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THE QUATEDRAL LOOP AND ITS MULTIPLICATION GROUP

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Abstract. The quatedral loop hybridizes the structures of the eight-element quaternion and dihedral groups. These three loops share the same character table. The quatedral loop provides a natural example of a nonassociative loop with a well-behaved Frobenius-Schur indicator. The aim of the current paper is to survey the properties of the quatedral loop, and to identify its multiplication group. The identification is facilitated by considering the loop as a superloop, and its multiplication group as a supergroup. Thus, the paper provides a brief summary of these concepts, and an illustration of their application.

1. Introduction. Among the loops of order 8, the quaternion group Q_8 has 6 elements of order 4, while the dihedral group D_4 has 2 elements of order 4. The loop which is the topic of this paper also has 8 elements, 4 of which have order 4. As a hybridization of the quaternion group Q_8 and dihedral group D_4, it is described as the *quatedral loop.*

The quatedral loop plays a significant role within the combinatorial character theory of finite loops and quasigroups [7], [10, Ch. 6]. This theory forms a direct extension of the ordinary character theory of finite groups, without invoking modules. Since the inception of the theory, no explicit answers to the following basic questions seem to have appeared in the literature (although some experts have privately been aware of answers to the first):

Problem 1.1. Is it possible for a loop which is not associative to have the same character table as a group?

Problem 1.2. Is it possible for a loop which is not associative to have a well-behaved Frobenius-Schur indicator?

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Now, as shown by Theorem 5.11 and Equation (5.6) in this paper, it transpires that the quatedral loop, which is not a group, has exactly the same character table as the quaternion group Q_8 and dihedral group D_4, and that its unique non-linear character has Frobenius-Schur indicator 0. In other words, the quatedral loop at once provides natural positive answers to Problems 1.1 and 1.2.

The main purpose of this paper is to provide a survey of the key properties of the quatedral loop, which include its decomposition as a superloop. The quatedral loop was actually discovered as a consequence of an investigation of superloops, superquasigroups and their multiplication groups [5]. Superloop considerations appear in the proof of the main new result of this paper, Theorem 5.12, which identifies the multiplication group of the quatedral loop.

1.1. Plan of the paper. Section 2 provides requisite background material on magmas, quasigroups, and loops. In view of the subsequent discussion of superstructures, more care than usual is devoted to the relationship between quasigroups and magmas. For other topics not treated in Section 2, readers may consult [1, 10, 11].

Since the multiplication group of the quatedral loop is given as a wreath product of groups, Section 3 provides a summary of the wreath product construction, initially at the level of monoid actions. This treatment is based on [11, p.40], which follows the account of the group-theoretical version given in [4, §15]. (Many group-theoretical treatments, particularly those insisting on left actions, involve inverses, and thus cannot be extended to monoid actions.)

Section 4 introduces supergroups and superquasigroups, including the treatment of the multiplication supergroups of superquasigroups. The quatedral loop itself appears in Section 5. Its previously known properties (from [5]) are surveyed, and the new results, which include the identification of the multiplication group, are presented.

1.1.1. *Notational remark.* Algebraic or diagrammatic notation, where functions follow their arguments (either on the line or as a superfix), is taken as the default option throughout the paper. This convention, which is followed by classics of non-associative algebra such as [1], and also implemented in GAP [3], mitigates the inevitable proliferation of brackets, and enables formulas to be read in natural order from left to right without backtracking or threading.

2. Quasigroups and loops.

2.1. Magmas and combinatorial quasigroups.

2.1.1. *Magmas.*

DEFINITION 2.1. A *magma* M or $(M, \cdot)$ is a (possibly empty) set M that is equipped with a binary operation

$$M \times M \to M; (m, m') \mapsto m \cdot m' \tag{2.1}$$

which by default may be described as *multiplication*.

REMARK 2.2. (a) It is often convenient to denote the product $m \cdot m'$ in (2.1) simply by juxtaposition, as mm'.

(b) Magmas are sometimes described as *binars*, or as *groupoids* in Ore's terminology (now abrogated by its alternative use in category theory).

2.1.2. *Right and left multiplications.* Currying the binary operation in a magma yields families of unary operations that are parametrized by the elements of the (underlying set of the) magma.

Definition 2.3. Consider an element q of a magma $(M,*)$.

(a) The function
$$R_*(q)\colon M \to M; x \mapsto xq \tag{2.2}$$
is known as *right multiplication* by q. Thus $xR_*(q) = x * q$.

(b) The function
$$L_*(q)\colon M \to M; x \mapsto qx \tag{2.3}$$
is known as *left multiplication* by q. Thus $xL_*(q) = q * x$.

The right and left multiplications of a magma $(M,\cdot)$, in which the multiplication is often denoted simply by juxtaposition, may be written simply as $R(q)$ and $L(q)$.

Definition 2.4. An element e of a magma $(M,*)$ is said to be an *identity element* if $R_*(e)\colon M \to M$ and $L_*(e)\colon M \to M$ fix each element of M.

Note that a magma has at most one identity element.

2.1.3. *Combinatorial quasigroups.*

Definition 2.5. A (*combinatorial*) *quasigroup* Q or $(Q,\cdot)$ is a magma in which the equation $x_1x_2 = x_3$ has a unique solution x_k in Q, with $k \in \{1,2,3\} \smallsetminus \{i,j\}$, for each two-element subset $\{i,j\}$ of $\{1,2,3\}$ and for each choice x_i, x_j of elements of Q.

Lemma 2.6. *Let $(Q,*)$ be a combinatorial quasigroup.*

(a) *For each $q \in Q$, the right multiplication* (2.2) *is bijective.*
(b) *For each $q \in Q$, the left multiplication* (2.3) *is bijective.*

Proof. It will suffice to prove (a). For the injectivity, suppose that $x'R_*(q) = x''R_*(q)$ for $x', x'' \in Q$. Then both x' and x'' are solutions x to the equation $x * q = x' * q$. The equality $x' = x''$ then follows from the uniqueness condition of Definition 2.5.

For the surjectivity, consider an element y of Q. The existence condition presented in Definition 2.5 implies that there is a solution x in Q to the equation $x * q = y$. Then $y = xR_*(q)$ lies in the image of $R_*(q)\colon Q \to Q$. ■

Lemma 2.6 has a converse.

Proposition 2.7. *A magma is a (combinatorial) quasigroup if and only if all its left and right multiplications are bijective.*

Proof. The forward (only if) direction is just Lemma 2.6. Conversely, suppose that the multiplications in a magma $(M,\cdot)$ are all bijective. We then have the following scheme of solutions to the various equations of Definition 2.5:
$$x_1 = x_3R(x_2)^{-1}, \quad x_2 = x_3L(x_1)^{-1},$$
$$x_3 = x_1R(x_2) = x_2L(x_1),$$

all of which are unique. ■

2.2. Equational quasigroups and loops.

Definition 2.8. A(n *equational*) *quasigroup* Q or $(Q,*,/,\backslash)$ is a set Q equipped with three magma structures:

- *multiplication* $(Q,*)$;
- *right division* $(Q,/)$;
- *left division* $(Q,\backslash)$,

such that the labelled identities

$$\mathsf{w}\backslash(\mathsf{w}*\mathsf{v}) \overset{\text{(IL)}}{=} \mathsf{v} \overset{\text{(IR)}}{=} (\mathsf{v}*\mathsf{w})/\mathsf{w} \tag{2.4}$$

$$\mathsf{w}*(\mathsf{w}\backslash\mathsf{v}) \overset{\text{(SL)}}{=} \mathsf{v} \overset{\text{(SR)}}{=} (\mathsf{v}/\mathsf{w})*\mathsf{w} \tag{2.5}$$

hold.

Remark 2.9. Note the symmetry of the (unlabelled) identities 2.4 and 2.5 about the axis through the central elements v.

Proposition 2.10. *Consider a (possibly empty) set Q.*

(a) *If $(Q,*)$ is a combinatorial quasigroup, then its magma structure augments to the structure $(Q,*,/,\backslash)$ of an equational quasigroup.*
(b) *Suppose that $(Q,*,/,\backslash)$ is an equational quasigroup. Then its magma reduct $(Q,*)$ is a combinatorial quasigroup.*

Proof. (a) By Lemma 2.6, the multiplications of $(Q,*)$ are bijective. For elements x,y of Q, define

$$x/y = xR(y)^{-1} \quad \text{and} \quad x\backslash y = yL(x)^{-1}\,.$$

The identities of 2.4 and 2.5 are then immediate.

(b) The identity (IR) says that the right multiplication $R_*(\mathsf{w})$ in $(Q,*)$ is injective. The identity (SR) says that the right multiplication $R_*(\mathsf{w})$ is surjective. Thus the right multiplications of $(Q,*)$ are bijective. Symmetrically, the left multiplications of $(Q,*)$ are bijective (compare Remark 2.9). Proposition 2.7 then shows that the magma $(Q,*)$ is a combinatorial quasigroup. ■

Thanks to Proposition 2.10, one may normally omit the qualifications "combinatorial" and "equational" when referring to quasigroups, or to the objects of this final definition:

Definition 2.11. A quasigroup is a *loop* if it has an identity element.

2.3. Multiplication groups. By Proposition 2.7, the right and left multiplications of a quasigroup $(Q,\cdot)$ lie inside the group $Q!$ of permutations of the underlying set Q, i.e., the group of bijective mappings from Q to Q.

Definition 2.12. Let $(Q,\cdot)$ be a quasigroup. Then its *multiplication group* $\operatorname{Mlt} Q$ or $\operatorname{Mlt}(Q,\cdot)$ is the subgroup

$$\langle R(q), L(q) | q \in Q\rangle_{Q!} \tag{2.6}$$

of $Q!$ generated by all the right and left multiplications of $(Q,\cdot)$.

3. Wreath products.

3.1. The wreath product of monoid actions. For sets B and M, recall that the direct power M^B is the set $\mathbf{Set}(B,M)$ of functions $s\colon B \to M$ from B to M.

DEFINITION 3.1. Consider (right) monoid actions (A,M) and (B,N).

(a) The set A is described as the set or space of *microstates.*
(b) The monoid M is described as the *microscopic monoid.*
(c) The set B is described as the set or space of *macrostates.*
(d) The monoid N is described as the *macroscopic monoid.*
(e) The set $A \times B$ is described as the (*system*) *state space.*
(f) The *wreath product* monoid $M \wr (B,N)$ is defined to have the underlying set $M^B \times N$, equipped with
$$(s,n)\cdot(t,n') = (b \mapsto bs \cdot bnt, nn') \tag{3.1}$$
as a monoid product.
(g) The *wreath product action*
$$(A,M) \wr (B,N) = \big(A \times B, M \wr (B,N)\big)$$
is the action of the wreath product monoid on the system state space with
$$(a,b)\cdot(s,n) = (a \cdot bs, bn) \tag{3.2}$$
for $a \in A$, $b \in B$, $s \in M^B$ and $n \in N$.

The terminology of Definition 3.1 is justified as follows.

LEMMA 3.2. *Consider the context of Definition 3.1.*

(a) *The product* (3.1) *is associative.*
(b) *The product* (3.1) *has an identity element. Its first component is the constant map to the identity element of M, while its second component is the identity element of N.*
(c) *The wreath product action is a monoid action.*
(d) *If (A,M) and (B,N) are group actions, then the same is true of their wreath product $(A,M) \wr (B,N)$.*
(e) *If (A,M) and (B,N) are faithful group actions, then the wreath product action of $M \wr (B,N)$ on $A \times B$ is faithful [4, Satz I.15.3].*
(f) *If (A,M) and (B,N) are transitive group actions, the wreath product action of $M \wr (B,N)$ on $A \times B$ is transitive [4, Satz I.15.3].*

3.2. The dihedral group of degree 4**.** In terms of quasigroup theory, the dihedral group D_4 of degree 4 and order 8 is defined algebraically as the multiplication group $\mathrm{Mlt}\,(\mathbf{Z}/_4, -)$ of the quasigroup of integers (or integer residues) modulo 4 under the non-associative operation of subtraction [10, Ex. 2.2], [11, p.56].

Again, writing C_2 for (the regular permutation representation of) the multiplicative group $C_2 = \big(\{\pm 1\}, \cdot\big)$, the (defining permutation representation of the) group D_4 also appears as $C_2 \wr C_2$, namely as a Sylow 2-subgroup of the symmetric group S_4 [2], [4, Hilfsatz I.15.5].

In preparation for the description of the multiplication group of the quatedral loop to be given in Theorem 5.12, it is worthwhile to relate these two descriptions of the dihedral action D_4, obtaning a similarity between the multiplication group action of $\mathrm{Mlt}\,(\mathbf{Z}/_4, -)$ and the wreath product action of $C_2 \wr C_2$ on $C_2 \times C_2$.

The identification of the sets is

$$\mathbf{Z}/_4 \to C_2 \times C_2; 0 \mapsto (1,1)\,,\, 1 \mapsto (1,-1)\,,\, 2 \mapsto (-1,1)\,,\, 3 \mapsto (-1,-1)\,.$$

This assignment corresponds to the respective binary representations

$$00, 01, 10, 11$$

when the cyclic group C_2 is written additively on the set $\{0, 1\}$. The identification of the generating permutations is

$$\begin{aligned} R_{-}(1) = (0\ 3\ 2\ 1) &\mapsto \big((1,1)\ (-1,-1)\ (-1,1)\ (1,-1)\big)\,; \\ L_{-}(1) = (0\ 1)(2\ 3) &\mapsto \big((1,1)\ (1,-1)\big)\big((-1,1)\ (-1,-1)\big)\,. \end{aligned}$$

Thus, if the image of $R_{-}(1)$ is the element (r, x) for some $r\colon C_2 \to C_2$ and $x \in C_2$, a first application of (3.2) implies solution of the equation

$$(1,1) \cdot (r,x) = (1 \cdot 1r, 1x) \stackrel{!}{=} (-1,-1)\,,$$

whence $x = -1$ and $r\colon 1 \mapsto -1$. Completion of the assignment requires the satisfaction of

$$(-1,-1) \cdot (r,-1) = \big(-1 \cdot (-1)r, (-1)(-1)\big) \stackrel{!}{=} (-1,1)\,,$$

so that $r\colon -1 \mapsto 1$, and $R_{-}(1)$ is represented by $\left(\begin{Bmatrix} 1 \mapsto -1 \\ -1 \mapsto 1 \end{Bmatrix}, -1\right)$. In turn, $L_{-}(1)$ is represented by $\left(\begin{Bmatrix} 1 \mapsto 1 \\ -1 \mapsto 1 \end{Bmatrix}, -1\right)$ in $C_2 \wr C_2$.

Remark 3.3. The identification between the sets $\mathbb{Z}/_4$ and $C_2 \times C_2$ is not arbitrary. For example, one might attempt the identification

$$\mathbf{Z}/_4 \to C_2 \times C_2; 0 \mapsto (1,1)\,,\, 1 \mapsto (-1,1)\,,\, 2 \mapsto (-1,-1)\,,\, 3 \mapsto (1,-1)$$

based on the popular view of D_4 as the group of symmetries of the square

$$\{\,(1, \pm 1), (-1, \pm 1)\,\}\ ,$$

trying to realize the rotation

$$\big((1,1)\ (-1,1)\ (-1,-1)\ (1,-1)\big)$$

counterclockwise by a right angle. The first application of (3.2) would imply solution of the equation

$$(1,1) \cdot (r,x) = (1 \cdot 1r, 1x) \stackrel{!}{=} (-1,1)\,,$$

leading to $x = 1$ and $r\colon 1 \mapsto -1$. The next application of (3.2) would then imply solution of the equation

$$(-1,1) \cdot (r,1) = \big(-1 \cdot 1r, 1\big) \stackrel{!}{=} (1,-1)\,,$$

which is impossible.

4. Superquasigroups.

4.1. Supersets and parity.

DEFINITION 4.1. Consider a set S.

(a) The set S becomes a *superset* when it is equipped with a specified disjoint union decomposition
$$S = S_0 \uplus S_1 \tag{4.1}$$
in which the respective uniands are identified as the *even part* S_0 and *odd part* S_1.
(b) Elements x of S_0 are described as having *even parity*: $|x| = 0$.
(c) Elements x of S_1 are described as having *odd parity*: $|x| = 1$.
(d) If T_0, T_1 are respective subsets of the even and odd uniands S_0, S_1 from (4.1), then $T = T_0 \uplus T_1$ is said to be a *supersubset* or *subsuperset* of S.

REMARK 4.2. (a) Equivalently, one may define a superset S to be the domain of a *parity function*
$$p_S \colon S \to \mathbf{Z}/_2; x \mapsto |x| \, . \tag{4.2}$$
Then, we have S_r as the inverse image $p_S^{-1}\{\, r \,\}$ for $r = 0, 1$. Note the consistency with Definition 4.1(c),(d).

(b) A categorical approach to supersets (from which we refrain in this paper) would start from (4.2), regarding it as an object of the slice category of sets over $\mathbf{Z}/_2$.

4.2. Supermagmas and supergroups.

DEFINITION 4.3. (a) Suppose that (the underlying set of) a magma S is a superset $S = S_0 \uplus S_1$. Suppose that whenever x and y are elements of S, then their product $x \cdot y$ has
$$|x \cdot y| = |x| + |y| \tag{4.3}$$
with addition modulo 2. Then S is said to be a *supermagma*.

(b) A *supergroup* is a group whose magma reduct is a supermagma.

REMARK 4.4. The condition (4.3) of Definition 4.3 says that the parity function (4.2) is a magma homomorphism to the additive group of residues modulo 2. Thus, within a supermagma, the even part forms a submagma.

LEMMA 4.5. *Let S be a supermagma.*

(a) *Each idempotent element of S is even.*
(b) *If a supermagma has an identity element, then it is even.*

Proof. It suffices to note the instance $|e| = |e| + |e|$ of (4.3) for each idempotent element e of S. ∎

EXAMPLE 4.6. Consider the symmetric group $S_n = X!$ on the set $X = \{\, 0, \dots, n-1 \,\}$ of finite cardinality n. It becomes a supergroup $S_n = A_n \uplus A_n(0\ 1)$. The even subgroup is the alternating group A_n.

4.3. Superquasigroups. In §2.2, it was noted that a quasigroup Q may be defined as a set carrying three compatible magma structures: multiplication $(Q,\cdot)$, right division $(Q,/)$, and left division $(Q,\backslash)$. In particular, a group is Boolean (i.e., elementary abelian of exponent 2) if and only if all three of these magma structures coincide.

DEFINITION 4.7. (a) A *superquasigroup* Q is a set Q carrying a superset structure whose parity function

$$p_Q\colon Q \to \mathbb{Z}/_2; x \mapsto |x|$$

is a quasigroup homomorphism from Q to the Boolean group $(\mathbb{Z}/_2,+)$. In particular, the domain Q of the quasigroup homomorphism p_Q is a quasigroup.

(b) A *superloop* is a superquasigroup which is a loop.

Within the context of Definition 4.7, Lemma 4.5 has two immediate consequences.

LEMMA 4.8. (a) *The identity of a superloop is even.*

(b) *An idempotent superquasigroup has no odd elements.*

4.4. Superfunctions between supersets.

DEFINITION 4.9. Suppose that $f\colon T \to T'; t \mapsto t^f$ is a function from (the underlying set of) a superset T to (the underlying set of) a superset T'. Its graph $\mathrm{Gr}\, f = \{\, (t, t^f) \mid t \in T \,\}$ is (the underlying set of) a supersubset of the superproduct $T \widehat{\times} T'$ of T with T'.

(a) The *even part* f_0 of f is the function $f_0\colon T \to T'$ with graph $\mathrm{Gr}\, f_0 = \big(\mathrm{Gr}\, f\big)_0$.
(b) The *odd part* f_1 of f is the function $f_1\colon T \to T'$ with graph $\mathrm{Gr}\, f_1 = \big(\mathrm{Gr}\, f\big)_1$.
(c) The decomposition of (the graph of) f into the disjoint parts

$$f_0|_{T_0}\colon T_0 \to T'_0\,; \quad f_0|_{T_1}\colon T_1 \to T'_1\,; \tag{4.4}$$

$$f_1|_{T_0}\colon T_0 \to T'_1\,; \quad f_1|_{T_1}\colon T_1 \to T'_0 \tag{4.5}$$

makes f a *superfunction*.
(d) The function f is *even* if $\mathrm{Gr}\, f_1$ is empty.
(e) The function f is *odd* if $\mathrm{Gr}\, f_0$ is empty.

4.5. Multiplication supergroups. The multiplication group of (the underlying quasigroup of) a superquasigroup becomes a supergroup.

LEMMA 4.10. *Suppose that $Q = Q_0 \uplus Q_1$ is a superquasigroup.*

(a) *If q is an element of Q_0, then $R(q)\colon Q \to Q$ and $L(q)\colon Q \to Q$ are even functions.*
(b) *If q is an element of Q_1, then $R(q)\colon Q \to Q$ and $L(q)\colon Q \to Q$ are odd functions.*
(c) *The set $\{\, R(q), L(q) \mid q \in Q \,\}$ has a decomposition*

$$\{\, R(q), L(q) \mid q \in Q \,\} = \{\, R(q), L(q) \mid q \in Q_0 \,\} \uplus \{\, R(q), L(q) \mid q \in Q_1 \,\}$$

as a superset.

Proof. (a) Suppose that q is even. Consider an element x of Q.

(i) If x is even, then xq and qx are even. Thus

$$R(q)\colon Q_0 \to Q_0 \text{ and } L(q)\colon Q_0 \to Q_0\,.$$

(ii) If x is odd, then xq and qx are odd. Thus

$$R(q)\colon Q_1 \to Q_1 \text{ and } L(q)\colon Q_1 \to Q_1\,.$$

By comparison with (4.4) from Definition 4.9, it is then apparent that $R(q)$ and $L(q)$ are even functions.

(b) Suppose that q is odd. Consider an element x of Q.

(i) If x is even, then xq and qx are odd. Thus

$$R(q)\colon Q_0 \to Q_1 \text{ and } L(q)\colon Q_0 \to Q_1\,.$$

(ii) If x is odd, then xq and qx are even. Thus

$$R(q)\colon Q_1 \to Q_0 \text{ and } L(q)\colon Q_1 \to Q_0\,.$$

By comparison with (4.5) from Definition 4.9, it is then apparent that $R(q)$ and $L(q)$ are odd functions.

(c) now follows from (a) and (b). ∎

THEOREM 4.11. *Suppose that $Q = Q_0 \uplus Q_1$ is a superquasigroup. The multiplication group* $\mathrm{Mlt}\, Q$ *of the quasigroup Q becomes a supergroup*

$$\mathrm{MLT}\, Q = (\mathrm{MLT}\, Q)_0 \uplus (\mathrm{MLT}\, Q)_1 \tag{4.6}$$

in which $(\mathrm{MLT}\, Q)_0$ *is the set of even functions in* $\mathrm{Mlt}\, Q$*, while* $(\mathrm{MLT}\, Q)_1$ *is the set of odd functions in* $\mathrm{Mlt}\, Q$.

Proof. It must be shown that the sets occuring in the decomposition (4.6) account for all the elements of the multiplication group. In other words, for each function

$$f\colon (Q_0 \uplus Q_1) \to (Q_0 \uplus Q_1)$$

in the multiplication group, either $\mathrm{Gr}\, f_0$ is empty or $\mathrm{Gr}\, f_1$ is empty. If Q_1 is empty, then $f|_{Q_0}\colon Q_0 \to Q_0$ from (4.4) is the only option, so $\mathrm{Mlt}\, Q = (\mathrm{MLT}\, Q)_0$ and the matter is settled.

Otherwise, suppose that the parity homomorphism $p\colon Q \to (\mathbb{Z}/_2, +)$ is surjective. Then by the results of [10, §2.2], there is a well-defined surjective homomorphism

$$\mathrm{Mlt}\, p\colon \mathrm{Mlt}\, Q \to \mathrm{Mlt}\, (\mathbb{Z}/_2, +) \cong (\mathbb{Z}/_2, +)$$

of groups, extending the surjective parity map

$$\{\, R(q), L(q) \mid q \in Q \,\} \to \mathbf{Z}/_2; R(q) \mapsto |q|, L(q) \mapsto |q|$$

of the superset

$$\{\, R(q), L(q) \mid q \in Q \,\} = \{\, R(q), L(q) \mid q \in Q_0 \,\} \uplus \{\, R(q), L(q) \mid q \in Q_1 \,\}$$

from Lemma 4.10(c) [10, (2.12)]. Thus $(\mathrm{MLT}\, Q)_r = (\mathrm{Mlt}\, p)^{-1}\{\, r \,\}$ for $r \in \mathbb{Z}/_2$, verifying (4.6) in this case. ∎

DEFINITION 4.12. If Q is a superquasigroup, the supergroup $\mathrm{MLT}\, Q$ of Theorem 4.11 will be described as the *multipication supergroup* of the superquasigroup Q.

Remark 4.13. Suppose that Q is a superquasigroup of finite order. By Definition 2.12, the multiplication group of Q is a group of permutations of the set Q. It is important to note that the parity of an element of the multiplication supergroup $\mathrm{MLT}\,Q$ does not have to match its parity as a permutation (even or odd in the usual sense) according to Example 4.6. For instance, the even permutation (5.7) below is an odd element of the multiplication supergroup in which it appears.

5. The quatedral loop and its multiplication group.

5.1. The quatedral (super)loop.

Definition 5.1. The *quatedral (super)loop* S is the loop given by the multiplication table

$$
\begin{array}{c||cc|cc||cc|cc||}
S & 1 & i & -1 & -i & e & ie & -e & -ie \\
\hline\hline
1 & 1 & i & -1 & -i & e & ie & -e & -ie \\
i & i & -1 & -i & 1 & ie & e & -ie & -e \\
\hline
-1 & -1 & -i & 1 & i & -e & -ie & e & ie \\
-i & -i & 1 & i & -1 & -ie & -e & ie & e \\
\hline\hline
e & e & ie & -e & -ie & 1 & -i & -1 & i \\
ie & ie & -e & -ie & e & -i & -1 & i & 1 \\
\hline
-e & -e & -ie & e & ie & -1 & i & 1 & -i \\
-ie & -ie & e & ie & -e & i & 1 & -i & -1 \\
\hline\hline
\end{array} \,. \qquad (5.1)
$$

Its decomposition as a superloop is $\{1, i, -1, -i\} \uplus \{e, ie, -e, -ie\}$.

The loop is not commutative, since in the table (5.1), we have

$$(ie)i = -e \neq e = i(ie)\,.$$

The loop is not associative, since no nonabelian group of order 8 has 4 elements of order 4. More directly, we have

$$[e(ie)]i = [-i]i = 1 \neq -1 = e[-e] = e[(ie)i]$$

to break the associative law.

The quatedral loop is listed as `NilpotentLoop(8,116)` in the `LOOPS` package [9] of `GAP` [3]: compare Theorem 5.4(b) below. We have

$$1 \mapsto 1\,,\ 2 \mapsto e\,,\ 3 \mapsto i\,,\ 4 \mapsto ie\,,\ 5 \mapsto -1\,,\ 6 \mapsto -e\,,\ 7 \mapsto -i\,,\ 8 \mapsto -ie$$

as the assignment of elements from the representation in `GAP` [12].

5.2. Properties of the quatedral loop. Recall that the *inner mapping group* of a loop is the stabilizer of its identity element within its multiplication group.

Proposition 5.2. *In the quatedral loop S with multiplication table* (5.1), *the subgroup*

$$\langle (i\ \ -i), (ie\ \ -ie), (e\ \ -e)\rangle_{S!} \qquad (5.2)$$

of $S!$ generated by the given transpositions is the inner multiplication group $\mathrm{Inn}\,S$.

Proof. Using the notation of [1, IV.1(1.5)],[10, §2.8], we have

$$\begin{aligned} T(i) &= R(i)L(i)^{-1} = (ie \ \ -ie)\,, \\ T(ie) &= R(ie)L(ie)^{-1} = (i \ \ -i)\,, \quad \text{and} \\ R(i,ie) &= R(i)R(ie)R\big(i(ie)\big)^{-1} = R(i)R(ie)R(e)^{-1} \\ &= (e \ \ -e)(i \ \ -i)(ie \ \ -ie)\,. \end{aligned}$$

Other inner mappings lie within the displayed group (5.2). ∎

COROLLARY 5.3. *The multiplication group of the quatedral loop S has order* 64.

Proof. The group (5.2), the stabilizer of 1 in the loop multiplication group, has order 8. Since the full multiplication group acts transitively on the loop of order 8, it has order $8 \times 8 = 64$. ∎

THEOREM 5.4. *[5, Th. 4.22] Consider the quatedral loop S along with its multiplication table* (5.1).

(a) *The (Hasse diagram of the) subloop lattice is*

$$\begin{tikzcd}
 & \boxed{S} & & \\
\boxed{\langle i\rangle} \arrow[ur] & \boxed{\langle ie\rangle} \arrow[u] & \boxed{\langle -1\rangle \times \langle e\rangle} \arrow[ul] & \\
 & \boxed{\langle -1\rangle} \arrow[ul] \arrow[u] \arrow[ur] & \langle -e\rangle \arrow[u] & \langle e\rangle \arrow[ul] \\
 & & \boxed{\langle 1\rangle} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

(where the normal subloops are boxed). In particular, each proper subloop is a group.

(b) *The quatedral loop S is nilpotent of class* 2, *with coincident derived loop and center* $Z(S) = \langle -1\rangle$.

(c) *The central quotient $S/Z(S)$ of the loop is the Boolean group C_2^2 of order* 4.

(d) *The loop is power-associative, but not di-associative.*

(e) *The loop does not have the right or left inverse property. In particular, it does not have the right or left Bol property.*

5.3. Conjugacy classes of the quatedral loop. Recall that the (*loop*) *conjugacy classes* of a loop are the orbits of its inner mapping group.

PROPOSITION 5.5. *The list*

$$\Gamma_1 = \{\,1\,\}\,, \Gamma_2 = \{\,-1\,\}\,, \Gamma_3 = \{\,\pm e\,\}\,, \Gamma_4 = \{\,\pm i\,\}\,, \Gamma_5 = \{\,\pm ie\,\} \tag{5.3}$$

displays the loop conjugacy classes of the quatedral loop.

Proof. This result is an immediate consequence of Proposition 5.2. ∎

Now, recall that *A-loops* are loops where each inner mapping is an automorphism. Certainly, groups are A-loops. On the other hand, we have the following.

LEMMA 5.6. *The quatedral loop is not an A-loop.*

Proof. Note $[i(ie)]^\eta = e^\eta = -e \neq e = i(ie) = i^\eta(ie)^\eta$ for the inner mapping $\eta = (e\ e)$. ∎

The following definitions extend the usual group definitions [6, p.263]. They become more subtle for loops which are not A-loops.

DEFINITION 5.7. Consider a finite, power-associative loop Q.

(a) An element x of Q is *real* if it is conjugate to its inverse x^{-1}.
(b) A conjugacy class of Q is *real* if each of its elements is real.

LEMMA 5.8. *In the context of Definition 5.7, each element x of order 2 in Q is real.*

Each conjugacy class of the quaternion group Q_8 and dihedral group D_4 is real. Thus, the following proposition displays a further analogy between these groups and the quatedral loop.

PROPOSITION 5.9. *Each conjugacy class of the quatedral loop S is real.*

Proof. Each non-identity element of the quatedral loop has order 2 or 4. In the first case, the reality is an immediate consequence of Lemma 5.8. In the second case, it may be observed from the multiplication table (5.1) that the inverse of each element x of order 4 is $-x$. The relevant conjugacy classes from (5.3), namely Γ_3 and Γ_4, are then both of the required form $\{\pm x\}$. ∎

REMARK 5.10. In the combinatorial character theory of quasigroups [7], [10, Ch. 6], a more general version of conjugacy class “reality” is available. The quasigroup conjugacy classes of a finite quasigroup Q with multiplication group G are the orbits of G in its diagonal action on the direct square $Q \times Q$ of Q. An element x of a loop with identity element 1 may then be described as *real* if the pairs $(1, x)$ and $(x, 1)$ lie in the same quasigroup conjugacy class. If Q is a finite power-associative loop, this condition holds for an element x of Q that is real in the sense of Definition 5.7. Indeed, if $x\alpha = x^{-1}$ for some $\alpha \in \operatorname{Inn} Q$, then $(1, x)\alpha R(x) = (1, x^{-1})R(x) = (x, 1)$.

5.4. Character table of the quatedral loop.

THEOREM 5.11. *[5, Th. 4.23] The character table of the quatedral loop coincides with the character table of the dihedral and quaternion groups.*

Proof. By Theorem 5.4(b)(c), the abelianization of the loop S of order 8 is the Boolean group C_2^2, which is also the abelianization of Q_8 and D_4. Thus, each of these three loops has 4 linear characters, leaving room for just one non-linear character, of dimension 2.

The loop conjugacy class decomposition (5.5) of the quatedral loop exactly matches the conjugacy class decompositions of the groups Q_8 and D_4. By [8, Th. 5.2] or [10, Th. 7.10], it then follows that all three loops have the same character table. ∎

The common character table of D_4, Q_8 and S is

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
λ_0	1	1	1	1	1
λ_1	1	1	-1	1	-1
λ_2	1	1	1	-1	-1
λ_3	1	1	-1	-1	1
ρ	2	-2	0	0	0

The distinction between the dihedral and quaternion groups emerges when one computes the *Frobenius-Schur indicator*

$$\iota_\rho = \frac{1}{8}\sum_{x\in G}\rho\left(x^2\right)$$

of the nonlinear character ρ [6, §23].

For the quaternion group, we have

$$\iota_\rho = \frac{1}{8}\Big\{\overbrace{2}^{1}+\overbrace{2}^{-1}+\overbrace{0\times 2}^{\text{other order }2}+\overbrace{6\times(-2)}^{\text{order }4}\Big\} = \frac{1}{8}\{2+2-12\} = -1\,, \tag{5.4}$$

reflecting the faithful (one-dimensional) representation $Q_8 \hookrightarrow \mathbb{H}$ of Q_8 over the division ring $\mathbb{H}$ of quaternions.

For the dihedral group, we have

$$\iota_\rho = \frac{1}{8}\Big\{\overbrace{2}^{1}+\overbrace{2}^{z}+\overbrace{4\times 2}^{\text{other order }2}+\overbrace{2\times(-2)}^{\text{order }4}\Big\} = \frac{1}{8}\{2+2+8-4\} = 1 \tag{5.5}$$

with the central involution z, reflecting the faithful real representation of the group D_4 by symmetries of the square convex hull of $\{\pm(1,0),\pm(0,1)\}$.

Now, for the quatedral loop, with the conjugacy class decomposition as given in the proof of Theorem 5.11, we have

$$\iota_\rho = \frac{1}{8}\Big\{\overbrace{2}^{1}+\overbrace{2}^{-1}+\overbrace{2\times 2}^{\text{other order }2}+\overbrace{4\times(-2)}^{\text{order }4}\Big\} = \frac{1}{8}\{2+2+4-8\} = 0\,, \tag{5.6}$$

interpolating exactly between the two values (5.4), (5.5) of ι_ρ for the groups. At the time of writing, there is no meaningful interpretation of this computation within the known theory of linear loop and quasigroup representations extending the theory of linear group representations, as presented, say, in [10, Chs. 10–12].

5.5. The multiplication group.

THEOREM 5.12. *Consider the quatedral loop S as a loop and superloop.*

(a) *The even part* $(\mathrm{MLT}\,S)_0$ *of the multiplication supergroup of the quatedral (super)loop is* $C_2^2 \wr C_2$.

(b) *The full multiplication group* $\mathrm{Mlt}\,S$ *of* S *is* $C_2 \wr C_2^2$.

Proof. (a) Inspection of the multiplication table (5.1) reveals that the multiplication supergroup has

$$L(-1) = (1\ \ -1)(i\ \ -i)(e\ \ -e)(ie\ \ -ie)$$

as an even element. From Proposition 5.2, it follows that

$$\langle (1\ \ -1), (i\ \ -i), (ie\ \ -ie), (e\ \ -e)\rangle_{S!} \cong C_2^4$$

is a subgroup of the even part of $\mathrm{MLT}\,S$. By Corollary 5.3, we have $\left|(\mathrm{MLT}\,S)_0\right| = 32$. Amongst the groups of order 32, with a faithful transitive permutation representation of degree 8, containing C_2^4 as a subgroup, the only example is $C_2^2 \wr C_2$ [2].

(b) The even subgroup $C_2^2 \wr C_2$ of $\mathrm{MLT}\,S$ has 19 involutions [2], and is contained as a subgroup of index 2 in $\mathrm{Mlt}\,S$. Now $\mathrm{Mlt}\,S$ is a group of order 64, with a faithful transitive permutation representation of degree 8, containing $C_2^2 \wr C_2$ as a subgroup. The only groups with these properties are $C_2^2 \wr C_4$ and $C_2 \wr C_2^2$, the former only containing 19 involutions [2]. In other words, $C_2^2 \wr C_4$ would have no room for any odd involutions. Thus, since $\mathrm{MLT}\,S$ does contain the odd involution

$$\begin{aligned} T(i)R(e) &= (ie\ \ -ie)(1\ e)(-1\ \ -e)(i\ ie\ \ -i\ \ -ie) \\ &= (1\ e)(-1\ \ -e)(i\ ie)(-i\ \ -ie)\,, \end{aligned} \tag{5.7}$$

the only possibility for $\mathrm{Mlt}\,S$ is $C_2 \wr C_2^2$. ∎

In order to give a concrete interpretation of Theorem 5.12(b), along the lines of the identification of $\mathrm{Mlt}\big(\mathbb{Z}/_4, -\big)$ with the wreath product $C_2 \wr C_2$ in Section 3.2, we implement C_2 and C_2^2 as the groups

C_2	1	-1
1	1	-1
-1	-1	1

and

C_2^2	1	i	e	ie
1	1	i	e	ie
i	1	1	ie	e
e	e	ie	1	i
ie	ie	e	i	1

in their regular representations. For $x \in C_2^2$, we interpret the element $\pm x$ of S as $(\pm 1, x)$ in the state space $C_2 \times C_2^2$ of the wreath product. The form $\langle 1^s, i^s, e^s, (ie)^s\rangle$ will be used to describe a function $s\colon C_2^2 \to C_2$. We then have the implementations of the five members of a generating set of $\mathrm{Mlt}\, S$ as follows:

$$\begin{aligned}
R(i) &= \big((1,1)\ (1\ i)\ (-1,1)\ (-1,i)\big)\big((1,e)\ (1\ ie)\ (-1,e)\ (-1,ie)\big) \\
&= \big(\langle 1,-1,1,-1\rangle\,,i\big)\,; \\
L(i) &= \big((1,1)\ (1\ i)\ (-1,1)\ (-1,i)\big)\big((1,e)\ (1\ ie)\big)\big((-1,e)\ (-1,ie)\big) \\
&= \big(\langle 1,-1,1,1\rangle\,,i\big)\,; \\
R(e) &= \big((1,1)\ (1,e)\big)\big((-1,1)\ (-1,e)\big)\big((1,i)\ (1\ ie)\ (-1,i)\ (-1,ie)\big) \\
&= \big(\langle 1,1,1,-1\rangle\,,e\big)\,; \\
R(ie) &= \big((1,1)\ (1\ ie)\ (-1,1)\ (-1,-ie)\big)\big((1,e)\ (-1\ i)\ (-1,e)\ (1,i)\big) \\
&= \big(\langle 1,1,-1,-1\rangle\,,ie\big)\,; \\
L(ie) &= \big((1,1)\ (1\ ie)\ (-1,1)\ (-1,-ie)\big)\big((1,e)\ (-1\ i)\big)\big((-1,e)\ (1,i)\big) \\
&= \big(\langle 1,-1,-1,-1\rangle\,,ie\big)\,,
\end{aligned}$$

noting $R(e) = L(e)$.

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