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REPRESENTATION THEORY FOR VARIETIES OF COMTRANS ALGEBRAS AND LIE TRIPLE SYSTEMS

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For a variety of comtrans algebras over a commutative ring, representations of algebras in the variety are identified as modules over an enveloping algebra. In particular, a new, simpler approach to representations of Lie triple systems is provided.

Keywords: Comtrans algebra; module; Lie triple system.

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1. Introduction

Comtrans algebras are unital modules over a commutative ring R, equipped with two basic trilinear operations: a *commutator* $[x, y, z]$ satisfying the *left alternative identity*

$$[x, x, y] = 0, \tag{1.1}$$

and a *translator* $\langle x, y, z\rangle$ satisfying the *Jacobi identity*

$$\langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle = 0, \tag{1.2}$$

such that together the commutator and translator satisfy the *comtrans identity*

$$[x, y, x] = \langle x, y, x\rangle. \tag{1.3}$$

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Comtrans algebras were introduced in [9] to answer a problem from differential geometry, asking for the algebraic structure in the tangent bundle corresponding to the coordinate n-ary loop of an $(n+1)$-web (cf. [1]). The role played by comtrans algebras is analogous to the role played by the Lie algebra of a Lie group. A comtrans algebra is said to be *abelian* if its commutator and translator are zero. Thus abelian comtrans algebras are essentially just R-modules. Comtrans algebras arise naturally in many different contexts [3–8, 10].

The class $\mathbf{CT}_R$ or $\mathbf{CT}$ of all comtrans algebras over a commutative ring R forms a variety in the sense of universal algebra. Within this class, a set of comtrans algebra identities specifies a subvariety $\mathbf{V}_R$ or $\mathbf{V}$ (for example, the variety $\mathbf{A}_R$ of abelian comtrans algebras over R, or the variety $\mathbf{L}_R$ of Lie triple systems over R). Each variety $\mathbf{V}_R$ then becomes (the class of objects of) a bicomplete category whose morphisms are the homomorphisms between the comtrans algebras (cf. [12, Theorems IV, 2.1.3 and 2.2.3]).

Let E be an algebra in a variety $\mathbf{V}$ of comtrans algebras over a commutative ring R. A *representation of* E *in* $\mathbf{V}$ is defined as an R-module in the slice category $\mathbf{V}/E$, i.e., an object of the category $\mathbf{A}_R \otimes \mathbf{V}/E$. For a field F, representations of a comtrans algebra E in the full class $\mathbf{CT}_F$ were studied in [6]. The representations were identified as modules over a certain universal enveloping algebra $U(E)$ (see Sec. 2 below). The current paper extends that result to the case of general commutative rings, and obtains a corresponding characterization for representations of comtrans algebras in each subvariety of $\mathbf{CT}_R$.

Section 3 recalls the correspondence between split extensions and representations. Section 4 examines monic comtrans algebras, where the commutator and translator agree. For a monic comtrans algebra E, Theorem 4.2 identifies the universal enveloping algebra as the even part of the tensor algebra $T(E)$. Based on the framework of monic algebras, Sec. 5 then offers a new treatment of the representation theory for Lie triple systems, which may be contrasted with [2]. The approach taken in Secs. 4 and 5 is as explicit as possible. Section 6 gives a general identification of the universal enveloping algebra for an arbitrary variety of comtrans algebras, using a differential calculus within the language of comtrans algebras. As a sample application of this calculus, Sec. 7 specifies a universal enveloping algebra for comtrans algebras defined by bilinear forms (compare [4, 5, 7]).

For concepts and conventions of algebra that have not otherwise been explained in this paper, readers are referred to [12]. Note that the tensor products and tensor algebras appearing, always in the context of modules over a given commutative ring R, are to be taken in the category of R-modules.

2. Universal Enveloping Algebras

Let $\mathbf{V}_R$ be a variety of comtrans algebras over a commutative ring R. For a member E of $\mathbf{V}_R$, let $E_{\mathbf{V}}[X]$ or $E[X]$ denote the coproduct of E in $\mathbf{V}_R$ with the free $\mathbf{V}_R$-algebra on a singleton $\{X\}$. For x, y in E, there are R-module homomorphisms

$K_{\mathbf{V}}(x, y)$ or

$$K(x, y) : E[X] \to E[X]; z \mapsto [z, x, y], \tag{2.1}$$

$R_{\mathbf{V}}(x, y)$ or

$$R(x, y) : E[X] \to E[X]; z \mapsto \langle z, x, y\rangle, \tag{2.2}$$

and $L_{\mathbf{V}}(x, y)$ or

$$L(x, y) : E[X] \to E[X]; z \mapsto \langle y, x, z\rangle. \tag{2.3}$$

These endomorphisms of $E_R[X]$, or their restrictions to endomorphisms of E alone, are known, respectively, as the *commutative*, *right*, and *left adjoint maps*. The *universal enveloping algebra* $U_{\mathbf{V}}(E)$ of E is defined as the R-subalgebra generated by

$$\{L(x, y), K(x, y), R(x, y) \mid x, y \in E\} \tag{2.4}$$

in the endomorphism ring of the R-module $E[X]$. Containment of varieties induces quotients of the corresponding universal enveloping algebras.

Proposition 2.1. *Let R be a commutative ring. Let $\mathbf{V}$ be a variety of comtrans algebras over R, and let $\mathbf{W}$ be a subvariety of $\mathbf{V}$. Let E be a member of $\mathbf{W}$. Then $U_{\mathbf{W}}(E)$ is a quotient of $U_{\mathbf{V}}(E)$.*

Proof. Since $E_{\mathbf{W}}[X]$ is a member of $\mathbf{V}$, a surjective homomorphism θ is specified uniquely from $E_{\mathbf{V}}[X]$ to $E_{\mathbf{W}}[X]$ by the requirements of mapping from X in $E_{\mathbf{V}}[X]$ to X in $E_{\mathbf{W}}[X]$, and by restricting to the identity on E. Then for x, y in E, one has $XK_{\mathbf{V}}(x, y)\theta = [X, x, y]\theta = [X\theta, x\theta, y\theta] = [X, x, y] = XK_{\mathbf{W}}(x, y)$, etc., so $K_{\mathbf{V}}(x, y) \mapsto K_{\mathbf{W}}(x, y)$, $L_{\mathbf{V}}(x, y) \mapsto L_{\mathbf{W}}(x, y)$, $R_{\mathbf{V}}(x, y) \mapsto R_{\mathbf{W}}(x, y)$ induce the required surjective homomorphism $U_{\mathbf{V}}(E) \to U_{\mathbf{W}}(E)$. □

Proposition 2.2. *Within the universal enveloping algebra $U(E)$ of a comtrans algebra E, one has*

$$L(x, x) - K(x, x) + R(x, x) = 0. \tag{2.5}$$

Proof. Apply the left-hand side of (2.5) to an element z of $E[X]$ and simplify by consecutive use of (1.1), (1.3), and (1.2). □

In view of Proposition 2.2, it will be convenient to define the *middle adjoint map*

$$M(x, y) : E[X] \to E[X]; z \mapsto [y, x, z]. \tag{2.6}$$

The comtrans identity gives

$$M(x, x) = 0$$

and

$$M(x,y) + K(x,y) = L(x,y) + R(x,y),$$

so the universal enveloping algebra may be generated by

$$\{L(x,y), M(x,y), R(x,y) \mid x, y \in E\} \tag{2.7}$$

in place of (2.4). This allows the following extension of [6, Theorem 4.5].

Theorem 2.3. *Let E be a comtrans algebra over a commutative ring R. Let*

$$V = (E \otimes E) \oplus (E \wedge E) \oplus (E \otimes E). \tag{2.8}$$

Then $U_{\mathbf{CT}}(E)$ is isomorphic to the tensor algebra on V.

Under the isomorphism of Theorem 2.3, an element

$$(v_{-3} \otimes v_{-2}) \oplus (v_{-1} \wedge v_1) \oplus (v_2 \otimes v_3)$$

of V corresponds to

$$L(v_{-3}, v_{-2}) + M(v_{-1}, v_1) + R(v_2, v_3)$$

in the endomorphism ring of $E[X]$. Primitive elements of the respective summands of (2.8) are then written as $L(x,y) = x \otimes y$, $M(x,y) = x \wedge y$, and $R(x,y) = x \otimes y$ for x, y in E, with

$$K(x,y) = L(x,y) - M(x,y) + R(x,y).$$

These conventions are followed in Secs. 6 and 7.

3. Representations and Modules

This section briefly recalls the relationship between comtrans algebra representations and modules over the universal enveloping algebra in the full variety of all comtrans algebras. Suppose that E is a comtrans algebra over a commutative ring R. A $U_{\mathbf{CT}}(E)$-module V furnishes a comtrans algebra structure $V \rtimes E$ on the module $V \oplus E$, where the V-component of the commutator $[v_1 \oplus e_1, v_2 \oplus e_2, v_3 \oplus e_3]$ is

$$v_1 K(e_2, e_3) - v_2 K(e_1, e_3) + v_3 M(e_2, e_1), \tag{3.1}$$

while the V-component of the translator $\langle v_1 \oplus e_1, v_2 \oplus e_2, v_3 \oplus e_3 \rangle$ is

$$v_1 R(e_2, e_3) - v_2 [L(e_1, e_3) + R(e_3, e_1)] + v_3 L(e_2, e_1). \tag{3.2}$$

Further, the projection $\pi : V \oplus E \to E$ becomes a comtrans algebra homomorphism $\pi : V \rtimes E \to E$, an R-module in the slice category $\mathbf{CT}/E$. Conversely, given an

R-module $\pi : A \to E$ in the slice category $\mathbf{CT}/E$, define $V = \pi^{-1}\{0\}$, and consider the zero section $\Delta : E \to A$. Then an action of $U_{\mathbf{CT}}(E)$ on V is defined by

$$vK(e_1, e_2) = [v, e_1\Delta, 0e_2\Delta],$$
$$vR(e_1, e_2) = \langle v, e_1\Delta, e_2\Delta\rangle,$$

and

$$vL(e_1, e_2) = \langle e_2\Delta, e_1\Delta, v\rangle$$

for v in V and e_1, e_2 in E. These correspondences are part of an equivalence between the category of $U_{\mathbf{CT}}(E)$-modules and the category $\mathbf{A} \otimes \mathbf{CT}/E$ of representations of E [6, Theorem 3.10].

4. Monic Algebras

A comtrans algebra is *monic* if it satisfies the identity

$$[x, y, z] = \langle x, y, z\rangle;$$

in other words, its commutators and translators are equal. The variety of monic comtrans algebras over a ring R will be written as $\mathbf{M}_R$, or simply $\mathbf{M}$. Within $\mathbf{M}$, the commutative and right adjoint maps coincide, as do the middle and left adjoint maps. Furthermore, the left adjoint maps may be expressed in terms of the right adjoint maps.

Proposition 4.1. *Within the variety of monic comtrans algebras,*

$$L(x, y) = R(x, y) - R(y, x). \tag{4.1}$$

Proof. For E in $\mathbf{M}_R$, the Jacobi identity in $E_{\mathbf{M}}[X]$ implies

$$\begin{aligned} 0 &= \langle X, y, x\rangle + \langle y, x, X\rangle + \langle x, X, y\rangle \\ &= \langle X, y, x\rangle + \langle y, x, X\rangle - \langle X, x, y\rangle \\ &= X(R(y, x) + L(x, y) - R(x, y)). \end{aligned}$$

Then (4.1) follows. □

Theorem 4.2. *Let E be a monic comtrans algebra over a ring R. Then the universal enveloping algebra $U_{\mathbf{M}}(E)$ is isomorphic to the even part $T(E)_0 = T(E \otimes E)$ of the tensor algebra $T(E)$ on E.*

Proof. Consider the R-algebra homomorphism $s : T(E)_0 \to U_{\mathbf{M}}(E)$ extending the linear map $E \otimes E \to U_{\mathbf{M}}(E); e_1 \otimes e_2 \mapsto R(e_1, e_2)$. By (4.1), s surjects. In order to show that s injects, a monic comtrans algebra structure $T(E)_0 \rtimes E$ will be built on the module $T(E)_0 \oplus E$ to yield an R-module $\pi : T(E)_0 \rtimes E \to E$ in $\mathbf{M}/E$. The $T(E)_0$-component of the commutator $[v_1 \oplus e_1, v_2 \oplus e_2, v_3 \oplus e_3]$ is defined as

$$v_1 \otimes (e_2 \otimes e_3) - v_2 \otimes (e_1 \otimes e_3) + v_3 \otimes (e_2 \otimes e_1 - e_1 \otimes e_2). \tag{4.2}$$

It is then straightforward to verify the left alternativity and Jacobi identity in the monic algebra $T(E)_0 \rtimes E$. Since

$$[1 \oplus 0, 0 \oplus e_2, 0 \oplus e_3] = (e_2 \otimes e_3) \oplus 0,$$

s is injective. □

Given Theorem 4.2, the various general observations on comtrans algebra representation theory from [6] specialize as follows.

Corollary 4.3. *Let E be a monic comtrans algebra over a ring R.*

(a) *Let V be a $T(E)_0$-module. Then the comtrans algebra structure $V \rtimes E$ given by the commutator*

$$\begin{aligned}[v_1 \oplus e_1, v_2 \oplus e_2, v_3 \oplus e_3] &= \big(v_1 \cdot (e_2 \otimes e_3) - v_2 \cdot (e_1 \otimes e_3) \\ &\quad + v_3 \cdot (e_2 \otimes e_1 - e_1 \otimes e_2)\big) \oplus [e_1, e_2, e_3]\end{aligned} \tag{4.3}$$

with the projection $\pi : V \oplus E \to E$ forms a representation $\pi : V \rtimes E \to E$ of E in $\mathbf{M}$.

(b) *Let $\pi : A \to E$ be a representation of E in* $\mathbf{M}$, *with zero section $\Delta : E \to A$. Let $V = \pi^{-1}\{0\}$. Then V becomes a $T(E)_0$-module specified by the basic actions*

$$v \cdot e_1 \otimes e_2 = [v, e_1\Delta, e_2\Delta]$$

for $v \in V$ and $e_1, e_2 \in E$.

5. Lie Triple Systems

A *Lie triple system* is a monic comtrans algebra that satisfies the identity

$$\begin{aligned}[x_1, x_2, [x_3, x_4, x_5]] &= [[x_1, x_2, x_3], x_4, x_5] + [x_3, [x_1, x_2, x_4], x_5] \\ &\quad + [x_3, x_4, [x_1, x_2, x_5]].\end{aligned} \tag{5.1}$$

The variety of Lie triple systems over a commutative ring R is written as $\mathbf{L}_R$, or simply $\mathbf{L}$. For an R-module E, the expression

$$[x, y] = x \otimes y - y \otimes x$$

will be used for binary commutators in the tensor algebra $T(E)$.

Theorem 5.1. *Let E be a Lie triple system over a commutative ring R. Let J be the ideal of $T(E \otimes E)$ generated by the union of the sets*

$$\{[x, y, z] \otimes t - [[x, y], z] \otimes t \mid x, y, z, t \in E\} \tag{5.2}$$

and

$$\{x \otimes [y, z, t] - x \otimes [[y, z], t] \mid x, y, z, t \in E\}. \tag{5.3}$$

Then $U_{\mathbf{L}}(E)$ is isomorphic to $T(E \otimes E)/J$.

Remark 5.2. Theorem 5.1 may be compared to the results of [2].

Let V be a $T(E \otimes E)/J$-module. The proof of Theorem 5.1 depends on computational lemmas that identify the respective V-components of the four repeated commutators from (5.1) in the monic algebra $V \rtimes E$ defined by (4.3). The proof is completed by Propositions 5.7 and 5.8 below, making use of the same general observations from [6] that were used in Sec. 4, but this time in the reverse direction.

Lemma 5.3. *In the algebra $V \rtimes E$, the V-component of*

$$[v_1 \oplus e_1, v_2 \oplus e_2, [v_3 \oplus e_3, v_4 \oplus e_4, v_5 \oplus e_5]]$$

is

$$v_1 \cdot e_2 \otimes [e_3, e_4, e_5], \tag{5.4}$$

$$-v_2 \cdot e_1 \otimes [e_3, e_4, e_5], \tag{5.5}$$

$$-v_3 \cdot e_4 \otimes e_5 \otimes [e_1, e_2], \tag{5.6}$$

$$+v_4 \cdot e_3 \otimes e_5 \otimes [e_1, e_2], \tag{5.7}$$

$$+v_5 \cdot [e_3, e_4] \otimes [e_1, e_2]. \tag{5.8}$$

Lemma 5.4. *In the algebra $V \rtimes E$, the V-component of*

$$[[v_1 \oplus e_1, v_2 \oplus e_2, v_3 \oplus e_3], v_4 \oplus e_4, v_5 \oplus e_5]$$

is

$$v_1 \cdot e_2 \otimes e_3 \otimes e_4 \otimes e_5, \tag{5.9}$$

$$-v_2 \cdot e_1 \otimes e_3 \otimes e_4 \otimes e_5, \tag{5.10}$$

$$-v_3 \cdot [e_1, e_2] \otimes e_4 \otimes e_5, \tag{5.11}$$

$$-v_4 \cdot [e_1, e_2, e_3] \otimes e_5, \tag{5.12}$$

$$-v_5 \cdot [[e_1, e_2, e_3], e_4]. \tag{5.13}$$

Lemma 5.5. *In the algebra $V \rtimes E$, the V_0-component of*

$$[v_3 \oplus e_3, [v_1 \oplus e_1, v_2 \oplus e_2, v_4 \oplus e_4], v_5 \oplus e_5]$$

is

$$-v_1 \cdot e_2 \otimes e_4 \otimes e_3 \otimes e_5, \tag{5.14}$$

$$+v_2 \cdot e_1 \otimes e_4 \otimes e_3 \otimes e_5, \tag{5.15}$$

$$+v_3 \cdot [e_1, e_2, e_4] \otimes e_5, \tag{5.16}$$

$$+v_4 \cdot [e_1, e_2] \otimes e_3 \otimes e_5, \tag{5.17}$$

$$+v_5 \cdot [[e_1, e_2, e_4], e_3]. \tag{5.18}$$

Lemma 5.6. *In the algebra $V \rtimes E$, the V-component of*

$$[v_3 \oplus e_3, v_4 \oplus e_4, [v_1 \oplus e_1, v_2 \oplus e_2, v_5 \oplus e_5]]$$

is

$$-v_1 \cdot e_2 \otimes e_5 \otimes [e_3, e_4], \tag{5.19}$$

$$+v_2 \cdot e_1 \otimes e_5 \otimes [e_3, e_4], \tag{5.20}$$

$$+v_3 \cdot e_4 \otimes [e_1, e_2, e_5], \tag{5.21}$$

$$-v_4 \cdot e_3 \otimes [e_1, e_2, e_5], \tag{5.22}$$

$$+v_5 \cdot [e_1, e_2] \otimes [e_3, e_4]. \tag{5.23}$$

Proposition 5.7. *Let E be a Lie triple system. Suppose that V is a $T(E \otimes E)/J$-module. Then the comtrans algebra structure $V \rtimes E$ given by the commutator*

$$[v_1 \oplus e_1, v_2 \oplus e_2, v_3 \oplus e_3] = (v_1 \cdot (e_2 \otimes e_3) - v_2 \cdot (e_1 \otimes e_3) + v_3 \cdot (e_2 \otimes e_1 - e_1 \otimes e_2)) \oplus [e_1, e_2, e_3], \tag{5.24}$$

along with the projection $\pi : V \oplus E \to E$, forms a representation $\pi : V \rtimes E \to E$ of E in the variety $\mathbf{L}$ of Lie triple systems.

Proof. Given Corollary 4.3, it remains to check that the commutator (5.24) satisfies the identity (5.1). Using Lemmas 5.3–5.6, it will be shown that the respective coefficients of the module elements v_i on each side of the identity (5.1) are congruent modulo J.

Coefficients of v_1. Using (5.4), (5.9), (5.14), and (5.19), it must be shown that modulo J,

$$e_2 \otimes [e_3, e_4, e_5] = e_2 \otimes e_3 \otimes e_4 \otimes e_5 - e_2 \otimes e_4 \otimes e_3 \otimes e_5 - e_2 \otimes e_5 \otimes [e_3, e_4].$$

Since the right-hand side reduces to $e_2 \otimes [[e_3, e_4], e_5]$, the equality follows from the containment of (5.3) in J.

Coefficients of v_2. The argument is similar to that used for v_1.

Coefficients of v_3. Using (5.6), (5.11), (5.16), and (5.21), it must be shown that modulo J,

$$-e_4 \otimes e_5 \otimes [e_1, e_2] = -[e_1, e_2] \otimes e_4 \otimes e_5 + [e_1, e_2, e_4] \otimes e_5 + e_4 \otimes [e_1, e_2, e_5]$$

or

$$[e_1, e_2, e_4] \otimes e_5 + e_4 \otimes [e_1, e_2, e_5] = [e_1, e_2] \otimes e_4 \otimes e_5 - e_4 \otimes e_5 \otimes [e_1, e_2].$$

The right-hand side of this latter equation may be rewritten as

$$[e_1, e_2] \otimes e_4 \otimes e_5 - e_4 \otimes [e_1, e_2] \otimes e_5 + e_4 \otimes [e_1, e_2] \otimes e_5 - e_4 \otimes e_5 \otimes [e_1, e_2]$$

or $[[e_1, e_2], e_4] \otimes e_5 + e_4 \otimes [[e_1, e_2], e_5]$, which is congruent modulo J to $[e_1, e_2, e_4] \otimes e_5 + e_4 \otimes [e_1, e_2, e_5]$ as required.

Coefficients of $\boldsymbol{v}_4$. The argument is similar to that used for v_3.

Coefficients of $\boldsymbol{v}_5$. Using (5.8), (5.13), (5.18), and (5.23), it must be shown that modulo J,

$$[e_3, e_4] \otimes [e_1, e_2] = -[[e_1, e_2, e_3], e_4] + [[e_1, e_2, e_4], e_3] + [e_1, e_2] \otimes [e_3, e_4]$$

or

$$[[e_1, e_2], [e_3, e_4]] = [[e_1, e_2, e_3], e_4] + [e_3, [e_1, e_2, e_4]].$$

Modulo J, the right-hand side of the latter equation is congruent to $[[[e_1, e_2], e_3], e_4] + [e_3, [[e_1, e_2], e_4]]$, which does expand and recollect to $[[e_1, e_2], [e_3, e_4]]$ as required. □

Proposition 5.8. *Let $\pi : A \to E$ be a representation of E in $\mathbf{L}$, with zero section $\Delta : E \to A$. Let $V = \pi^{-1}\{0\}$. Then V becomes a $T(E \otimes E)/J$-module specified by the basic actions*

$$v \cdot e_1 \otimes e_2 = [v, e_1\Delta, e_2\Delta]$$

for $v \in V$ and $e_1, e_2 \in E$.

Proof. By the results of Sec. 4, V is certainly a $T(E \otimes E)$-module. It remains to be shown that the module is annihilated by J. Consider the instance of the Lie triple system identity (5.1) with $x_1 = v_1 \oplus e_1$ and $x_i = 0 \oplus e_i$ for $i > 1$. Then the examination of the coefficients of v_1 from the proof of Proposition 5.7 shows that (5.3) annihilates V. Now consider the instance of the Lie triple system identity (5.1) with $x_3 = v_3 \oplus e_1$ and $x_i = 0 \oplus e_i$ for $i \neq 3$. This time, the examination of the coefficients of v_3 from the proof of Proposition 5.7 shows that (5.2) annihilates V. □

6. Varieties of Comtrans Algebras

Let $\mathbf{V}$ be a variety of comtrans algebras over a commutative ring R, defined within the full variety $\mathbf{CT}_R$ of all comtrans algebras over R by a set

$$\{f_i(x_1, \ldots, x_{n_i}) = 0 \mid i \in I\}$$

of identities indexed by a set I. Suppose that E is a member of $\mathbf{V}$. As described for the case of $\mathbf{V} = \mathbf{CT}$ in [6], the category $\mathbf{A}_R \otimes \mathbf{V}/E$ of representations of E in $\mathbf{V}$ is equivalent to the category of modules over the universal enveloping algebra $U_{\mathbf{V}}(E)$. Since $\mathbf{V}$ is a subvariety of $\mathbf{CT}$, Proposition 2.1 shows that the algebra $U_{\mathbf{V}}(E)$ is a quotient of the algebra $U_{\mathbf{CT}}(E)$ specified in Theorem 2.3. In order to specify $U_{\mathbf{V}}(E)$, it remains to determine the ideal $J_{\mathbf{V}}$ or J such that

$$U_{\mathbf{V}}(E) \cong U_{\mathbf{CT}}(E)/J.$$

This is achieved by a form of differential calculus, similar to that used for quasigroups in [11]. We use notational conventions analogous to those of classical differential calculus.

Consider the comtrans algebra $U_{\mathbf{CT}}(E) \rtimes E$ with structure given by (3.1) and (3.2). Suppose that $f(x_1, \ldots, x_n)$ is a word in the language of comtrans algebras over R. Then for $1 \leq i \leq n$, there are elements

$$f_{x_i}(x_1, \ldots, x_n), \quad \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n), \quad \text{or just} \quad f_{x_i}, \quad \frac{\partial f}{\partial x_i} \tag{6.1}$$

of $U_{\mathbf{CT}}(E)$ such that

$$f(dx_1 \oplus x_1, \ldots, dx_n \oplus x_n) = \sum_{i=1}^{n} dx_i \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) \oplus f(x_1, \ldots, x_n) \tag{6.2}$$

for dx_i in $U_{\mathbf{CT}}(E)$ and x_i in E. As in calculus, it may be convenient to write *df* for the $U_{\mathbf{CT}}(E)$-component of (6.2). The algebra elements (6.1) are known as the *partial derivatives* of f with respect to x_i.

There are rules to compute the partial derivatives, analogous to the familiar rules of calculus. Constants from R differentiate to zero. For variables x_i and x_j,

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

For elements c_1 and c_2 of F, and words f_1, f_2, *linearity* means

$$\frac{\partial}{\partial x_i}(c_1 f_1 + c_2 f_2) = c_1 \frac{\partial f_1}{\partial x_i} + c_2 \frac{\partial f_2}{\partial x_i}.$$

Now consider the words $u = u(x_1, \ldots, x_l)$, $v = v(y_1, \ldots, y_m)$, and $w = w(z_1, \ldots, z_n)$, with mutually disjoint sets $\{x_1, \ldots, x_l\}$, $\{y_1, \ldots, y_m\}$, and $\{z_1, \ldots, z_n\}$ of arguments.

Proposition 6.1 (Commutator rules). *If*

$$f(x_1, \ldots, x_l, y_1, \ldots, y_m, z_1, \ldots, z_n) = [u, v, w],$$

then

$$\begin{aligned} & f_{x_i}(x_1, \ldots, x_l, y_1, \ldots, y_m, z_1, \ldots, z_n) \\ & \quad = u_{x_i}(x_1, \ldots, x_l) K(v(y_1, \ldots, y_m), w(z_1, \ldots, z_n)), \end{aligned} \tag{6.3}$$

$$\begin{aligned} & f_{y_j}(x_1, \ldots, x_l, y_1, \ldots, y_m, z_1, \ldots, z_n) \\ & \quad = -v_{y_j}(y_1, \ldots, y_m) K(u(x_1, \ldots, x_l), w(z_1, \ldots, z_n)), \end{aligned} \tag{6.4}$$

and (*with abbreviated notation*)

$$f_{z_k} = w_{z_k} M(v, u) \tag{6.5}$$

for $1 \le i \le l, 1 \le j \le m,$ *and* $1 \le k \le n$.

Proof. Apply the commutator definition (3.1) to the computation of $df \oplus f = [du \oplus u, dv \oplus v, dw \oplus w]$. □

Proposition 6.2 (Translator rules). *If*

$$f(x_1, \ldots, x_l, y_1, \ldots, y_m, z_1, \ldots, z_n) = \langle u, v, w \rangle,$$

then

$$f_{x_i} = u_{x_i} R(v, w), \tag{6.6}$$

$$f_{y_j} = -v_{y_j}(L(u, w) + R(w, u)), \tag{6.7}$$

and

$$f_{z_k} = w_{z_k} L(v, u) \tag{6.8}$$

for $1 \le i \le l, 1 \le j \le m,$ *and* $1 \le k \le n$.

Proof. Use the translator definition (3.2) to produce the computation of $df \oplus f = \langle du \oplus u, dv \oplus v, dw \oplus w \rangle$. □

If the words $u, v,$ or w share a repeated argument, then the derivative of f with respect to that argument is just the sum of the individual derivatives as given by the commutator rules (6.3)–(6.5) or the translator rules (6.6)–(6.8). Thus, for example,

$$\frac{\partial}{\partial y}[y, x, y] = \frac{\partial}{\partial y}\langle y, x, y \rangle = L(x, y) + R(x, y) = M(x, y) + K(x, y).$$

Theorem 6.3. *Let R be a commutative ring. Let $\mathbf{V}$ be a variety of comtrans algebras over R, defined within the full variety $\mathbf{CT}_R$ of all comtrans algebras over R by identities*

$$\{f_i(x_1, \ldots, x_{n_i}) = 0 \,|\, i \in I\}$$

indexed with a set I. Let E be a member of $\mathbf{V}$. Let J be the ideal of $U_{\mathbf{CT}}(E)$ generated by the set

$$\left\{ \frac{\partial f_i}{\partial x_j}(e_1, \ldots, e_{n_i}) \,\middle|\, i \in I, 1 \le j \le n_i, e_1, \ldots, e_{n_i} \in E \right\}.$$

Then $U_{\mathbf{V}}(E)$ is the quotient of $U_{\mathbf{CT}}(E)$ by J.

For monic algebras or Lie triple systems, Theorems 4.2 and 5.1 may be viewed as special cases of Theorem 6.3. Another example appears in the final section below.

7. Form Algebras

An important application of comtrans algebras is the algebraization of bilinear forms directly on a space, without the exponential blowup of dimension exhibited by Clifford algebras, for example. Suppose that β is a bilinear form on a module E over a commutative ring R. Then a comtrans algebra $\mathrm{CT}(E,\beta)$ is defined on E by

$$[x, y, z] = y\beta(x, z) - x\beta(y, x)$$

and

$$\langle x, y, z\rangle = y\beta(z, x) - x\beta(y, z)$$

(see [7]). This comtrans algebra has the properties that

$$[z, y, x] + \langle y, z, x\rangle \in zR \tag{7.1}$$

and

$$\langle x, y, z\rangle \in xR + yR. \tag{7.2}$$

If R is a field, then a comtrans algebra of dimension at least 3 over R is obtained as $\mathrm{CT}(E,\beta)$ from a bilinear form β on E if and only if the conditions (7.1) and (7.2) are satisfied [7, Theorem 4.1].

The conditions (7.1) and (7.2) imply comtrans algebra identities.

Proposition 7.1. *Let β be a bilinear form on a module E over a commutative ring R. Then $\mathrm{CT}(E,\beta)$ satisfies the identities*

$$[[z, y, x] + \langle y, z, x\rangle, z, u] = 0 \tag{7.3}$$

and

$$[[\langle x, y, z\rangle, x, y], \langle y, x, y\rangle, u] = 0. \tag{7.4}$$

Proof. The identity (7.3) follows from (7.1) and left alternativity. For (7.4), note that (7.2) and left alternativity imply

$$[\langle x, y, z\rangle, x, y] \in [y, x, y]R.$$

The comtrans identity allows the containment to be rewritten as

$$[\langle x, y, z\rangle, x, y] \in \langle y, x, y\rangle R,$$

from which (7.4) follows by left alternativity. □

Definition 7.2. Members of the variety **P** of comtrans algebras that satisfy the identities (7.3) and (7.4) are known as *pre-formed algebras*.

For a vector space E over a commutative ring R, equipped with a bilinear form β, Theorem 7.5 below specifies the universal enveloping algebra of $\mathrm{CT}(E,\beta)$ in the variety $\mathbf{P}_R$ of pre-formed algebras over R. The proof requires a couple of lemmas, derived using the calculus of Sec. 6.

Lemma 7.3. *In* $\mathrm{CT}(E, \beta)$, *set*

$$c(x, y, z, u) = [[z, y, x] + \langle y, z, x\rangle, z, u].$$

Then

$$\begin{aligned} c_x &= (K(z, y) - R(z, y))K(z, u), \\ c_y &= (R(z, x) - K(z, x))K(z, u), \quad \textit{and} \\ c_z &= (\beta(y, x) - \beta(x, y) - M(y, x))K(z, u), \end{aligned}$$

while $c_u = 0$.

Lemma 7.4. *In* $\mathrm{CT}(E, \beta)$, *set*

$$s(x, y, z, u) = [[\langle x, y, z\rangle, x, y], \langle y, x, y\rangle, u].$$

Then

$$\begin{aligned} s_x &= [R(y, z) + \beta(y, z)] \cdot K(x, y)[\beta(y, y)K(x, u) - \beta(x, y)K(y, u)], \\ s_y &= -(\beta(z, x) + L(x, z) + R(z, x))K(x, y)[\beta(y, y)K(x, u) - \beta(x, y)K(y, u)], \\ s_z &= L(y, x) \cdot K(x, y)[\beta(y, y)K(x, u) - \beta(x, y)K(y, u)], \end{aligned}$$

while $s_u = 0$.

Theorem 7.5. *Let R be a commutative ring. Let E be an R-module, equipped with a bilinear form β. Let $V = (E \otimes E) \oplus (E \wedge E) \oplus (E \otimes E)$. Let J be the ideal of $T(V)$ generated by the sets*

$$\{(K(a, x) - R(a, x))K(a, u) \mid a, x, u \in E\}, \tag{7.5}$$

$$\{(\beta(x, y) - \beta(y, x) - M(x, y))K(z, u) \mid x, y, z, u \in E\}, \tag{7.6}$$

$$\{L(y, x)K(x, y)[\beta(y, y)K(x, u) - \beta(x, y)K(y, u)] \mid x, y, u \in E\}, \tag{7.7}$$

$$\begin{aligned} \{[\beta(z, x) + L(x, z) + R(z, x)]K(x, y)[\beta(y, y)K(x, u) \\ - \beta(x, y)K(y, u)] \mid x, y, z, u \in E\}, \end{aligned} \tag{7.8}$$

and

$$\{[\beta(y, z) + R(y, z)]K(x, y)[\beta(y, y)K(x, u) - \beta(x, y)K(y, u)] \mid x, y, z, u \in E\}. \tag{7.9}$$

Then $U_{\mathbf{P}}(\mathrm{CT}(E, \beta))$ is isomorphic to the quotient of the tensor algebra on V by J.

Proof. In the notation of Lemmas 7.3 and 7.4, the defining identities (7.3) and (7.4) for **P** are written as

$$c(x, y, z, u) = 0 \quad \text{and} \quad s(x, y, z, u) = 0.$$

The desired result then follows as a relatively direct application of Theorem 6.3. The generating set (7.5) corresponds to the derivatives c_x and c_y. The generating set (7.6) corresponds to the derivative c_z, while (7.7) corresponds to s_z and (7.8) corresponds to s_y. Finally, (7.9) corresponds to s_x. □

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