

QUASIPRIMITIVITY AND QUASIGROUPS

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It is well known that Q is a simple quasigroup if and only if $\text{Mlt } Q$ acts primitively on Q . Here we show that Q is a simple quasigroup if and only if $\text{Mlt } Q$ acts quasiprimatively on Q , and that Q is a simple right quasigroup if and only if $\text{RMlt } Q$ acts quasiprimatively on Q .

A *quasigroup* is set with a single binary operation, denoted by juxtaposition, such that in $xy = z$, knowledge of any two of x, y and z specifies the third uniquely. A *right quasigroup* is a set with a single binary operation whose right translations biject.

The *multiplication group*, $\text{Mlt } Q$, of a quasigroup Q is the subgroup of the group of all bijections on Q generated by right and left translations, that is $\text{Mlt } Q := \langle R(x), L(x) : x \in Q \rangle$, where $R(x)$ (respectively, $L(x)$) is right (respectively, left) translation by x . The *right multiplication group*, $\text{RMlt } Q$, of a right quasigroup Q is the subgroup of the group of all bijections on Q generated by right translations, that is $\text{RMlt } Q := \langle R(x) : x \in Q \rangle$. A quasigroup Q is called *type 1* if $\text{RMlt } Q = \text{Mlt } Q$. For example, commutative quasigroups are type 1; so too are finite simple Moufang loops [4].

A permutation group G on a set Q acts *primitively* on Q if the only G -invariant partitions of Q are the two trivial partitions $\{Q\}$ and $\{\{x\} : x \in Q\}$. Of course, if G acts primitively on Q then each nontrivial normal subgroup of G is transitive on Q . In [5], Praeger used this fact to generalise the notion of primitivity: a permutation group G on a set Q acts *quasiprimatively* on Q if each non-trivial normal subgroup of G is transitive on Q . This definition is useful because, as Praeger proved [5], there is an O’Nan-Scott type theorem classifying all quasiprimitive permutation groups as one of eight types.

Given a quasigroup Q , there exist two binary operations $/, \backslash$ on Q such that $(xy)/y = x$, $(x/y)y = x$, $x \backslash (xy) = y$, and $x(x \backslash y) = y$. Conversely, an algebra with three binary operations satisfying these four identities is a quasigroup — as defined at the beginning of this paper — under any one of these operations [2]. Similarly, right quasigroups are axiomatised by the first two of the four quasigroup identities above. A

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congruence on a (right) quasigroup Q is an equivalence relation V on Q such that if x_1Vy_1 and x_2Vy_2 , then $x_1x_2fVy_1y_2f$, where f is any one of the (two) three binary operations on Q . Q is *simple* if its only congruences are the trivial congruence and the improper congruence. We record the following well known fact [2]:

PROPOSITION 1. *A quasigroup Q is simple if and only if $\text{Mlt } Q$ acts primitively on Q .*

THEOREM 2. *A quasigroup Q is simple if and only if $\text{Mlt } Q$ acts quasiprimatively on Q .*

PROOF: ($\rightarrow$) Quasiprimitivity is a generalisation of primivity.

($\leftarrow$) Let V be a nontrivial congruence on Q . There is a corresponding normal subgroup $\text{Nor}(V) := \{g \in \text{Mlt } Q : \forall q \in Q, (q, qg) \in V\}$ of $\text{Mlt } Q$. Now pick $x \in Q$. The subset $x\text{Nor}(V)$ of Q is contained in the congruence class of x . But since $\text{Mlt } Q$ acts quasiprimatively on Q , $x\text{Nor}(V) = Q$. Thus V is improper and Q is simple. $\square$

An important research program from the theory of quasigroups is to determine which permutation group actions are the actions of a multiplication group of a quasigroup, and which permutation groups are not [2, 4]. The following corollary to Theorem 2 advances this program.

COROLLARY 3. *An imprimitive, quasiprimitive action is not a multiplication group action.*

For examples of imprimitive, quasiprimitive actions see [5]. While every symmetric group, under its natural action, can be realised as a multiplication group of some quasigroup, [3, Proposition 4.2] shows that the actions of symmetric groups of odd prime power degree bigger than three on unordered pairs cannot be multiplication group actions. Corollary 3 above expands on this theme: it is the first example of a *general class* of permutation actions — that is, the quasiprimitive, imprimitive actions — that cannot be multiplication group actions. The following result gives a unilateral version of Theorem 2.

THEOREM 4. *Q is a simple right quasigroup if and only if $\text{RMlt } Q$ acts quasiprimatively on Q .*

PROOF: /

) Given a right quasigroup epimorphism $f : Q \rightarrow M$, there is a group epimorphism $F : \text{RMlt } Q \rightarrow \text{RMlt } M$; $R(x) \rightarrow R(xf)$. Pick $x \in Q$. The subset $x\text{Ker } F$

is in the congruence $(\ker f)$ class of x . But since $\text{RMlt } Q$ acts quasiprimitively on Q , $x \text{ Ker } F = Q$. Hence, $\ker f$ is improper, and Q is simple. $\square$

COROLLARY 5. *If Q is a type 1 simple quasigroup, then Q is a simple right quasigroup.*

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