union), with $\pi = \sum_{h\in H} \big(hR \to \{h\}\big)$. Defining

$$x * y = x(x^\pi * y^\pi \to x^\pi)^R \tag{5.6}$$

for x, y in E gives the bundle

$$R\Lambda\colon E \to H \tag{5.7}$$

of a left normal band $(E, *)$. The constructions Γ of (5.4) and Λ of (5.7) extend to functors providing an equivalence

$$\widehat{H} \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} (\mathbf{Ln}, H) \tag{5.8}$$

between the presheaf category $\widehat{H}$ for the fixed semilattice H and the comma or slice category $(\mathbf{Ln}, H)$ of left normal bands over H (cf. [25, §II.6]). The equivalence (5.8) is an algebraic analogue of the topological equivalence (5.3).

5.2. **Extending dualities to semilattice representations.**

5.2.1. *Semicolon categories.* The algebraic equivalence (5.8) may be extended. Suppose that $\mathbf{C}$ is a category whose objects are small categories and whose morphisms are functors. Let $\mathbf{D}$ be a category. Define a *semicolon* category $(\mathbf{C}; \mathbf{D})$ as follows. Its objects are covariant functors $R \colon C \to \mathbf{D}$ from an object C of $\mathbf{C}$ to $\mathbf{D}$. Given two such objects $R \colon C \to \mathbf{D}$ and $R' \colon C' \to \mathbf{D}$, a morphism $(\sigma, f) \colon R \to R'$ is a pair consisting of a $\mathbf{C}$-morphism $f \colon C \to C'$ and a natural transformation $\sigma \colon R \to fR'$. The composition of morphisms in $(\mathbf{C}; \mathbf{D})$ is defined by $(\sigma, f)(\tau, g) = (\sigma(f\tau), fg)$ [10, §0.1], [27, Ex. V.2.5(b)]. (Note that Mac Lane used the name "supercomma" and the symbol $\dot{\downarrow}$ in place of the semicolon.)

For a concrete category $\mathbf{D}$, let $(\mathbf{SL}; \mathbf{D}^{\mathsf{op}})'$ denote the full subcategory of $(\mathbf{SL}; \mathbf{D}^{\mathsf{op}})$ comprising the functor $\emptyset \to \mathbf{D}^{\mathsf{op}}$ along with functors from nonempty semilattices to the full subcategory of $\mathbf{D}^{\mathsf{op}}$ that consists of nonempty $\mathbf{D}$-objects. Then the equivalence (5.8) may be extended to the equivalence

$$(\mathbf{SL}; \mathbf{Set}^{\mathsf{op}})' \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Ln} \ . \tag{5.9}$$

Consider a left normal band homomorphism $F \colon E \to E'$, with semilattice replica $f \colon H \to H'$. Define $R = E\Gamma \colon H \to \mathbf{Set}$ and $R' = E'\Gamma \colon H' \to \mathbf{Set}$. A natural transformation $\varphi \colon R \to fR'$ is defined by its components

$$\varphi_h \colon hR \to hfR'; x \mapsto xF \tag{5.10}$$

at elements h of H. The $(\mathbf{SL}; \mathbf{Set}^{\mathsf{op}})'$-morphism $F\Gamma \colon E\Gamma \to E'\Gamma$ is then defined as the pair (φ, f). Conversely, for such a pair forming an $(\mathbf{SL}; \mathbf{Set}^{\mathsf{op}})'$-morphism, a left normal band homomorphism $f = (\varphi, f)\Gamma \colon R\Gamma \to R'\Gamma$ is defined as the coproduct or disjoint union of the components (5.10).

5.2.2. *Strongly irregular varieties.* The fundamental equivalence (5.9) may be lifted to other contexts. Recall that a variety $\mathbf{V}$ of finitary algebras is *strongly irregular* if the $\mathbf{V}$-algebra structure reduces to a left trivial semigroup [32, §4.8]. For example, the variety of lattices is strongly irregular by virtue of the derived operation

$$x * y = x + (x \times y)\,, \tag{5.11}$$

since absorption implies $x * y = x$. The *regularization* $\widetilde{\mathbf{V}}$ of any variety $\mathbf{V}$ of finitary algebras is defined to be the variety of algebras that satisfy each regular identity of $\mathbf{V}$. (An identity is *regular* if it involves exactly the same set of arguments on each side [30, p.13], [33, p.47].) Then Płonka's Theorem describing regularizations of strongly irregular varieties of algebras without nullary operations [30, 239], [31], [32, 4.8 and 7.1] may be formulated as follows.

Theorem 5.3. *The equivalence* (5.9) *lifts to an equivalence*

$$(\mathbf{SL};\mathbf{V}^{\mathsf{op}})' \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \widetilde{\mathbf{V}} \tag{5.12}$$

when $\mathbf{V}$ *is a strongly irregular variety of algebras whose type contains no constants.*

For an operation ω in the context of Theorem 5.3, the analogue of (5.6) is

$$\dots x_i \dots \omega = \dots x_i(\dots x_i^{\pi} \dots \omega \to x_i^{\pi})^R \dots \omega$$

for elements x_i of $E = R\Lambda$.

5.2.3. *Continuous representations.* Consider a dual equivalence

$$\mathbf{A} \underset{E}{\overset{D}{\rightleftarrows}} \mathbf{X} \tag{5.13}$$

between complete and cocomplete categories. Let $\widetilde{\mathbf{A}}$ be the category of representations of meet semilattices in $\mathbf{A}$. Such representations are often implicitly identified with the corresponding bundles $R\Lambda$. For a $\mathfrak{B}$-space G, a representation $R\colon G \to \mathbf{X}$ is said to be $\mathfrak{B}$-*continuous* if

$$gR = \varprojlim \big(R\colon GK \cap g^{\leq} \to \mathbf{X}\big) \tag{5.14}$$

for each element g of G. Here, the limit on the right hand side of (5.14) is the limit of the restriction of R to the upwardly-directed ordered subset of G consisting of compact elements below g. A category $\widetilde{\mathbf{X}}$ is then defined to be the full subcategory of $(\mathfrak{B};\mathbf{X}^{\mathsf{op}})$ consisting of $\mathfrak{B}$-continuous representations of $\mathfrak{B}$-spaces in $\mathbf{X}$. As for the case of $\widetilde{\mathbf{A}}$, such representations are often identified with the corresponding bundles $R\Lambda$.

5.2.4. *Lifting dualities.* Consider a dual equivalence (5.13). Then there is a dual equivalence

$$\widetilde{\mathbf{A}} \underset{\widetilde{E}}{\overset{\widetilde{D}}{\rightleftarrows}} \widetilde{\mathbf{X}} \tag{5.15}$$

for suitable functors $\widetilde{D}$ and $\widetilde{E}$ [11, Th. 4.4], [12, Th. 4.3].

The functor $\widetilde{D}$ is defined as taking a representation $R\colon H \to \mathbf{A}$ to a $\mathfrak{B}$-continuous representation $HC \to \mathbf{X}$ of the semilattice dual HC of H. This $\widetilde{\mathbf{X}}$-object, as a continuous representation, sends a character θ of H to the dual space $\big[\varinjlim(R\colon \theta^{-1}\{1\} \to \mathbf{A})\big]D$ of the colimit of the corresponding restriction. For further details of the functor $\widetilde{D}$, see [12, Prop. 4.1].

Now consider an object $R\colon G \to \mathbf{X}$ of $\widetilde{\mathbf{X}}$. Consider the natural transformation component $\kappa_G\colon GF \to GK$ of (2.4), along with the embedding $j\colon GK \hookrightarrow G$. Then at the object level, the functor $\widetilde{E}$ takes the continuous representation $R\colon G \to \mathbf{X}$ to the representation

$$GF \xrightarrow{\kappa_G} GK \xrightarrow{j} G \xrightarrow{R} \mathbf{X} \xrightarrow{E} \mathbf{A}$$

of the meet semilattice GF. For further details of the functor $\widetilde{E}$, see [12, Prop. 4.2].

6. Duality for quasilattices

6.1. **Quasilattices as Płonka sums.**

Definition 6.1. [2, Lemma 1] A *quasilattice* $(Q,+,\times)$ is an algebra with binary operations $+$ of *join* and $\times$ of *meet*, such that $(Q,+)$ and $(Q,\times)$ are semilattices, and the identities

$$[(x+y)\times z]+[y\times z]=(x+y)\times z\,,$$
$$[(x\times y)+z]\times[y+z]=(x\times y)+z$$

are satisfied.

Recall that a *stammered semilattice* $(Q,\cdot,\cdot)$ is an algebra with two binary operations, each of which is the multiplication of a semilattice $(Q,\cdot)$ [30, 327].

Lemma 6.2. *The class of (stammered) semilattices coincides with the class of quasilattices in which the regular identity*

$$x+y=x\times y$$

is satisfied.

For the following, compare [2, p.184, Footnote (3)]

Proposition 6.3. *The class of lattices coincides with the class of quasilattices in which the irregular identity*

$$x + (x \times y) = x$$

is satisfied.

One obtains the following result as a consequence.

Proposition 6.4. [2, Lemma 3] *An algebra* $(Q, +, \times)$ *with binary operations* $+$ *and* $\times$ *is a quasilattice if and only if it is a Płonka sum of lattices.*

In view of Proposition 6.4 and §5.2.2, we may consider the variety of quasilattices to be the regularization $\widetilde{\mathbf{L}}$ of the strongly irregular variety $\mathbf{L}$ of lattices. In particular, a quasilattice $(X, +, \times)$ is a Płonka sum over its stammered semilattice replica, in the sense of Lemma 6.2.

6.2. **Duality for quasilattices.** Application of the lifting machinery of §5.2 to the Hartonas-Dunn duality for lattices now yields our main theorem.

Theorem 6.5. *The duality*

$$\mathbf{L} \underset{\Delta}{\overset{F}{\rightleftarrows}} \mathbf{LP}$$

of (4.7)*, between lattices and lattice pairings, extends to a duality*

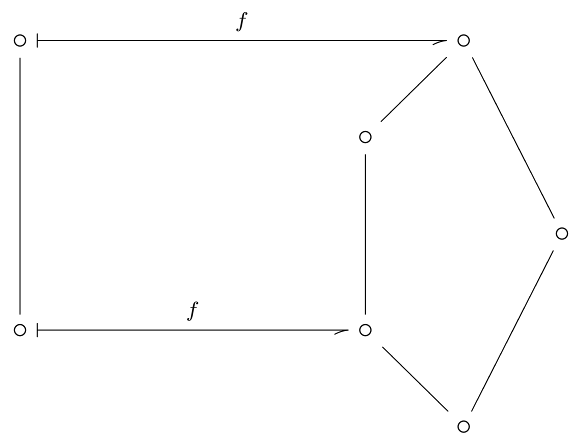
$$\widetilde{\mathbf{L}} \underset{\widetilde{\Delta}}{\overset{\widetilde{F}}{\rightleftarrows}} \widetilde{\mathbf{LP}} \tag{6.1}$$

between quasilattices and $\mathfrak{B}$*-continuous representations of* $\mathfrak{B}$*-spaces in the category* $\mathbf{LP}$ *of lattice pairings.*

Thus the spaces dual to quasilattices are the total spaces or bundles of sheaves of lattice pairings over $\mathfrak{B}$-spaces.

6.3. **An example.** Theorem 6.5 may be illustrated by the 7-element quasilattice $(Q, +, \times)$ given as the Płonka sum of the 2-element lattice $(\mathbf{2}, +, \times)$ and the non-modular 5-element lattice $(N_5, +, \times)$ displayed in (6.2). The semilattice replica is the 2-element semilattice.

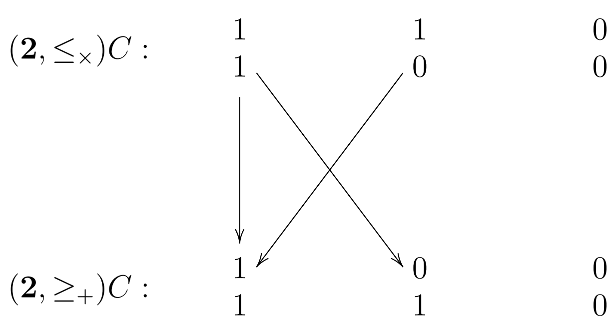
(6.2)



The Płonka sum (6.2) may be summarized by the lattice homomorphism $f\colon \mathbf{2} \to N_5$.

6.3.1. *The dual of the two-element lattice.* The full lattice polarity $(\mathbf{2}, +, \times)F$ dual to $(\mathbf{2}, +, \times)$ is

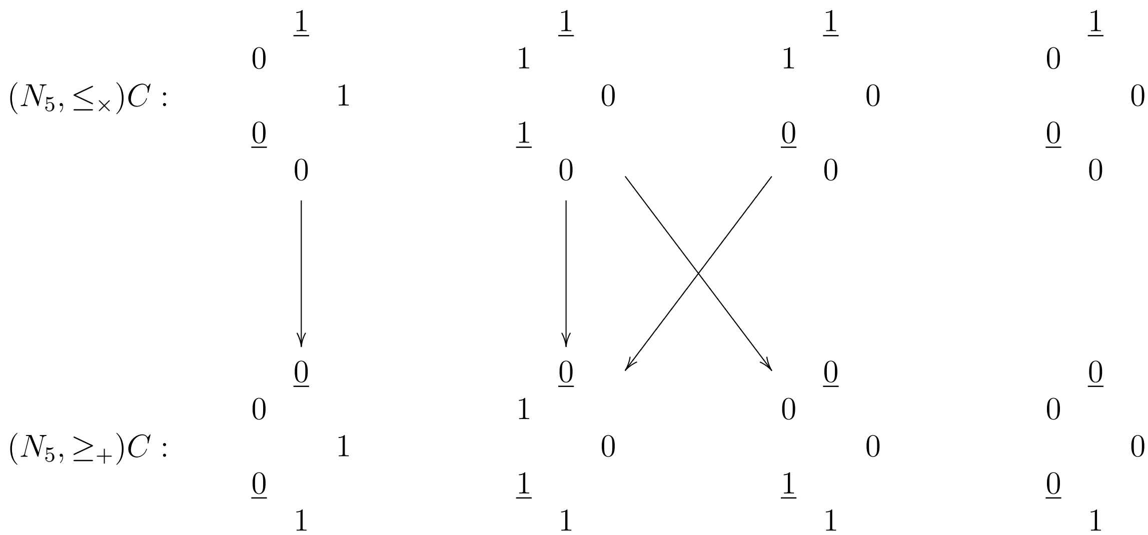
(6.3)



Here, for each of the semilattice reducts $(\mathbf{2}, \leq_\times)$ and $(\mathbf{2}, \geq_+)$ of $(\mathbf{2}, +, \times)$, the rows of (6.3) display the three characters by showing their respective images of the elements of $\mathbf{2}$ in their relative positions on the Hasse diagram of $\mathbf{2}$ that appears on the left of (6.2).

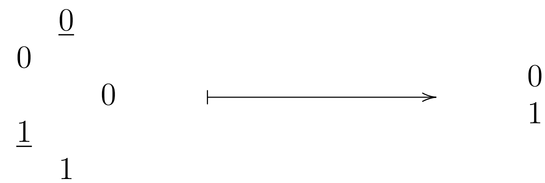
6.3.2. *The dual of the five-element non-modular lattice.* The following diagram shows a reduced and decorated picture of the lattice polarity

$(N_5, +, \times)\mathcal{F}$ dual to the non-modular 5-element lattice $(N_5, +, \times)$.
(6.4)



The decoration, consisting of the underlining of certain elements, is irrelevant for the interpretation of (6.4) as the lattice polarity $(N_5, +, \times)\mathcal{F}$ corresponding to $(N_5, +, \times)$. The reduction consists of the suppression of the two constant characters. Thus for each of the semilattice reducts $(N_5, \leq_\times)$ and $(N_5, \geq_+)$ of $(N_5, +, \times)$, the rows of (6.4) display the non-constant characters by showing the respective images of the elements of N_5 in their relative positions on the Hasse diagram that appears on the right of (6.2). The polarity relations that are not displayed in the reduced diagram (6.4) relate the constant character 1 of $(N_5, \leq_\times)$ to each non-zero character of $(N_5, \geq_+)$, together with relations from each non-zero character of $(N_5, \leq_\times)$ to the constant character 1 of $(N_5, \geq_+)$.

6.3.3. *The dual of the seven-element quasilattice.* The 7-element quasilattice $(Q, +, \times)$ is the Płonka sum determined by the lattice homomorphism $f\colon \mathbf{2} \to N_5$ of (6.2). Thus the non-trivial part of the dual object $(Q, +, \times)\widetilde{\mathcal{F}}$ provided by Theorem 6.5 comprises the **LP**-morphism $f\mathcal{F}\colon N_5\mathcal{F} \to \mathbf{2}\mathcal{F}$. Following (4.4), the **LP**-morphism $f\mathcal{F}$ is given by the underscores that decorate (6.4): Each character displayed in either of the rows of (6.4) is taken to the corresponding character of $\mathbf{2}$ given by the ordered pair of underlined elements appearing within that character. For example, one has



as part of the mapping from $(N_5, \geq_+)C$ to $(\mathbf{2}, \geq_+)C$.

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