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Quasigroups, right quasigroups and category coverings

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Dedicated to the memory of Alan Day

Abstract. The category of modules over a fixed quasigroup in the category of all quasigroups is equivalent to the category of representations of the fundamental groupoid of the Cayley diagram of the quasigroup in the category of abelian groups. The corresponding equivalent category of coverings, and the generalization to the right quasigroup case, are also described.

0. Introduction

The theory of quasigroup modules, or quasigroup representation theory, is equivalent to the representation theory of quotients of group algebras of certain groups associated with quasigroups; namely the stabilizers in the so-called universal multiplication groups (cf. [9, 336]). Fundamental groupoids originally were used as invariants of topological spaces. (See e.g. [1, 6.5.10]). It is also a fact that covering spaces of a topological space can be classified by their fundamental groupoids (see e.g. [1, §9]). Generalizations of fundamental groupoids and coverings in the directed graph case are given in [4, pp. 67 and 97]. Another interpretation of quasigroup modules, namely as representations of the fundamental groupoid on the Cayley diagram [9, 213] of the quasigroup in the category of abelian groups, is given here. The equivalent coverings are obtained using [4, 30]. Generalizations to right quasigroups in the sense of [8] (compare the "right groupoids" of [3]) involve the path category (cf. [4, §3]) of the Cayley diagram of the right quasigroup.

1. Quasigroups and groupoids

A quasigroup can be considered either as a not-necessarily finite Latin square or as a not-necessarily associative group (not necessarily containing an identity element).

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DEFINITION 1.1. [9, 116]. A *quasigroup* Q is a set Q with three binary operations $\cdot$, $/$, and $\backslash$ called respectively *multiplication*, *right division*, and *left division* such that these operations satisfy the following axioms:

(ER): $(x/y)\cdot y = x$;

(UR): $(x\cdot y)/y = x$;

(EL): $x\cdot(x\backslash y) = y$;

(UL): $x\backslash(x\cdot y) = y$.

A *right quasigroup* Q is a set Q with two binary operations $\cdot$, and $/$, satisfying (ER) and (UR). (The names (ER), (UR), (EL) and (UL) stand respectively for Existence of a solution involving Right division, Uniqueness of the solution involving Right division, and similarly for Left division.)

DEFINITION 1.2. [Cf. 5, II.6]. Let C be a category and let c be an object of C. The *comma category* of C over c has as its objects all C-morphisms $f\colon c' \to c$ and as its morphisms from $f\colon c' \to c$ to $g\colon c'' \to c$ all C-morphisms $\theta\colon c' \to c''$ such that the diagram

$$\begin{array}{ccc} c' & \xrightarrow{\theta} & c'' \\ f\big\downarrow & & \big\downarrow g \\ c & \xrightarrow[1_c]{} & c \end{array}$$

commutes. This category will be denoted by C/c.

An example of a comma category is $\mathfrak{Q}/q$, the variety of all quasigroups $\mathfrak{Q}$ over a fixed quasigroup Q.

DEFINITION 1.3. [Cf. 6]. Let Q be a quasigroup and $\mathfrak{Q}$ be the variety of all quasigroups. A *Q-module* in $\mathfrak{Q}$ is an abelian group in $\mathfrak{Q}/Q$ (the comma category of $\mathfrak{Q}$ over Q), i.e. an object $A \to Q$ of $\mathfrak{Q}/Q$ equipped with $\mathfrak{Q}/Q$-morphisms $0_Q\colon Q \to A$, $-\colon A \to A$, and $+\colon A \times_Q A \to A$ such that the abelian group identity diagrams commute. A Q-module morphism $f\colon A \to B$ between Q-modules in $\mathfrak{Q}$ is a $\mathfrak{Q}/Q$-morphism such that $+f = (f \times_Q f)+$, $-f = f-$, and $0_Q f = 0_Q$. The category $\mathfrak{A} \otimes (\mathfrak{Q}/Q)$ of Q-modules in $\mathfrak{Q}$ has Q-modules in Q as its objects and Q-morphisms between them as its morphisms.

An object of $\mathfrak{A} \otimes (\mathfrak{Q}/Q)$ can be considered as a quasigroup A which has a self-centralizing congruence α such that $A^{\alpha} \cong Q$ (via a natural isomorphism) [9, 317 and 318].

DEFINITION 1.4. Let Q be a quasigroup. Then the *Cayley diagram* Cay (Q) of Q is a directed graph with vertex set Q, and labelled arcs. For each x and y in Q, there is an arc $\langle x, R(y), xy \rangle$ from x to xy, labelled $R(y)$, and an arc $\langle x, L(y), yx \rangle$ from x to yx, labelled $L(y)$.

Quasigroups can be considered as generalizations of groups. Another generalization of a group in the categorical sense is a groupoid.

DEFINITION 1.5. A *groupoid* is a category such that all its morphisms are invertible.

DEFINITION 1.6. The *fundamental groupoid* on a directed graph X, denoted $\pi(X)$, is the free groupoid on the graph X, i.e. the codomain of a graph map $i: X \to \pi(X)$, such that for every groupoid C and graph map $j: X \to G$, there exists a unique groupoid map $\tau: \pi(X) \to G$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & \pi(X) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \tau} \\ G & \xrightarrow[1_G]{} & G \end{array}$$

commutes.

An easy characterization of the fundamental groupoid $\pi(X)$ on a graph X has been given in [4, Ch. 8].

DEFINITION 1.7. Given two categories C_1 and C_2, the *category* $C_1^{C_2}$ *of representations* of C_2 into C_1 has all functors $P: C_2 \to C_1$ as its objects and all natural transformations between them as its morphisms.

Notice that if C_2 is a groupoid, then groupoid representations of C_2 into C_1 are the same as category representations of C_2 into C_1. The category $\mathfrak{A}^{\pi CayQ}$ for example is the category of all representations of the fundamental groupoid on the Cayley diagram of Q into the category $\mathfrak{A}$ of abelian groups. Another example is the category $\mathfrak{A}^G$ of representations of a group G into the category of abelian groups.

2. Abelian coverings

Coverings of the fundamental groupoid on a graph X arise naturally in analogy with coverings of a topological space.

DEFINITION 2.1. Let G and G' be two groupoids. Let $V(G)$ denote the set of objects (vertex set) of G, and let G_{j*} denote the sets of all morphisms in G with source $j \in V(G)$. G' *covers* G if there exists a category map $\varphi: G' \to G$ such that for every $i \in V(G')$, the restriction $\varphi_i: G'_{i*} \to G_{i\varphi *}$ of φ is bijective.

DEFINITION 2.2. An *abelian covering* of the fundamental groupoid on the Cayley diagram of a quasigroup Q is a covering map $\varphi: E \to \pi CayQ$ such that:

(i) For every q in Q, the inverse image of q under φ in E, viz. E^q, is an abelian group.

(ii) For every morphism a from q to s in $\pi CayQ$, the map E^q: $E^q \to E^s$; $v \mapsto w$ (here w is the target of the unique cover of a with starting point v) is an isomorphism.

DEFINITION 2.3. The *category of abelian coverings* of the fundamental groupoid on the Cayley diagram of a fixed quasigroup Q is the subcategory of the comma category of groupoids $\mathfrak{G}$ over $\pi CayQ$ with objects all abelian covers of $\pi CayQ$ and morphisms all $\mathfrak{G}/\pi CayQ$ morphisms $\theta: E_1 \to E_2$ from $\varphi_1: E_1 \to \pi CayQ$ to $\varphi_2: E_2 \to \pi CayQ$ such that for every q in Q, the restriction $\theta|_{E_1^q}$ of θ to E_1^q is a homomorphism of abelian groups into E_2^q. We will denote this category by $AbCovQ$.

An easy characterization of an element of $AbCovQ$ is given by the following proposition.

PROPOSITION 2.4. *Let Q be a non-empty quasigroup. Suppose we are given a covering $\varphi: E \to \pi CayQ$ of $\pi CayQ$ such that the following statements are true:*

(i) *there exists an r in Q such that the inverse image of r under φ in E, namely $E^r = \{v \in E \mid v\varphi = r\}$, is an abelian group;*

(ii) *for every morphism a from r to r in $\pi CayQ$, the map E^a: $E^r \to E^r$; $v \mapsto w$ (here w is the unique cover of a with starting point v) is a homomorphism. Then φ is an object of $AbCovQ$.*

Proof. (i) We claim that $\forall q \in Q$, E^q is an abelian group. Let $b \in [\pi CayQ]_{rq}$, then $E^b E^{b^{-1}}$: $E^r \to E^r$. If $\bar{b}$ is the unique cover of b with starting point $v \in E^r$ and target $v' \in E^q$, while $\bar{b}^{-1}$ is the unique cover of b^{-1} with starting point v' and target $v'' \in E^r$, then $\bar{b}\bar{b}^{-1}$ is the unique cover of $bb^{-1} = 1_r = 1_v\varphi$, so that $v'' = v$, i.e. E^b is a bijection. Defining $vE^b \cdot wE^b = (v \cdot w)E^b$ in E^q makes E^q an abelian group. This product is well-defined, since if $v_1E^b = v_2E^c$, $w_1E^b = w_2E^c$ for $c \in [\pi CayQ]_{rq}$, then $(v_1 \cdot w_1)E^b = v_1E^b \cdot w_1E^b = v_2E^c \cdot w_2E^c = (v_2 \cdot w_2)E^c$, which proves (i).

To prove (ii) of Definition 2.2, from (i) it is clear that E^b is an isomorphism

$\forall b \in [\pi Cay Q]_{rq}$. Let $a = b^{-1}d$ where $b \in [\pi Cay Q]_{rq}$, and $d \in [\pi Cay Q]_{rs}$. Then $E^a = E^{b^{-1}d} = E^{b^{-1}}E^d$ is an isomorphism, since $E^{b^{-1}}$ and E^d are. □

Notice also that $a\varphi^{-1} = \{(m, n) \in E^q \times E^s | mE^a = n\}$ is an abelian group by the isomorphism E^a, i.e. these pairs form a subgroup of $E^q \times E^r$ isomorphic with E^q and E^r.

3. Multiplication groups

The structure of a quasigroup implies that the mappings $R_Q(q): Q \to Q;\ x \mapsto xq$ and $L_Q(q): Q \to Q;\ x \mapsto qx$ are permutations of the set Q for each $q \in Q$.

DEFINITION 3.1. If P is a subquasigroup of the quasigroup Q, then the *relative multiplication group* $\mathrm{Mlt}_Q P$ of P in Q is the permutation group generated by $\{R_Q(p), L_Q(p) | p \in P\}$. In the case $P = Q$, we simply call the permutation group the (*combinatorial*) *multiplication group* Mlt Q, and we may write $R(q)$ for $R_Q(q)$.

An important relative multiplication group is $\mathrm{Mlt}_{\tilde{Q}}\, Q$, where $\tilde{Q} = Q * I$, the coproduct of Q with the free quasigroup I on one generator x in the variety $\mathfrak{Q}$ of all quasigroups.

DEFINITION 3.2. *The universal multiplication group* $U(Q, \mathfrak{Q})$ of Q in $\mathfrak{Q}$ is the relative multiplication group of Q in $\tilde{Q}$. We will use the notation $\tilde{R}(q)$ and $\tilde{L}(q)$ respectively for $R_{\tilde{Q}}(q)$ and $L_{\tilde{Q}}(q)$ for $q \in Q$.

THEOREM 3.3. *Given a quasigroup Q, the universal multiplication group $\tilde{G} = U(Q, \mathfrak{Q})$ is the free group on $\{\tilde{R}(q), \tilde{L}(q) | q \in Q\}$.*

Proof. In the Cayley graph $Cay(\tilde{Q})$, consider the subgraph $(x\tilde{G})$ consisting of all vertices lying in the orbit $x\tilde{G}$ of x under $\tilde{G}$, and of all arcs between these vertices labelled $\tilde{R}(q)$ or $\tilde{L}(q)$ for some q in Q. Note that $(x\tilde{G})$ is (weakly) connected. If there is a circuit in $(x\tilde{G})$ starting at a vertex $x\tilde{E}(p_1, \ldots, p_m)$, its labels form a product $\tilde{F}(q_1, \ldots, q_n)$ s.t. $x\tilde{E}(p_1, \ldots, p_m)\tilde{F}(q_1, \ldots, q_n) = x\tilde{E}(p_1, \ldots, p_m)$. By [9, 236], it follows that $\tilde{F}(q_1, \ldots, q_n) = 1$.

Because $\tilde{F}(q_1, \ldots, q_n)$ is a product of labels, we can assume $\tilde{F}(q_1, \ldots, q_n) = \Pi_{i=1}^{S} \widetilde{m_{j_i k_i}}(h_i)$ where $h_i \in \{q_1, \ldots, q_n\}, j_i = R$ or $L, k_i = \pm 1$, with the following conventions:

if $j_i = R, k_i = 1$, *then* $\tilde{m}_{R1} = \tilde{R}(h_i)$;

if $j_i = R, k_i = -1$, *then* $\tilde{m}_{R-1} = \tilde{R}^{-1}(h_i)$;

if $j_i = L, k_i = 1$, *then* $\tilde{m}_{L1} = \tilde{L}(h_i)$;

if $j_i = L, k_i = -1$, *then* $\tilde{m}_{L-1} = \tilde{L}^{-1}(h_i)$.

Suppose the circuit is not trivial. Then we can assume further than $S \geq 1$ (since $S = 0 \Rightarrow \tilde{F} = 1$), and $\tilde{F}$ is in "reduced form", i.e. there is no i such that $j_i = j_{i+1}$, $k_i = -k_{i+1}$, and $h_i = h_{i+1}$. From $\tilde{F} = 1$, we have $x\tilde{F}(q_i, \ldots, q_n) = x$. Now x is in "normal form" [3, 2.1] in $Q * I$, so that there exists a reduction chain

$$U = x\tilde{F} \to U_1 \to U_2 \to \cdots \to U_k = x$$

[3, T2.2]. Now, $Q = \langle q \in Q | q_1 b q_2 = q_3 \text{ if } q_1 b q_2 = q_3 \text{ in } Q \rangle$ is a set of closed relations for Q with $b \in \{\cdot, /, \backslash\}$ (in the sense of [3, 1.3]), so that $Q * I = \langle q \in Q, x | q_1 b q_2 = q_3$ if $q_1 b q_2 = q_3$ in $Q \rangle$.

Define a relation $\leq$ on the set of "components" [3, 1.2] of $x\tilde{F}$, by $z_1 \leq z_2$ if z_1 is a component of z_2. Let m be the "minimal" component of $x\tilde{F}$ such that the elementary operation $U \to U_1$ occurs within it. Since the "reduced" form of $\tilde{F}$ is not 1, the length of m, $\ell(m) \geq 2$ (if $\ell(m) = 1$, m is a generator, so that the elementary operation is on a generator, a contradiction). So the operation occurs at $m = x \prod_{i=1}^{t} \widetilde{m_{j_i k_i}}(h_i)$, $j \geq 1$, not at $x \prod_{i=1}^{t-1} \widetilde{m_{j_i k_i}}(h_i)$, i.e. involving h_t. In cases (i)–(iv) of elementary reductions [3, 2.1], we will have a contradiction since $j_{t-1} = j_t$ and $k_{t-1} = -k_t$. Cases (v) and (vi) are out of consideration since h_j is already in normal form. In case (vii), if we replace $(x \prod_{i=1}^{t-1} \widetilde{m_{j_i k_i}}(h_i)) \cdot h_t$ by z, then $z = q$ for some $q \in Q$ so that $x\tilde{F}(q_1, \ldots, q_n) = q \prod_{i=\ell+1}^{S} \widetilde{m_{j_i k_i}}(h_i) \in Q$, but $x \in Q$, a contradiction. Hence the circuit is trivial.

A left action of $\tilde{G}$ on $(x\tilde{G})$ is defined by letting $\tilde{F}(q_1, \ldots, q_n)$ in $\tilde{G}$ send the arc $\langle x\tilde{E}(p_1, \ldots, p_m), \tilde{D}(q), x\tilde{E}(p_1, \ldots, p_m)\tilde{D}(q) \rangle$ to $\langle x\tilde{F}(q_1, \ldots, q_n)^{-1}\tilde{E}(p_1, \ldots, p_m)$, $\tilde{D}(q), x\tilde{F}(q_1, \ldots, q_n)^{-1}\tilde{E}(p_1, \ldots, p_m)\tilde{D}(q) \rangle$, where $\tilde{D}(q)$ denotes $\tilde{R}(q)$ or $\tilde{L}(q)$. Suppose that a vertex $x\tilde{E}(p_1, \ldots, p_m)$ is fixed by an element $\tilde{F}(q_1, \ldots, q_n)$ of $\tilde{G}$. Then $x\tilde{E}(p_1, \ldots, p_m) = x\tilde{F}(q_1, \ldots, q_n)^{-1}\tilde{E}(p_1, \ldots, p_n)$, whence $x\tilde{F}(q_1, \ldots, q_n) = x$. By [9, 236] with $m = 0$, we have $\tilde{F}(q_1, \ldots, q_n) = 1$. Thus no non-identity element of $\tilde{G}$ leaves a vertex of $(x\tilde{G})$ fixed.

Now suppose that an arc $\langle x\tilde{E}(p_1, \ldots, p_m), \tilde{R}(q), x\tilde{E}(p_1, \ldots, p_m)\tilde{R}(q) \rangle$ of $(x\tilde{G})$ is inverted by $\tilde{F}(q_1, \ldots, q_n)$ in $\tilde{G}$, so that $x\tilde{F}(q_1, \ldots, q_n)^{-1}\tilde{E}(p_1, \ldots, p_m) = x\tilde{E}(p_1, \ldots, p_m)\tilde{R}(q)$ and $x\tilde{F}(q_1, \ldots, q_n)^{-1}\tilde{E}(p_1, \ldots, p_m)\tilde{R}(q) = x\tilde{E}(p_1, \ldots, p_m)$. Then $x\tilde{E}(p_1, \ldots, p_m)\tilde{R}(q)^2 = x\tilde{E}(p_1, \ldots, p_m)$, whence $\tilde{R}(q)^2 = 1$ by [9, 236]. In particular $xq \cdot q = x$. Consider the quasigroup $(Q, \cdot, /, \backslash)$ defined on the set of rationals Q by $r \cdot s = 2r + s$, $r/s = (r - s)/2$, and $r\backslash s = s - 2r$ for r, s in Q. Define $f: Q \to Q$; $q \mapsto 0$. Since $\{0\}$ is a subquasigroup of $(Q, \cdot, /, \backslash)$, f is a quasigroup morphism. The image of $x = (xq)q$ in $\tilde{Q}$ under $f * (x \mapsto 1)$; $\tilde{Q} \to Q$ is $1 = (1.0) \cdot 0 = 4$, an impossibility. Thus no arc of $(x\tilde{G})$ labelled $\tilde{R}(q)$ is inverted by an element of $\tilde{G}$. A flipping argument [9, 115] shows that no arc labelled $\tilde{L}(q)$ is inverted. Thus $\tilde{G}$ acts freely on $(x\tilde{G})$ (in the sense of [7, I.3.3]). The quotient graph $\tilde{G}\backslash(x\tilde{G})$ is a bouquet of circles labelled with the elements of $\tilde{R}(Q) U \tilde{L}(Q)$. By the Reidemeister Theorem [7, Theorem 1.4] it follows that $\tilde{G}$ is the free group on $\tilde{R}(Q) U \tilde{L}(Q)$. □

Let e be a fixed element of Q. Then the category $\mathfrak{A}^{\tilde{G}_e}$ of representations of the stabilizer group $\tilde{G}_e$ of e in the universal multiplication group $\tilde{G} = U(Q, \mathfrak{Q})$ is equivalent to the category $\mathfrak{A} \otimes (\mathfrak{Q}/Q)$ of Q-modules in $\mathfrak{Q}$ [9, 236].

4. The equivalence of representations

The aim of this section is to present the key result showing how the category of Q-modules in $\mathfrak{Q}$ is equivalent to both the category of representations of the fundamental groupoid on the Cayley diagram of Q and the category of abelian coverings of Q.

PROPOSITION 4.1. *The stabilizer $\tilde{G}_e$ is the vertex group of $\pi CayQ$ at the vertex e.*

Proof. Take $y = [y_1, y_2, \ldots, y_n] \in [\pi CayQ]_{ee}$, a loop at e.
Then

$$y_i = \begin{cases} \langle e_i, R(e_i \backslash e_{i+1}), e_{i+1} \rangle & \text{or} \\ \langle e_i, R^{-1}(e_{i+1} \backslash e_i), e_{i+1} \rangle & \text{or} \\ \langle e_i, L(e_{i+1} / e_i), e_{i+1} \rangle & \text{or} \\ \langle e_i, L^{-1}(e_i / e_{i+1}), e_{i+1} \rangle, \end{cases}$$

where $e_1 = e_{n+1} = e$.

We can denote y_i by $\langle e_i, m_{j_i k_i}(e_i, e_{i+1}), e_{i+1} \rangle$, where $j_i = R$ or L and $k_i = \pm 1$, with the following conventions:

if $j_i = R, k_i = 1,$ then $m_{j_ik_i}(e_i, e_{i+1}) = R(e_i \backslash e_{i+1});$

if $j_i = R, k_i = -1,$ then $m_{j_ik_i}(e_i, e_{i+1}) = R^{-1}(e_{i+1} \backslash e_i);$

if $j_i = L, k_i = 1,$ then $m_{j_ik_i}(e_i, e_{i+1}) = L(e_{i+1}/e_i);$

if $j_i = L, k_i = -1,$ then $m_{j_ik_i}(e_i, e_{i+1}) = L^{-1}(e_i/e_{i+1}).$

Let $f: [\pi CayQ]_{ee} \to \tilde{G}$ be defined by $f([y_1, y_2, \ldots, y_n]) = \Pi_{i=1}^n \widetilde{m_{j_ik_i}}(e_i, e_{i+1}),$ where $\widetilde{m_{j_ik_i}}(e_i, e_{i+1}) \in \tilde{G}$ with the following conventions:

if $j_i = R, k_i = 1,$ then $\widetilde{m_{j_ik_i}}(e_i, e_{i+1}) = \tilde{R}(e_i \backslash e_{i+1});$

if $j_i = R, k_i = -1,$ then $\widetilde{m_{j_ik_i}}(e_i, e_{i+1}) = \tilde{R}^{-1}(e_{i+1} \backslash e_i);$

if $j_i = L, k_i = 1,$ then $\widetilde{m_{j_ik_i}}(e_i, e_{i+1}) = \tilde{L}(e_{i+1}/e_i);$

if $j_i = L, k_i = -1,$ then $\widetilde{m_{j_ik_i}}(e_i, e_{i+1}) = \tilde{L}^{-1}(e_i/e_{i+1}).$

Clearly f is a well-defined mapping, since if $[y_1, y_2, \ldots, \hat{y}_i, \hat{y}_{i+1}, y_{i+2}, \ldots, y_n]$ is a simple reduction (i.e. $y_i = y_{i+1}^{-1}$) of y (where $\hat{y}_i$ means y_i is omitted), then $y_i = \langle e_i, m_{j_ik_i}(e_i, e_{i+1}), e_{i+1} \rangle,$ and $y_{i+1} = \langle e_{i+1}, m_{j_{i+1}k_{i+1}}(e_{i+1}, e_{i+2}), e_{i+2} \rangle$ where $e_{i+2} = e_i$ and $m_{j_ik_i}(e_i, e_{i+1}) = m_{j_{i+1}k_{i+1}}(e_{i+1}, e_{i+2})^{-1}.$ Hence $f[(y_1, y_2, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots, y_n]) = \Pi_{t \in \{1,2,\ldots,\hat{i},\widehat{i+1},i+2,\ldots,n\}} \widetilde{m_{j_ik_i}}(e_t, e_{t+1}) = \Pi_{t=1}^n \widetilde{m_{j_ik_i}}(e_t, e_{t+1}) = f(y),$ where $\widetilde{m_{j_ik_i}}(e_i, e_{i+1}) = \widetilde{m_{j_{i+1}k_{i+1}}}(e_{i+1}, e_{i+2})^{-1}.$ We can restrict the codomain of f to be $\tilde{G}_e$, since $e\, \Pi_{i=1}^n \widetilde{m_{j_ik_i}}(e_i, e_{i+1}) = e.$ Now f is also a homomorphism, since

$$
\begin{aligned}
f(x \cdot y) &= f([x_1, x_2, \ldots, x_m][y_1, y_2, \ldots, y_n]) \\
&= \prod_{i=1}^{m} \widetilde{n_{j_ik_i}}(e_i, e_{i+1}) \prod_{i=1}^{n} \widetilde{m_{j_{m+i}k_{m+i}}}(e_{m+i}, e_{m+i+1}) \\
&= \left(\prod_{i=1}^{m} \widetilde{n_{j_ik_i}}(e_i, e_{i+1}) \right) \left(\prod_{i=1}^{n} \widetilde{m_{j_{m+i}k_{m+i}}}(e_{m+i}, e_{m+i+1}) \right) \\
&= f(x)f(y),
\end{aligned}
$$

where

$$x_i = \langle e_i, n_{j_ik_i}(e_i, e_{i+1}), e_{i+1} \rangle$$

and

$$y_i = \langle e_{m+i}, m_{j_{m+i}k_{m+i}}(e_{m+i}, e_{m+i+1}), e_{m+i+1} \rangle, \quad \text{with } e_1 = e_{m+1} = e_{m+n+1}.$$

Now f is one to one: Suppose that x is a reduced path in $[\pi CayQ]_{ee}$ and $f(x) = f([x_1, x_2, \ldots, x_m]) = 1_{\tilde{Q}}$. Then $\Pi_{i=1}^{n} \widetilde{n_{j_ik_i}}(e_i, e_{i+1}) = 1_{\tilde{Q}}$ implies, by the freeness of $\tilde{F}$ on $\{\tilde{R}(q), \tilde{L}(q) | q \in Q\}$ [9, 238], that $\exists i \in \{1, 2, \ldots, n\}$. $\widetilde{n_{j_ik_i}}(e_i, e_{i+1}) = \widetilde{n_{j_{i+1}k_{i+1}}}(e_{i+1}, e_{i+2})^{-1}$. Applying r_Q [9, 333], we obtain $n_{j_ik_i}(e_i, e_{i+1}) = n_{j_{i+1}k_{i+1}}(e_{i+1}, e_{i+2})^{-1}$, i.e. $x_i = x_{i+1}^{-1}$, contradicting the reducedness of x. Hence $x = 1_{ee}$. The homomorphism f is also onto, since $\tilde{G}_e = \langle \tilde{T}_e(q), \tilde{R}_e(q, r), \tilde{L}_e(q, r) \rangle$ [9, 244], and $\tilde{T}_e(q) = f[\langle e, R(e \backslash q), q \rangle, \langle q, L^{-1}(q/e), e \rangle]$; $\tilde{R}_e(q, r) = f[\langle e, R(e \backslash q), q \rangle, \langle q, R(r), qr \rangle, \langle qr, R(e \backslash qr)^{-1}, e \rangle]$; $\tilde{L}_e(q, r) = f[\langle e, L(q/e), q \rangle, \langle q, L(r), rq \rangle, \langle rq, L(rq/e)^{-1}, e \rangle]$. Hence we conclude $\tilde{G}_e \cong [\pi CayQ]_{ee}$. □

Suppose we have a representation δ from $\tilde{G}_e$ to $\mathfrak{A}$. Then using the notation in [9, 247], where $\tilde{\rho}(e, q) = \tilde{R}(e \backslash e)^{-1} \tilde{R}(e \backslash q)$, we will define an element $\delta\alpha = P: \pi CayQ \to \mathfrak{A}$ of $\mathfrak{A}^{\pi CayQ}$ as follows:

$$\tilde{\rho}(e, q)\alpha : \delta(e) = M \to M \otimes \tilde{\rho}(e, q); m \mapsto m \otimes \tilde{\rho}(e, q)$$

[2, §10.1] or [9, §3.3].

If $a \in [\pi CayQ]_{qr}$, then a is written uniquely in the form $\tilde{\rho}(e, q)^{-1} g_e \tilde{\rho}(e, r)$ by taking $g_e = \tilde{\rho}(e, q) a \tilde{\rho}(e, r)^{-1} \in \tilde{G}_e = [\pi CayQ]_{ee}$ by Proposition 4.1, so that $a\alpha : M \otimes \tilde{\rho}(e, q) \to M \otimes \tilde{\rho}(e, r)$, $a\alpha = [\tilde{\rho}(e, q)\alpha]^{-1}(g_e\alpha)[\tilde{\rho}(e, r)\alpha]$, and $g_e\alpha = g_e$. On objects, $qP = M \otimes \tilde{\rho}(e, q)$, so that $eP = M \otimes \tilde{\rho}(e, e) = M \otimes 1 = M$, hence P is well-defined.

If $f: \delta_1 \to \delta_2$ is a morphism in $\mathfrak{A}^{\tilde{\mathfrak{G}}_e}$, i.e. a $\tilde{G}_e$-module homomorphism, then $f\alpha: \delta_1\alpha \to \delta_2\alpha$ is

$$M_1 \otimes \tilde{\rho}(e, q) \to M_2 \otimes \tilde{\rho}(e, q); \qquad m_1 \otimes \tilde{\rho}(e, q) \mapsto m_1 f \otimes \tilde{\rho}(e, q)$$

with the property that for every $g_e \in \tilde{G}_e$, $(m_1 \otimes \tilde{\rho}(e, q)) f\alpha g_e \delta_2 = (m_1 f \otimes \tilde{\rho}(e, q)) g_e \delta_2 = (m_1 f) g_e \otimes \tilde{\rho}(e, q) = (m_1 g_e \otimes \tilde{\rho}(e, q)) f\alpha = (m_1 \otimes \tilde{\rho}(e, q)) g_e \delta_1 f\alpha$, that is $f\alpha g_e \delta_2 = g_e \delta_1 f\alpha$, so tht $f\alpha$ is a natural transformation from $\delta_1\alpha$ to $\delta_2\alpha$. This leads us to the following proposition.

PROPOSITION 4.2. *The category map α gives a functor from $\mathfrak{A}^{\tilde{G}_e}$ to $\mathfrak{A}^{\pi CayQ}$.*

Given a representation $P: \pi CayQ \to \mathfrak{A}$, define $P\beta = P|_{[\pi CayQ]_{ee}}$.

PROPOSITION 4.3. *The map β can be extended to a functor from $\mathfrak{A}^{\pi CayQ}$ to $\mathfrak{A}^{\tilde{G}_e}$.*

Proof. Take a morphism $f: P_1 \to P_2$ in $\mathfrak{A}^{\pi CayQ}$ under β to $f\beta = f|_{P_1\beta}$. Clearly $f\beta$ is a morphism in $\mathfrak{A}^{\tilde{G}_e}$. □

LEMMA 4.4. *For each representation P of $\pi CayQ$ into $\mathfrak{A}$, there is a natural $\mathfrak{A}^{\pi CayQ}$-isomorphism $h_p : P(\beta\alpha) \to P$, a collection of $\mathfrak{A}$-morphisms, one for each object q of $\pi CayQ$:*

$$\begin{array}{ccccc} qPh_p : M \otimes \tilde{\rho}(e, q) & \xrightarrow{[\tilde{\rho}(e,q)\alpha]^{-1}} & M & \xrightarrow{\tilde{\rho}(e,q)P} & A_q \\ m \otimes \tilde{\rho}(e, q) & \mapsto & m & \mapsto & m\tilde{\rho}(e, q)P. \end{array}$$

Proof. For every $a \in [\pi CayQ]_{qr}$, the following diagram commutes:

$$\begin{array}{ccc} qP\beta\alpha & \xrightarrow{qPh_p} & qP \\ {\scriptstyle aP\beta\alpha}\downarrow & & \downarrow{\scriptstyle aP} \\ rP\beta\alpha & \xrightarrow[rPh_p]{} & rp \end{array},$$

i.e.

$$\begin{array}{ccc} M \otimes \tilde{\rho}(e, q) & \xrightarrow{qPh_p} & Aq \\ {\scriptstyle a\alpha = [\tilde{\rho}(e,q)\alpha]^{-1}(g_e\alpha)[\tilde{\rho}(e,q)]}\downarrow & & \downarrow{\scriptstyle a = [\tilde{\rho}(e,q)P]^{-1}g_e P[\tilde{\rho}(e,r)P]} \\ M \otimes \tilde{\rho}(e, r) & \xrightarrow[rPh_p]{} & Ar \end{array},$$

where $a = \tilde{\rho}(e, q)^{-1}g_e\tilde{\rho}(e, r)$, since $[m \otimes (e, q)]qPh_p a = (m\tilde{\rho}(e, q)P)(\tilde{\rho}(e, q)P)^{-1} \times g_e P\tilde{\rho}(e, r)P = mg_e P\tilde{\rho}(e, r)P = mg_e\alpha\tilde{\rho}(e, r)P = m(g_e\alpha)(\tilde{\rho}(e, r)\alpha)[\tilde{\rho}(e, r)\alpha]^{-1}\tilde{\rho}(e, r)P = [m \otimes \tilde{\rho}(e, q)]a\beta\alpha[rPh_p]$. The other direction is proved similarly. □

THEOREM 4.5. *The functors α and β give an equivalence between $\mathfrak{A}^{\tilde{G}_e}$ and $\mathfrak{A}^{\pi CayQ}$.*

Proof. It is obvious that $\alpha\beta = 1$. To show that $\beta\alpha$ is equivalent to 1, suppose $f: P_1 \to P_2$ is an $\mathfrak{A}^{\pi CayQ}$ morphism, i.e. $\forall q \in Q$, $qf: A_{1q} \to A_{2q}$.

Then the following diagram commutes:

$$\begin{array}{ccc} P_1\beta\alpha & \xrightarrow{P_1h_{P_1}} & P_1 \\ {\scriptstyle f\beta\alpha}\downarrow & & \downarrow{\scriptstyle f} \\ P_2\beta\alpha & \xrightarrow{P_2h_{P_2}} & P_2 \end{array},$$

since $[m \otimes \tilde{\rho}(e, q)](qf\beta\alpha)(qP_2h_{P_2}) = [m\tilde{\rho}(e, q)P_1(qf)(\tilde{\rho}(e, q)P_2)^{-1} \otimes \tilde{\rho}(e, q)]qP_2h_{P_2} = m\tilde{\rho}(e, q)P_1(q,f)(\tilde{\rho}(e, q)P_2)^{-1}\tilde{\rho}(e, q)P_2 = m\tilde{\rho}(e, q)P_1(qf) = [m \otimes \tilde{\rho}(e, q)](qP_1h_{P_1})(qf)$.

Analogously, the following diagram commutes:

$$\begin{array}{ccc} P_1 & \xrightarrow{P_1 h_{P_1}^{-1}} & P_1\beta\alpha \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\beta\alpha} \\ P_2 & \xrightarrow[P_2 h_{P_2}^{-1}]{} & P_2\beta\alpha \end{array}$$

Hence the theorem has been proved. □

Summing up all these results, we will have:

THEOREM 4.6. *The following categories are equivalent*:
(i) $\mathfrak{A}\otimes(\mathfrak{Q}/Q)$; (ii) $\mathfrak{A}^{\tilde{G}_e}$; (iii) $\mathfrak{A}^{\pi CayQ}$; (iv) $AbCovQ$.

Proof. The equivalence between $\mathfrak{A}\otimes(\mathfrak{Q}/Q)$ and $\mathfrak{A}^{\tilde{G}_e}$ is given in [9, 336]. Theorem 4.6 proves the equivalence between $\mathfrak{A}^{\tilde{G}_e}$ and $\mathfrak{A}^{\pi CayQ}$. For the equivalence between $\mathfrak{A}^{\pi CayQ}$ and $AbCovQ$, we just have to use the corresponding functors $\Lambda: \mathfrak{A}^{\pi CayQ} \to AbCovQ$ and $\Lambda': AbCovQ \to \mathfrak{A}^{\pi CayQ}$ as described in [4, 13.30]. □

If we start with an abelian group $A \to Q$ in $\mathfrak{Q}/Q$, then we will get a right $\tilde{G}_e$-module $M = \pi^{-1}(e), e \in Q$. Applying α, we will get a representation $P: \pi CayQ \to \mathfrak{A}$ where $A_q = M \otimes \tilde{\rho}(e, q)$, and $\tilde{\rho}(e, q)\alpha: M \to M \otimes \tilde{\rho}(e, q)$; $m \mapsto m \otimes \tilde{\rho}(e, q)$. If $a \in [\pi CayQ]_{qr}$, $a\alpha: M \otimes \tilde{\rho}(e, q) \to M \otimes \tilde{\rho}(e, r)$. Applying Λ, we have a covering groupoid $\varphi: E \to \pi CayQ$, where $V(E) = \amalg_{q\in Q} M_q = \{(v, q) | v \in M_q\} = \{(m \otimes \tilde{\rho}(e, q), q) | m \in \pi^{-1}(e)\}$. This vertex set is exactly the set $A = \bigcup_{q\in Q} M \otimes \tilde{\rho}(e, q)$, of [9, 336], and $V(\varphi): V(E) \to Q$ is $\pi: A \to Q$ with operations analogous to that of [9, 332].

5. The right quasigroup case

The structure of a right quasigroup implies that the mapping $R_Q(q)$ of Section 3 is a permutation of the underlying set Q at a right quasigroup Q. However, the corresponding $L_Q(q)$ need not be a permutation on Q. Let SQ denote the monoid of all mappings from the set Q into Q with composition as the binary operation.

DEFINITION 5.1. The submonoid of SQ generated by $\{R(q), R^{-1}(q), L(q) | q \in Q\}$ is called the *multiplication monoid* MQ, and the group generated by $\{R(q) | q \in Q\}$ contained in MQ is called the *right multiplication group* $RMltQ$.

DEFINITION 5.2. If P is a right subquasigroup of a right quasigroup Q, then the *relative multiplication monoid* $M_Q P$ of P in Q is the submonoid of MQ generated by $\{R_Q(p), R_Q(p)^{-1}, L_Q(p) | p \in P\}$.

The analogous construction of the $\mathscr{R}$-universal multiplication monoid $UM(Q, \mathscr{R})$ of Q in the variety $\mathscr{R}$ of all right quasigroups is the relative multiplication monoid of Q in $\tilde{Q} = Q * I$, the coproduct in $\mathscr{R}$ of Q with the free right quasigroup on a generator x. We will also have the analogous category $\mathfrak{A} \otimes \mathscr{R}/Q$ of Q-modules in R. Let $\pi: A \to Q$ be a Q-module in R. Then there is an $\mathscr{R}/Q$-morphism $-: \alpha \to A$, called *subtraction*, defined as the following composition:

$$-: A \times_Q A \xrightarrow{1 \times (-)} A \times_Q A \xrightarrow{+} A.$$

The kernel of $-: \alpha \to A$ is a congruence $(\alpha|\alpha)$ on α which is a centering congruence by which α centralizes itself (cf. [9, 315]). Also, we will get analogies to [9, 334] and [9, 336] as follows:

PROPOSITION 5.3. *Let Q be a right quasigroup appearing as a right subquasigroup of a right quasigroup A in R. Then $r_A: UM(Q, R) \to M_A Q$; $\tilde{F}(q_1, \ldots, q_n) \mapsto F_A(q_1, \ldots, q_n)$ is a monoid epimorphism from $\tilde{G} = UM(Q, R)$ onto $M_A Q$.*

PROPOSITION 5.4. *Let $\pi: A \to Q$ be a Q-module. Identify Q with its image in A under O. Then for elements a, b of A, one has:*

(i) $ab = a \cdot b\pi + a\pi \cdot b$;

(ii) $a/b = a/b\pi - [a\pi/b\pi \cdot b]/b\pi$. □

The next step is to define the Cayley diagram of a right quasigroup Q.

DEFINITION 5.5. Let Q be a right quasigroup. The *Cayley diagram* Cay Q of Q is a directed graph with vertex set Q and labelled arcs. For each x, y in Q, there is an arc $\langle x, R(y), xy \rangle$ from x to xy, an arc $\langle x, R^{-1}(y), x/y \rangle$ from x to x/y, and an arc $\langle x, L(y), yx \rangle$ from x to yx. These arcs are labelled $R(y)$, $R^{-1}(y)$ and $L(y)$ respectively.

We are able to generate a category $\vec{P}CayQ$ from $CayQ$ by taking the set Q as objects and "reduced" paths of $CayQ$ as morphisms. Here "reducedness" means applying all possible equations $\langle x, R(y), xy \rangle \langle xy, R^{-1}(y), x \rangle = 1$ and $\langle x, R^{-1}(y), x/y \rangle \langle x/y, R(y), x \rangle = 1$. Consider the groupoid generated by $\{\langle x, R(y), xy \rangle | x, y \in Q\}$, which is a subcategory of $\vec{P}CayQ$. It consists of components C_i for i in an index set I. For each i, pick a representative $\bar{q}_i$ in $V(C_i)$. Then for each $q \in V(\vec{P}CayQ)$, q is connected to exactly one element $\bar{q}$ of the set $\{\bar{q}_i | i \in I\}$ of representatives by a sequence of simple arcs only using the R's and their inverses. Let $\rho(\bar{q}, q)$ be the reduced path from $\bar{q}$ to q given by a sequence of labels $R^{\pm 1}(x_j)$, where $x_j \in Q$. Take $\rho(\bar{q}, \bar{q}) = 1$, the empty path at $\bar{q}$. Let $\tilde{\rho}(\bar{q}, q)$ be the element corresponding to $\rho(\bar{q}, q)$ in $UM(Q, R)$. Then given a Q-module $\pi: A \to Q$ in $\mathscr{R}$, we will get a representation $P_A = \pi\alpha'$ of $\vec{P}CayQ$ into the category $\mathfrak{A}$ of abelian groups as follows: $qP_A = \{(m, \bar{q}, q): m \in \pi^{-1}(q\varepsilon)\}$, $\rho(\bar{q}, q)\alpha': (m, \bar{q}, \bar{q}) \mapsto (m, \bar{q}, q)$. If $a \in [\vec{P}CayQ]_{qr}$, then a can be rewritten uniquely as $a = \rho(\bar{q}, q)^{-1}[\rho(\bar{q}, q)a\rho(\bar{r}, r)^{-1}]\rho(\bar{r}, r)$, so that $a\alpha' = \rho(\bar{q}, q)\alpha^{-1}[\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1}]\rho(\bar{r}, r)\alpha$, where $\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1} \in [UM(Q, R)]_{\bar{q}\bar{r}}$ acts via $M_A Q$ as in Proposition 5.3. Clearly $\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1}$ is a homomorphism of abelian groups.

If $f: (\pi_1: A_1 \to Q) \to (\pi_2: A_2 \to Q)$ is a morphism in $\mathfrak{A} \otimes \mathscr{R}/Q$, then clearly $f[\pi_1^{-1}(q)] \subseteq \pi_2^{-1}(\bar{q})$, so that $f\alpha': P_1 = \pi_1\alpha' \to \pi_2\alpha' = P_2$, defined by $qP_1 \to qP_2$; $(m, \bar{q}, q) \mapsto (f(m), \bar{q}, q)$, is an abelian group homomorphism.

PROPOSITION 5.6. *The assignment α' is a functor from $\mathfrak{A} \otimes \mathscr{R}/Q$ to $\mathfrak{A}^{\vec{P}CayQ}$.*

Proof. Clearly for every $a \in [\vec{P}CayQ]_{qr}$, the diagram

$$\begin{array}{ccc} qP_1 & \xrightarrow{qf\alpha'} & qP_2 \\ {\scriptstyle aP_1}\downarrow & & \downarrow{\scriptstyle aP_2} \\ rP_1 & \xrightarrow[rf\alpha']{} & rP_2 \end{array}$$

commutes, since $(m, \bar{q}, q)(qf\alpha')(aP_2) = (f(m), \bar{q}, q)(aP_2) = (mf[\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1}]\pi_2, \bar{r}, r) = (m[\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1}]\pi_1 f, \bar{r}, r) = (m[\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1}]\pi_1, \bar{r}, r)(rf\alpha) = (m, \bar{q}, q) \times (aP_1)(rf\alpha)$. The other conditions to be a functor are satisfied by α' trivially. □

Suppose on the other hand that we have a representation P from $\vec{P}CayQ$ into the category $\mathfrak{A}$ of abelian groups. Let $P\beta': A \to Q$, where

$$\begin{aligned} V(a) &= \mathop{\overset{\circ}{\bigcup}}_{\bar{q} \in \{\bar{q}_i | i \in I\}} \coprod_{q \in [\bar{q}]} \bar{q}P \\ &= \{(m, \bar{q}, q) | q \in [\bar{q}], \bar{q} \in \{\bar{q}_i | i \in I\}\}. \end{aligned}$$

Here $[\bar{q}]$ is the path component of q in the groupoid generated by $\{\langle x, R(y), xy\rangle | x, y \in Q\}$, and $(m, \bar{q}, q)P\beta' = q$. Define $(m, \bar{q}, q) \cdot (n, \bar{r}, r) = [m\rho(\bar{q}, q)\langle q, R(r), qr\rangle\rho(\bar{q}r, qr)^{-1} + n\rho(\bar{r}, r)\langle r, L(q), qr\rangle\rho(\bar{q}r, qr)^{-1}, \bar{q}r, qr]$ and $(m, \bar{q}, q)/(n, \bar{r}, r) = ([m\rho(\bar{q}, q) - n\rho(\bar{r}, r)\langle r, L(q/r), q\rangle]\langle q, R^{-1}(r), q/r\rangle\rho(\overline{q/r}, q/r)^{-1}, \overline{q/r}, q/r)$. Also define maps $0_Q: Q \to A$; $q \mapsto (0, \bar{q}, q)$ ("zero"), $-: A \to A$; $(m, \bar{q}, q) \mapsto (-m, \bar{q}, q)$ ("negation"), and $+: A \times_Q A \to A$; $((m, \bar{q}, q), (n, \bar{q}, q)) \mapsto (m + n, \bar{q}, q)$ ("addition"). Then it is easy to see that $P\beta'$ is a Q-module in R.

Let $f: P_1 \to P_2$ be a morphism in $\mathfrak{A}^{\vec{P}CayQ}$, i.e. $\forall q \in Q$, $qf: qP_1 \to qP_2$ is given such that, for every $a \in [\vec{P}CayQ]_{qr}$, the diagram

$$\begin{array}{ccc} qP_1 & \xrightarrow{qf} & qP_2 \\ {\scriptstyle aP_1}\downarrow & & \downarrow{\scriptstyle aP_2} \\ rP_1 & \xrightarrow[rf]{} & rP_2 \end{array}$$

commutes. Define

$$f\beta': (P_1\beta' = \pi_1: A_1 \to Q) \to (P_2\beta' = \pi_2: A_2 \to Q);$$
$$(m, \bar{q}, q) \mapsto (m\rho(\bar{q}, q)P_1 qf\rho(\bar{q}, q)^{-1}P_2, \bar{q}, q).$$

PROPOSITION 5.7. *The assignment* β' *gives a functor from* $\mathfrak{A}^{\vec{P}CayQ}$ *to* $\mathfrak{A} \otimes \mathcal{R}/Q$.

Proof. It is an easy exercise to show that $f\beta'$ is a right quasigroup homomorphism which commutes with O_Q, $-$, *and* $+$. Furthermore, it is trivial that $(f \cdot g)\beta' = f\beta' \cdot g\beta'$ and $1\beta' = 1$. □

LEMMA 5.8. *For each Q-module* $\pi: A \to Q$ *in* $\mathcal{R}$, *there exists an* $(\mathfrak{A} \otimes \mathcal{R}/Q)$-*natural isomorphism* $g'_\pi: (\pi: A \to Q) \to (\pi\alpha\beta: A' \to Q)$ *given by restrictions* $\pi^{-1}(q) \to \pi^{-1}(\bar{q}) \times \{\bar{q}\} \times \{q\}$; $m \mapsto (m\rho(\bar{q}, q)\pi^{-1}, \bar{q}, q)$. (*Note that indices have been omitted from the notation here for clarity.*)

Proof. The map $\pi g'_\pi$ is a right quasigroup homomorphism, since for $m \in \pi^{-1}(q)$ and $n \in \pi^{-1}(r)$ we have $mn \in \pi^{-1}(qr)$ and $m/n \in \pi^{-1}(q/r)$, so that $m\pi g'_\pi \cdot n\pi g'_\pi = (m\rho(\bar{q}, q)\pi^{-1}, \bar{q}, q) \cdot (n\rho(\bar{r}, r)\pi^{-1}, \bar{r}, r) = [m\rho(\bar{q}, q)\pi^{-1}\rho(\bar{q}, q)\pi\alpha\langle q, R(r), qr\rangle\rho(\bar{q}r, qr)^{-1}\pi\alpha + n\rho(\bar{r}, r)\pi^{-1}\rho(\bar{r}, r)\pi\alpha\langle r, L(q), qr\rangle\rho(\bar{q}r, qr)^{-1}\pi\alpha, \bar{q}r, qr] = ([m\langle q, R(r), qr\rangle + n\langle r, L(q), qr\rangle]\rho(\bar{q}r, qr)^{-1}\pi, \bar{q}r, qr) = (mn\rho(\bar{q}r, qr)\pi^{-1}, \bar{q}r, qr) = (mn)\pi g'_\pi$, using Proposition 5.4 (i). Similarly, using Proposition 5.4 (ii), one can show $m\pi g'_\pi / n\pi g'_\pi = (m/n)\pi g'_\pi$. Clearly g'_π is an $\mathfrak{A} \otimes \mathcal{R}/Q$ morphism, since it commutes with $+$, $-$, and 0_Q. An analogous result is valid for g'^{-1}_π. □

LEMMA 5.9. *Given a representation P from* $\vec{P}CayQ$ *to the category* $\mathfrak{A}$ *of abelian groups, there exists a natural isomorphism* $h'_p: P \to P\beta\alpha$, *defined by* $qP \to \bar{q}P \times \{\bar{q}\} \times \{q\}$, $m \mapsto (m\rho(\bar{q}, q)P^{-1}, \bar{q}, q)$.

Proof. Suppose $a \in [\vec{P}CayQ]_{qr}$. Then $a = \rho(\bar{q}, q)^{-1}[\rho(\bar{q}, q)a\rho(\bar{r}, r)^{-1}]\rho(\bar{r}, r)$, so that $a\beta\alpha = \rho(\bar{q}, q)\alpha^{-1}[\rho(\bar{q}, q)a\rho(\bar{r}, r)^{-1}]\rho(\bar{r}, r)\alpha$. The diagram

$$\begin{array}{ccc} qP & \xrightarrow{qPh'_p} & \bar{q}P \times \{\bar{q}\} \times \{q\} \\ {\scriptstyle aP}\downarrow & & \downarrow{\scriptstyle a\beta\alpha} \\ rP & \xrightarrow[rPh'_p]{} & \bar{r}P \times \{\bar{r}\} \times \{r\} \end{array}$$

commutes, since $t(qPh'_p)(a\beta\alpha) = (t\tilde{\rho}(\bar{q}, q)P^{-1}, \bar{q}, q)(a\beta\alpha) = (t\tilde{\rho}(\bar{q}, q)P^{-1}\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho} \times (\bar{r}, r)^{-1}, \bar{r}, r) = (t\tilde{\rho}(\bar{q}, q)P^{-1}(\rho(\bar{q}, q)a\rho(\bar{r}, r)^{-1})P, \bar{r}, r) = t(aP)(rPh'_p)$. The penultimate equality here holds since the action of $\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1}$ in $[UM(Q, R)]_{\bar{q}\bar{r}}$ is via $M_A Q$ by Proposition 5.3, which is the same as that of $\rho(\bar{q}, q)a\rho(\bar{r}, r)P$. Analogously, the diagram

$$\begin{array}{ccc} \bar{q}P \times \{\bar{q}\} \times \{\bar{q}\} & \xrightarrow{qPh_p'^{-1}} & qP \\ {\scriptstyle a\beta\alpha}\downarrow & & \downarrow{\scriptstyle a} \\ \bar{r}P \times \{\bar{r}\} \times \{\bar{r}\} & \xrightarrow[rPh_p'^{-1}]{} & rP \end{array}$$

commutes. □

THEOREM 5.10. *The functors α' and β' give an equivalence bewteen $\mathfrak{A} \otimes (\mathscr{R}/Q)$ and $\mathfrak{A}^{\vec{P}CayQ}$.*

Proof. If $f\colon (\pi_1\colon A_1 \to Q) \to (\pi_2\colon A_2 \to Q)$ is a morphism in $\mathfrak{A} \otimes \mathscr{R}/Q$, then the diagram

$$\begin{array}{ccc} \pi_1 & \xrightarrow{\pi_1 g'_{\pi_1}} & \pi_1\alpha\beta \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\alpha\beta} \\ \pi_2 & \xrightarrow[\pi_2 g'_{\pi_2}]{} & \pi_2\alpha\beta \end{array}$$

commutes, since $m(\pi_1 g'\pi_1)[f\alpha\beta] = (m\rho(\bar{q}, q)\pi_1^{-1}, \bar{q}, q)[f\alpha\beta] = (m\rho(\bar{q}, q)\pi_1^{-1}f_{\bar{q}}, \bar{q}, q) = (mf_q\rho(\overline{qf}, qf)\pi_2^{-1}, \bar{q}, q) = (mf_q\rho(\bar{q}, q)\pi_2^{-1}, \bar{q}, q) = mf(\pi_2 g'_{\pi_2})$. Analogously, the diagram

$$\begin{array}{ccc} \pi_1\alpha\beta & \xrightarrow{g'^{-1}_{\pi_1}} & \pi_1 \\ {\scriptstyle f\alpha\beta}\downarrow & & \downarrow{\scriptstyle f} \\ \pi_2\alpha\beta & \xrightarrow[g'^{-1}_{\pi_2}]{} & \pi_2 \end{array}$$

commutes. It is an easy exercise to show that for each morphism $f\colon P_1 \to P_2$, in $\mathfrak{A}^{\vec{P}CayQ}$, the diagram

$$\begin{array}{ccccc} P_1 & \xrightarrow{h'_{p1}} & P_1\beta\alpha & \xrightarrow{h'^{-1}_{p1}} & P_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\beta\alpha} & & \downarrow{\scriptstyle f} \\ P_2 & \xrightarrow[h'_{p2}]{} & P_2\beta\alpha & \xrightarrow[h'^{-1}_{p2}]{} & P_2 \end{array}$$

commutes. □

Note that if Q is a quasigroup, and $\pi: A \to Q$ is a Q-module in $\mathfrak{Q}$, then by forgetting the left division structure, we will have a Q-module $\pi: A \to Q$ in $\mathscr{R}$. The construction of α' and β' specializes in the case of quasigroups, in the sense that the following diagrams commute:

$$\begin{array}{ccc} \mathfrak{A} \otimes (\mathfrak{Q}/Q) & \xrightarrow{\tau\alpha} & \mathfrak{A}^{\pi CayQ} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathfrak{A} \otimes (\mathscr{R}/Q) & \xleftarrow[\beta']{} & \mathfrak{A}^{\vec{P}CayQ} \end{array} \qquad \begin{array}{ccc} \mathfrak{A} \otimes (\mathfrak{Q}/Q) & \xleftarrow{\beta\tau^{-1}} & \mathfrak{A}^{\pi CayQ} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathfrak{A} \otimes (\mathscr{R}/Q) & \xrightarrow[\alpha']{} & \mathfrak{A}^{\vec{P}CayQ} \end{array}$$

$$\begin{array}{ccc} \mathfrak{A} \otimes (\mathfrak{Q}/Q) & \xrightarrow{\tau\alpha} & \mathfrak{A}^{\pi CayQ} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathfrak{A} \otimes (\mathscr{R}/Q) & \xrightarrow[\alpha']{} & \mathfrak{A}^{\vec{P}CayQ} \end{array} \qquad \begin{array}{ccc} \mathfrak{A} \otimes (\mathfrak{Q}/Q) & \xleftarrow{\beta\tau^{-1}} & \mathfrak{A}^{\pi CayQ} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathfrak{A} \otimes (\mathscr{R}/Q) & \xleftarrow[\beta']{} & \mathfrak{A}^{\vec{P}CayQ} \end{array}$$

Here τ is the functor that gives the equivalences of $\mathfrak{A} \otimes (\mathfrak{Q}/Q)$ with $\mathfrak{A}^{\widetilde{G}e}$, while F and F' are forgetful functors.

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