RECURRENCES FOR TRI-RESTRICTED NUMBERS

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ABSTRACT. By analogy with Stirling numbers, tri-restricted numbers of the second kind count the number of partitions of a given set into a given number of parts, each part being restricted to at most three elements. Tri-restricted numbers of the first kind are then defined as elements of the matrix inverse to the matrix of tri-restricted numbers of the second kind. A new recurrence relation for the tri-restricted numbers of the second kind is presented, with a combinatorial proof. In answer to a problem that has remained open for several years, it is then shown by determinantal techniques that the tri-restricted numbers of the first kind also satisfy a recurrence relation. This relation is used to obtain a reciprocity theorem connecting the two kinds of tri-restricted number.

1. INTRODUCTION

For non-negative integers n and k, a *tri-restricted* k-partition of an n-set is a partition in which each of the k parts has at most 3 elements. The *tri-restricted number* $M_2^3(n,k)$ or $a_{n,k}$ *of the second kind* is defined to be the number of tri-restricted k-partitions of an n-set. By comparison, the Stirling number $S_2(n,k)$ of the second kind counts the total number of (unrestricted) k-partitions of an n-set [1, 2.66]. Then, just as Stirling numbers $S_1(k,n)$ of the first kind may be defined as elements of the matrix inverse to the matrix of Stirling numbers of the second kind, the *tri-restricted number* $M_1^3(k,n)$ *of the first kind* is the (k,n)-entry of the matrix inverse to the matrix $[a_{n,k}]$ of tri-restricted numbers of the second kind. The top left 6×6 corner of the matrix of tri-restricted numbers of the second kind is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 7 & 6 & 1 & 0 & 0 \\ 0 & 10 & 25 & 10 & 1 & 0 \\ 0 & 10 & 75 & 65 & 15 & 1 \end{bmatrix}, \tag{1.1}$$

while the corresponding part of the inverse matrix is

$$(1.2)\qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -5 & 11 & -6 & 1 & 0 & 0 \\ 10 & -45 & 35 & -10 & 1 & 0 \\ 35 & 175 & -210 & 85 & -15 & 1 \end{bmatrix} .$$

Thus $M_1^3(5,2) = -45$, for example. Unlike the case of Stirling numbers, successive subdiagonals of the matrix $[M_1^3(k,n)]$ do not alternate in sign — note the positivity of the bottom left entry of (1.2), for example. This anomalous sign behavior is an obstacle to any potential combinatorial interpretation of the "unsigned" tri-restricted numbers $(-1)^{k+n}M_1^3(k,n)$ of the first kind analogous to the interpretation of $(-1)^{k+n}S_1(k,n)$ as the number of permutations of k symbols with n cycles.

Tri-restricted numbers were originally defined analytically in [2][1], as part of a series of papers looking at the corresponding bi-restricted or *Bessel numbers* (coefficients in Bessel polynomials [3]), and the *multi-restricted numbers* [4]. Theorem 5.4 of [4] gave a recurrence relation for multi-restricted numbers of the second kind, analogous to two-term recurrence relations for the Stirling numbers of the second kind [1, 3.29(ii)] or Bessel numbers [4, (2.5)]. For the case of tri-restricted numbers, this recurrence relation specializes to the relation

$$(1.3)\qquad M_2^3(n+1,k+1) = M_2^3(n,k) + (k+1)M_2^3(n,k+1) - \binom{n}{3}M_2^3(n-3,k) .$$

Recalling the recurrence relation

$$(1.4)\qquad S_2(n+1,k+1) = S_2(n,k) + (k+1)S_2(n,k+1)$$

for Stirling numbers of the second kind, one may regard the third term of the right hand side of (1.3) as a correction appropriate to the additional restriction on the parts of the partitions counted by the tri-restricted numbers of the second kind.

The discovery of (1.3) naturally raised the problem of finding a recurrence relation for tri-restricted numbers of the first kind. However, repeated attempts to find such a recurrence based on (1.3) met with only limited success, even though (1.3) seems quite natural as a suitably modified version of (1.4). Thus the first result of the current paper, Theorem 2.1, presents a new recurrence relation

$$M_2^3(n+1,k+1) = \binom{n}{0}M_2^3(n,k) + \binom{n}{1}M_2^3(n-1,k) + \binom{n}{2}M_2^3(n-2,k)$$

for the tri-restricted numbers of the second kind. The proof of Theorem 2.1 is purely combinatorial. The main result, Theorem 3.3, then derives a corresponding

[1]In that paper, where the connection with Stirling numbers was still secondary, the "kinds" were interchanged from the current and now established usage. (But for criticism of this usage, see the conclusion of [7].) To minimize any possible confusion, here we avoid the notation $T(n,k)$ of [2].

recurrence relation

$$\begin{aligned} M_1^3(k,n) &= \binom{n}{0} M_1^3(k+1,n+1) \\ &+ \binom{n+1}{1} M_1^3(k+1,n+2) \\ &+ \binom{n+2}{2} M_1^3(k+1,n+3) \end{aligned}$$

for the tri-restricted numbers of the first kind.[2] One of the key ideas underlying the linear-algebraic proof of Theorem 3.3 is the use of Cramer's Rule to identify the tri-restricted numbers of the first kind as appropriately signed determinants of tri-restricted numbers of the second kind (compare [5, p. 157], [6]). As a consequence of Theorem 3.3, the final section of the paper derives a Reciprocity Theorem for Tri-restricted Numbers, namely

$$(-1)^{n+k} M_1^3(n,k) = M_2^3(-k,-n)\,,$$

as an analogue of the reciprocity relation $(-1)^{n+k} S_1(n,k) = S_2(-k,-n)$ for Stirling numbers [6] [7, (2.4)].

2. Recurrence relation of the second kind

By convention, set $M_2^3(n,k) = 0$ if $n < 0$ and $k \geq 0$.

Theorem 2.1. *For n, $k > 0$, one has*

$$M_2^3(n+1,k+1) = \binom{n}{0} M_2^3(n,k) + \binom{n}{1} M_2^3(n-1,k) + \binom{n}{2} M_2^3(n-2,k)\,. \tag{2.1}$$

Proof. The left hand side of (2.1) is the number of tri-restricted $(k+1)$-partitions of an $(n+1)$-set $S = \{s_0, \ldots, s_n\}$. Denote the element s_0 of S as *special.* The tri-restricted $(k+1)$-partitions of S are divided into three mutually disjoint types, according to whether the special element s_0 has 0, 1, or 2 companions in the partition. The respective terms on the right hand side of (2.1) then count the number of partitions of each of these types. The binomial coefficient $\binom{n}{c}$ counts the choices of c-sets $\{s_{i_j} \mid 1 \leq j \leq c, 1 \leq i_j \leq n\}$ of companions (for $c = 0, 1$ or 2), and for each such choice, the corresponding tri-restricted number $M_2^3(n-c,k)$ counts the number of tri-restricted k-partitions of the set $S \setminus \{s_0, s_{i_1}, \ldots, s_{i_c}\}$ of remaining elements. □

Corollary 2.2. *The recurrences* (1.3) *and* (2.1) *are equivalent.*

Proof. Along with the (redundant) boundary conditions

$$M_2^3(n,k) = \begin{cases} 1 & \text{if } n = k; \\ 0 & \text{if } n < k; \\ 0 & \text{if } n > 0, k \leq 0; \\ 0 & \text{if } n = 0, k < 0, \end{cases} \tag{2.2}$$

[2] Of course, if one wished to use this relation as a true recurrence for iterative generation of a table of tri-restricted numbers of the first kind, one could regard it as an expression for $M_1^3(k+1,n+3)$ in terms of $M_1^3(k,n)$, $M_1^3(k+1,n+1)$, and $M_1^3(k+1,n+2)$.

the recurrences (1.3) and (2.1) both serve to define the general tri-restricted numbers $M_2^3(n,k)$ of the second kind for general natural numbers n and k. Thus each recurrence, chosen as the defining recurrence, implies the other. □

3. Recurrence relation of the first kind

Throughout this section, k and n are integers with $k > n \geq 1$, while $a_{n,k}$ denotes the tri-restricted number of the second kind and the recurrence relation (2.1) provides

$$a_{n+1,k+1} = a_{n,k} + \binom{n}{1} a_{n,k} + \binom{n}{2} a_{n-1,k}. \tag{3.1}$$

Since the unimodular matrices of tri-restricted numbers of the first and second kinds are mutually inverse, Cramer's Rule specifies the tri-restricted number $M_1^3(j,m)$ of the first kind as the (j,m)-cofactor of the transposed matrix of tri-restricted numbers of the second kind. In other words, defining $b_{j,m}$ as the determinant of the submatrix of tri-restricted numbers of the second kind obtained by deleting the m-th row and j-th column, one has the relation

$$M_1^3(k,n) = (-1)^{k+n} b_{k,n}. \tag{3.2}$$

The following lemma gives a more compact form of $b_{k,n}$.

Lemma 3.1. *For $k > n \geq 1$,*

$$b_{k,n} = \begin{vmatrix} a_{n+1,n} & 1 & 0 & \dots & 0 \\ a_{n+2,n} & a_{n+2,n+1} & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{k-1,n} & a_{k-1,n+1} & a_{k-1,n+2} & \dots & 1 \\ a_{k,n} & a_{k,n+1} & a_{k,n+2} & \dots & a_{k,k-1} \end{vmatrix}. \tag{3.3}$$

Proof. The principal $k \times k$ submatrix of the matrix of tri-restricted numbers of the second kind is

$$\begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 \\ * & 1 & \ddots & \ddots & & & & & \vdots & 0 \\ \vdots & * & \ddots & \ddots & \ddots & & & & \vdots & 0 \\ \vdots & \ddots & \ddots & 1 & 0 & \ddots & & \dots & 0 & 0 \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & 1 & 0 & & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & * & a_{n+1,n} & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & * & a_{n+2,n} & a_{n+2,n+1} & 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & * & a_{k-1,n} & a_{k-1,n+1} & a_{k-1,n+2} & \dots & 1 & 0 \\ * & \dots & \dots & * & a_{k,n} & a_{k,n+1} & a_{k,n+2} & \dots & a_{k,k-1} & 1 \end{bmatrix}.$$

Deleting the n-th row and k-th column, and then taking the determinant, one obtains $b_{k,n}$ as

$$
(3.4)\qquad \begin{vmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
* & 1 & \ddots & \ddots & \ddots & & & & \vdots \\
\vdots & * & \ddots & \ddots & \ddots & & & & \vdots \\
\vdots & \ddots & \ddots & 1 & 0 & 0 & & \cdots & 0 \\
\vdots & \ddots & \ddots & * & a_{n+1,n} & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & * & a_{n+2,n} & a_{n+2,n+1} & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
\vdots & \ddots & \ddots & * & a_{k-1,n} & a_{k-1,n+1} & a_{k-1,n+2} & \cdots & 1 \\
* & \cdots & \cdots & * & a_{k,n} & a_{k,n+1} & a_{k,n+2} & \cdots & a_{k,k-1}
\end{vmatrix},
$$

which reduces to (3.3). □

The technical result is known as the *Sweeping Lemma*, for the way in which the change $n+1 \to n$ in the second index of the column elements is gradually swept down to the bottom. Note the accompanying change in the first indices, producing a longer consecutive run of these.

Lemma 3.2. *For $r > 2$,*

$$
(3.5)\qquad \begin{vmatrix}
a_{n+1,n} & 1 & 0 & 0 & \cdots \\
a_{n+2,n} & a_{n+2,n+1} & 1 & 0 & \cdots \\
\vdots & \vdots & & & \\
a_{n+r-1,n} & a_{n+r-1,n+1} & \cdots & & \\
a_{n+r+1,n+1} & a_{n+r+1,n+2} & \cdots & & \\
a_{n+r+2,n+1} & a_{n+r+2,n+2} & \cdots & & \\
\vdots & \vdots & & &
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a_{n+1,n} & 1 & 0 & 0 & \cdots \\
a_{n+2,n} & a_{n+2,n+1} & 1 & 0 & \cdots \\
\vdots & \vdots & & & \\
a_{n+r-1,n} & a_{n+r-1,n+1} & \cdots & & \\
a_{n+r,n} & a_{n+r,n+1} & \cdots & & \\
a_{n+r+2,n+1} & a_{n+r+2,n+2} & \cdots & & \\
\vdots & \vdots & & &
\end{vmatrix}.
$$

Proof. By the recurrence relation (3.1), each element $a_{n+r+1,n+s+1}$ in the $(n+r)$-th row of the left hand side can be replaced with a sum of three terms

$$
a_{n+r,n+s} + \binom{n+r}{1} a_{n+r-1,n+s} + \binom{n+r}{2} a_{n+r-2,n+s}.
$$

for any $s = 0, 1, 2, \ldots, r$, i.e. the $(s+1)$-th column of the left hand side becomes

$$\left|\begin{matrix} \vdots \\ a_{n+r-2,n+s} \\ a_{n+r-1,n+s} \\ a_{n+r,n+s} + \binom{n+r}{1} a_{n+r-1,n+s} + \binom{n+r}{2} a_{n+r-2,n+s} \\ a_{n+r+2,n+s+1} \\ \vdots \end{matrix}\right| .$$

Subtracting $\binom{n+r}{1}$ times the $(r-1)$-th row and $\binom{n+r}{2}$ times the $(r-2)$-th row from the r-th row then reduces this version of the left hand side to the right hand side. □

Theorem 3.3. *The tri-restricted numbers of the first kind satisfy the recurrence relation*

$$M_1^3(k,n) = \binom{n}{0} M_1^3(k+1,n+1) + \binom{n+1}{1} M_1^3(k+1,n+2) + \binom{n+2}{2} M_1^3(k+1,n+3) . \tag{3.6}$$

Proof. Applying the recurrences (3.1)

$$a_{n+3,n+1} = a_{n+2,n} + (n+2)a_{n+1,n} + \binom{n+2}{2}$$

and

$$a_{n+3,n+2} = a_{n+2,n+1} + (n+2) + 0 \tag{3.7}$$

to the first two terms in the second row of the determinant

$$b_{k+1,n+1} = \begin{vmatrix} a_{n+2,n+1} & 1 & 0 & 0 & \cdots & 0 \\ a_{n+3,n+1} & a_{n+3,n+2} & 1 & 0 & \cdots & 0 \\ a_{n+4,n+1} & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k,n+1} & a_{k,n+2} & a_{k,n+3} & a_{k,n+4} & \cdots & 1 \\ a_{k+1,n+1} & a_{k+1,n+2} & a_{k+1,n+3} & a_{k+1,n+4} & \cdots & a_{k+1,k} \end{vmatrix} \tag{3.8}$$

(along with the decomposition $1 = 1 + 0 + 0$), one obtains this determinant as a sum of

$$\begin{vmatrix} a_{n+2,n+1} & 1 & 0 & 0 & \cdots \\ a_{n+2,n} & a_{n+2,n+1} & 1 & 0 & \cdots \\ a_{n+4,n+1} & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} , \tag{3.9}$$

$$\begin{vmatrix} a_{n+2,n+1} & 1 & 0 & 0 & \cdots \\ (n+2)a_{n+1,n} & n+2 & 0 & 0 & \cdots \\ a_{n+4,n+1} & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} , \tag{3.10}$$

and

$$(3.11)\qquad \begin{vmatrix} a_{n+2,n+1} & 1 & 0 & 0 & \dots \\ \binom{n+2}{2} & 0 & 0 & 0 & \dots \\ a_{n+4,n+1} & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix},$$

i.e. (3.8) = (3.9) + (3.10) + (3.11). Applying the recurrence (3.1)

$$(3.12)\qquad a_{n+2,n+1} = a_{n+1,n} + (n+1)$$

(along with $1 = 1 + 0$) to the top row of (3.9), and simplifying, reduces it to a sum of

$$(3.13)\qquad \begin{vmatrix} a_{n+1,n} & 1 & 0 & 0 & \dots \\ a_{n+2,n} & a_{n+2,n+1} & 1 & 0 & \dots \\ a_{n+4,n+1} & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

and

$$(3.14)\qquad (n+1)\begin{vmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & a_{n+2,n+1} & 1 & 0 & \dots \\ 0 & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix},$$

i.e. (3.9) = (3.13) + (3.14). Applying the same recurrence (3.12) to (3.10), and again simplifying, reduces it to a sum of

$$(3.15)\qquad \begin{vmatrix} a_{n+1,n} & 1 & 0 & 0 & \dots \\ (n+2)a_{n+1,n} & n+2 & 0 & 0 & \dots \\ a_{n+4,n+1} & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

and

$$(3.16)\qquad (n+1)\begin{vmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & n+2 & 0 & 0 & \dots \\ 0 & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}.$$

Since (3.15) is zero, this sum is reduced to (3.16) alone, i.e. (3.10) = (3.16). The last determinant (3.11) reduces to $-\binom{n+2}{2}b_{k+1,n+3}$ in the form

$$(3.17)\qquad -\binom{n+2}{2}b_{k+1,n+3}\begin{vmatrix} a_{n+4,n+3} & 1 & 0 & \dots \\ a_{n+5,n+3} & a_{n+5,n+4} & 1 & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix}.$$

Repeated application of the Sweeping Lemma 3.2 reduces (3.13) to $b_{k,n}$. On the other hand, use of the recurrence (3.7) enables one to reconstitute (3.14) and (3.16) as

$$(3.18) \qquad (n+1)\begin{vmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & a_{n+3,n+2} & 1 & 0 & \dots \\ 0 & a_{n+4,n+2} & a_{n+4,n+3} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix},$$

namely as $\binom{n+1}{1} b_{k+1,n+2}$. Collecting this term along with (3.13) (as $b_{k,n}$) and (3.17) (as $-\binom{n+2}{2} b_{k+1,n+3}$) yields the recurrence

$$(3.19) \qquad b_{k+1,n+1} = b_{k,n} + \binom{n+1}{1} b_{k+1,n+2} - \binom{n+2}{2} b_{k+1,n+3}\,.$$

This reduces to the required form (3.6) on application of (3.2). □

4. Matrix extension and reciprocity

Using the recurrence relation (2.1) for arbitrary integers n and k, together with the boundary conditions (2.2) (which are redundant, but consistent with the recurrence relation), one may extend the matrix of tri-restricted numbers of the second kind from the SE quadrant to the whole plane. (Alternatively, as an irredundant boundary condition, one may take $M_2^3(n,k) = 0$ if n or k is zero together with $M_2^3(0,0) = 1$.) The central part of the doubly infinite matrix is displayed as follows:

1	0	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0	0
85	10	1	0	0	0	0	0	0	0	0	0	0
210	35	6	1	0	0	0	0	0	0	0	0	0
175	45	11	3	1	0	0	0	0	0	0	0	0
−35	10	5	2	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	1	3	1	0	0	0
0	0	0	0	0	0	0	0	7	6	1	0	0
0	0	0	0	0	0	0	0	10	25	10	1	0
0	0	0	0	0	0	0	0	10	75	65	15	1

(note that the row and column comprising the irredundant boundary condition have been ruled off). It appears that the NW quadrant contains a reflected version of the matrix of "unsigned" tri-restricted numbers of the first kind. Indeed, if one sets

$$(4.1) \qquad b_{k,n} = M_2^3(-n,-k),$$

then the recurrence relation (3.19) reduces to (2.1). Combining (4.1) with (3.2) then yields the *Reciprocity Theorem* for tri-restricted numbers (compare [7, (2.4)] for the corresponding theorem for Stirling numbers).

Theorem 4.1. *For integers n and k,*

$$(4.2) \qquad (-1)^{n+k} M_1^3(n,k) = M_2^3(-k,-n)\,.$$

Remark 4.2. In [7, pp. 417-8], Gessel is credited with showing how the reciprocity theorem for Stirling numbers reduces to a special case of Stanley's Reciprocity Theorem for Order Polynomials [8, Prop. 13.2(i)]. It appears that the anomalous sign behavior of the tri-restricted numbers of the first kind, precluding a combinatorial interpretation of the corresponding "unsigned" numbers, also precludes a reduction of Theorem 4.1 to a special case of Stanley's theorem.

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