QUANTUM QUASIGROUPS AND LOOPS

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Abstract. Quantum quasigroups and quantum loops are self-dual objects providing a general framework for the nonassociative extension of quantum group techniques. Bialgebra reducts of Hopf algebras are quantum loops, while sufficient conditions are given for quantum loop structure to augment to a Hopf algebra. The Moufang-Hopf algebras of Benkart *et al.*, the Hopf quasigroups and coquasigroups of Klim–Majid, and the coassociative H-bialgebras of Pérez-Izquierdo (for instance, the universal enveloping algebras of Sabinin algebras), all form quantum loops. Other quantum quasigroups offer natural nonassociative extensions of Hopf algebra constructions: quasigroup algebras, dual quasigroup algebras, and quantum couples of groups with quasigroups. Further examples include an algebra of rooted binary trees, and an algebra of skein polynomials.

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1. Introduction

Over the last few decades, there have been substantial developments in two parallel extensions of group theory: Hopf algebras ("quantum groups") on the one hand (compare [9], for example), and quasigroups ("nonassociative groups") on the other (compare [13], for example). More recently, there have been some initial moves toward a unification of these two topics [1, 4, 6], although none of the nonassociative objects proposed up to now have been able to maintain the self-duality of Hopf algebras.

The aim of the current paper is to introduce a broad, natural, and self-dual framework for the unification of Hopf algebras and quasigroups, namely quantum quasigroups and loops. A quantum quasigroup (Definition 3.1) is a bimagma (A, ∇, Δ), with a multiplication $\nabla\colon A\otimes A \to A$ and a comultiplication $\Delta\colon A \to A\otimes A$ that are mutually homomorphic, on which the two dual composites $(\Delta\otimes 1_A)(1_A\otimes\nabla)$ and $(1_A \otimes \Delta)(\nabla \otimes 1_A)$ are invertible.[1] Quantum loops (Definition 3.2) are biunital versions $(A, \nabla, \Delta, \eta, \varepsilon)$ of quantum quasigroups.

Proposition 4.1 shows that the bimonoid reduct $(A, \nabla, \Delta, \eta, \varepsilon)$ of a Hopf algebra $(A, \nabla, \Delta, \eta, \varepsilon, S)$ is a quantum loop, while Theorem 4.5 gives sufficient conditions for quantum loop structure to augment to a Hopf algebra. Corollary 4.2 shows that the Moufang-Hopf algebras of Benkart *et al.* [1] are quantum loops, while Corollary 4.3 identifies the Hopf quasigroups and coquasigroups of Klim and Majid [4] as quantum loops. Again, §4.3 explores the relationships between quantum loops and the H-bialgebras of Pérez-Izquierdo [6]. In particular, it is observed (in Example 4.4 and Corollary 4.9 respectively) that the universal enveloping algebras of Mal'tsev algebras and of Sabinin algebras (or "hyperalgebras") form quantum loops.

[1]Algebraic notation is used here and throughout the paper, with functions to the right of, or as superfixes to, their arguments. Thus compositions are read from left to right. These conventions minimize the proliferation of brackets.

Quantum quasigroups and loops are presented within the general context of a symmetric monoidal category. In the combinatorial setting of the category of sets under the Cartesian product, quantum loops just correspond to loops, while finite quantum quasigroups comprise quasigroups equipped with an ordered pair of automorphisms (§3.3). In categories of vector spaces under tensor product, among others, there are quantum quasigroup and loop extensions of familiar Hopf algebra constructions, such as group algebras (§3.4), dual group algebras (§3.5), and the quantum double of a group (§3.6). Quantum quasigroups in the symmetric monoidal category of abelian groups under the direct sum are studied in §3.7. These objects include an algebra of rooted binary trees (Example 3.43), and a "Conway algebra" of skein polynomials of links (Example 3.44).

Background material on quasigroups and loops, symmetric monoidal categories, magmas, bimagmas and Hopf algebras is provided in §2. For concepts and notational conventions not otherwise covered directly in the paper or its references, readers are advised to consult [14].

2. Background

2.1. **Quasigroups and loops.** Quasigroups may be defined combinatorially or equationally. Combinatorially, a *quasigroup* $(Q,\cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x,y,z in the equation $x\cdot y=z$ determines the third uniquely. A *loop* is a quasigroup Q with an *identity* element e such that $e\cdot x=x=x\cdot e$ for all x in Q. Groups are examples of loops. An important nonassociative example is provided by the set of nonzero octonions under multiplication. Immediate examples of quasigroups which are not loops are provided by abelian groups under subtraction.

Equationally, a quasigroup $(Q,\cdot,/,\backslash)$ is a set Q equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{llll} \text{(SL)} & x\cdot(x\backslash z)=z\,; & \text{(SR)} & z=(z/x)\cdot x\,; \\ \text{(IL)} & x\backslash(x\cdot z)=z\,; & \text{(IR)} & z=(z\cdot x)/x\,. \end{array} \tag{2.1}$$

If x and y are elements of a group $(Q,\cdot)$, the left division is given by $x\backslash y=x^{-1}y$, with $x/y=xy^{-1}$ as right division. For an abelian group considered as a combinatorial quasigroup under subtraction, the right division is addition, while the left division is subtraction.

2.2. **Symmetric monoidal categories.** The most general setting for the structures examined in this paper is a symmetric monoidal category (or "symmetric tensor category" — compare [16, Ch. 11]) $(\mathbf{V}, \otimes, \mathbf{1})$. The standard example is provided by the category $\underline{\underline{K}}$ of vector spaces over a field K. More general concrete examples are provided by varieties $\mathbf{V}$ of entropic (universal) algebras, algebras on which each (fundamental and derived) operation is a homomorphism (compare [3, 10]). These include the category $\mathbf{Set}$ of sets, the category of pointed sets, the category $\underline{\underline{R}}$ of (right) modules over a commutative, unital ring R, the category of commutative monoids, and the category of semilattices.

In a monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, there is an object $\mathbf{1}$ known as the *unit object*. For example, the unit object of $\underline{\underline{K}}$ is the vector space K. For objects A and B in a monoidal category, a *tensor product* object $A \otimes B$ is defined. For example, if U and V are vector spaces over K with respective bases X and Y, then $U \otimes V$ is the vector space with basis $X \times Y$, written as $\{x \otimes y \mid x \in X,\ y \in Y\}$. There are natural isomorphisms with components

$$\alpha_{A,B;C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)\,,\ \rho_A \colon A \otimes \mathbf{1} \to A\,,\ \lambda_A \colon \mathbf{1} \otimes A \to A$$

satisfying certain *coherence* conditions guaranteeing that one may as well regard these isomorphisms as identities [16, p.67]. Thus the bracketing of repeated tensor products is suppressed in this paper, although the natural isomorphisms ρ and λ are retained for clarity in cases such as the unitality diagram (2.2) below. In the vector space example, adding a third space W with basis Z, one has

$$\alpha_{U,V;W} \colon (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$$

for $z \in Z$, along with $\rho_U \colon x \otimes 1 \mapsto x$ and $\lambda_U \colon 1 \otimes x \mapsto x$ for $x \in X$.

A monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is *symmetric* if there is a given natural isomorphism with *twist* components $\tau_{A,B} \colon A \otimes B \to B \otimes A$ such that $\tau_{A,B}\tau_{B,A} = 1_{A \otimes B}$ [16, pp.67–8]. One uses $\tau_{U,V} \colon x \otimes y \mapsto y \otimes x$ with $x \in X$ and $y \in Y$ in the vector space example.

2.3. **Diagrams.** Let A be an object in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. Consider the respective *associativity* and *unitality diagrams*

(2.2)
$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1_A \otimes \nabla"] \arrow[d, "\nabla \otimes 1_A"'] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \otimes A \arrow[dr, "\nabla"] & A \otimes \mathbf{1} \arrow[l, "1_A \otimes \eta"'] \arrow[d, "\rho_A"] \\
\mathbf{1} \otimes A \arrow[u, "\eta \otimes 1_A"] \arrow[r, "\lambda_A"'] & A
\end{tikzcd}$$

in the category $\mathbf{V}$, the respective dual *coassociativity* and *counitality diagrams*

(2.3)
$$
\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "1_A \otimes \Delta"'] \\
A \otimes A \arrow[u, "\Delta \otimes 1_A"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \otimes A \arrow[r, "1_A \otimes \varepsilon"] \arrow[d, "\varepsilon \otimes 1_A"'] & A \otimes \mathbf{1} \\
\mathbf{1} \otimes A & A \arrow[l, "\lambda_A^{-1}"] \arrow[ul, "\Delta"'] \arrow[u, "\rho_A^{-1}"']
\end{tikzcd}
$$

in the category $\mathbf{V}$, the *bimagma diagram*

(2.4)
$$
\begin{tikzcd}
 & A \arrow[dr, "\Delta"] & \\
A \otimes A \arrow[ur, "\nabla"] \arrow[d, "\Delta \otimes \Delta"'] \arrow[dr, dashed] & & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[urr, dashed] \arrow[rr, "1_A \otimes \tau \otimes 1_A"'] & & A \otimes A \otimes A \otimes A \arrow[u, "\nabla \otimes \nabla"']
\end{tikzcd}
$$

in the category $\mathbf{V}$, the *biunital diagram*

(2.5)
$$
\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "\nabla"] & \mathbf{1} & & \mathbf{1} \arrow[ll, "1"'] \arrow[r, "\Delta"] \arrow[dl, "\eta"] & \mathbf{1} \otimes \mathbf{1} \arrow[d, "\eta \otimes \eta"] \\
A \otimes A \arrow[u, "\varepsilon \otimes \varepsilon"] \arrow[rr, "\nabla"] & & A \arrow[ul, "\varepsilon"] \arrow[rr, "\Delta"] & & A \otimes A
\end{tikzcd}
$$

in the category $\mathbf{V}$, and the *antipode diagram*

(2.6)
$$
\begin{tikzcd}
 & A \otimes A \arrow[rr, "S \otimes 1_A"] & & A \otimes A \arrow[dr, "\nabla"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbf{1} \arrow[rr, "\eta"'] & & A \\
 & A \otimes A \arrow[rr, "1_A \otimes S"'] & & A \otimes A \arrow[ur, "\nabla"'] &
\end{tikzcd}
$$

in the category $\mathbf{V}$, all of which are commutative diagrams. The arrow across the bottom of the bimagma diagram (2.4) makes use of the twist isomorphism $\tau_{A,A}$ or $\tau\colon A \otimes A \to A \otimes A$.

2.4. **Magmas and bimagmas.** This paragraph and its successor collect a number of basic definitions of various structures and homomorphisms between them.

Definition 2.1. Let $\mathbf{V}$ be a symmetric monoidal category.

(a.1) A *magma* in $\mathbf{V}$ is a $\mathbf{V}$-object A with a $\mathbf{V}$-morphism

$$\nabla \colon A \otimes A \to A$$

known as *multiplication*.

(a.2) Let A and B be magmas in $\mathbf{V}$. Then a *magma homomorphism* $f \colon A \to B$ is a $\mathbf{V}$-morphism such that the diagram

$$\begin{tikzcd} A \arrow[d, "f"'] & A \otimes A \arrow[l, "\nabla"'] \arrow[d, "f \otimes f"] \\ B & B \otimes B \arrow[l, "\nabla"] \end{tikzcd}$$

commutes.

(b.1) A *comagma* in $\mathbf{V}$ is a $\mathbf{V}$-object A with a $\mathbf{V}$-morphism

$$\Delta \colon A \to A \otimes A$$

known as *comultiplication*.

(b.2) Let A and B be comagmas in $\mathbf{V}$. A *comagma homomorphism* $f \colon A \to B$ is a $\mathbf{V}$-morphism such that the diagram

$$\begin{tikzcd} B \arrow[r, "\Delta"] & B \otimes B \\ A \arrow[u, "f"] \arrow[r, "\Delta"'] & A \otimes A \arrow[u, "f \otimes f"'] \end{tikzcd}$$

commutes.

(c.1) A *bimagma* (A, ∇, Δ) in $\mathbf{V}$ is a magma (A, ∇) and comagma (A, Δ) in $\mathbf{V}$ such that the bimagma diagram (2.4) commutes.

(c.2) Let A and B be bimagmas in $\mathbf{V}$. A *bimagma homomorphism* $f \colon A \to B$ is a magma and comagma homomorphism between bimagmas A and B.

Remark 2.2. (a) Commuting of the bimagma diagram (2.4) in a bimagma (A, ∇, Δ) means that

$$\Delta \colon (A, \nabla) \to \big(A \otimes A, (1_A \otimes \tau \otimes 1_A)(\nabla \otimes \nabla)\big)$$

is a magma homomorphism (commuting of the upper-left solid and dotted quadrilateral), or equivalently, that

$$\nabla \colon \big(A \otimes A, (\Delta \otimes \Delta)(1_A \otimes \tau \otimes 1_A)\big) \to (A, \Delta)$$

is a comagma homomorphism (commuting of the upper-right solid and dotted quadrilateral).

(b) If $\mathbf{V}$ is an entropic variety of universal algebras (compare [3, 10]), the comultiplication of a comagma in $\mathbf{V}$ may be written as

$$\Delta\colon A \to A \otimes A; a \mapsto \big((a^{L_1} \otimes a^{R_1}) \dots (a^{L_{n_a}} \otimes a^{R_{n_a}})\big)w_a \tag{2.7}$$

in a universal-algebraic version of the well-known *Sweedler notation*. In (2.7), the *tensor rank* of the image of a (or any such general element of $A \otimes A$) is the smallest arity n_a of the derived word w_a expressing the image (or general element) in terms of elements of the generating set $\{b \otimes c \mid b, c \in A\}$ for $A \otimes A$. A more compact but rather less explicit version of Sweedler notation, generally appropriate within any concrete monoidal category $\mathbf{V}$, is $a\Delta = a^L \otimes a^R$, with the understanding that the tensor rank of the image is not implied to be 1.

(c) As with quasigroups (§2.1), magma multiplication on an object A of a concrete monoidal category is often denoted by juxtaposition: $(a \otimes b)\nabla = ab$, or with $a \cdot b$ as an infix notation, for elements a, b of A.

Definition 2.3. Suppose that A is an object in a symmetric monoidal category $\mathbf{V}$.

(a) A magma (A, ∇) is *commutative* if $\tau\nabla = \nabla$. Thus if $\mathbf{V}$ is concrete, this may be written in the usual form $ba = ab$ for $a, b \in A$.
(b) A comagma (A, Δ) is *cocommutative* if $\Delta\tau = \Delta$. In Sweedler notation: $a^R \otimes a^L = a^L \otimes a^R$ for $a \in A$.
(c) A magma (A, ∇) is *associative* if the associativity diagram (2.2) commutes. In the concrete case, one often writes $ab \cdot c = a \cdot bc$, with $\cdot$ binding less strongly than juxtaposition, for a, b, c in A.
(d) A comagma (A, Δ) is *coassociative* if the coassociativity diagram (2.3) commutes. In Sweedler notation: $a^{LL} \otimes a^{LR} \otimes a^R = a^L \otimes a^{RL} \otimes a^{RR}$ for $a \in A$.

Remark 2.4. In a bimagma (A, ∇, Δ), the concepts of Definition 2.3 may be applied to the respective magma and comagma reducts of A.

2.5. **Unital structures and Hopf algebras.**

Definition 2.5. Let $\mathbf{V}$ be a symmetric monoidal category.

(a.1) A magma (A, ∇) in $\mathbf{V}$ is *unital* if it has a $\mathbf{V}$-morphism $\eta\colon \mathbf{1} \to A$ such that the unitality diagram (2.2) commutes.
(a.2) Let A and B be unital magmas in $\mathbf{V}$. Then a *unital magma homomorphism* $f\colon A \to B$ is a magma homomorphism such

that the diagram

$$\begin{tikzcd} A & \mathbf{1} \arrow[l, "\eta"'] \arrow[d, "1"] \\ B & \mathbf{1} \arrow[l, "\eta"] \arrow[from=1-1, to=2-1, "f"'] \end{tikzcd}$$

commutes.

(b.1) A comagma (A, Δ) in $\mathbf{V}$ is *counital* if it has a $\mathbf{V}$-morphism $\varepsilon\colon A \to \mathbf{1}$ such that the counitality diagram (2.3) commutes.

(b.2) Let A and B be comagmas in $\mathbf{V}$. Then a *counital comagma homomorphism* $f\colon A \to B$ is a comagma homomorphism such that the diagram

$$\begin{tikzcd} B \arrow[r, "\varepsilon"] & \mathbf{1} \\ A \arrow[u, "f"] \arrow[r, "\varepsilon"'] & \mathbf{1} \arrow[u, "1"'] \end{tikzcd}$$

commutes.

(c.1) A *biunital bimagma* $(A, \nabla, \Delta, \eta, \varepsilon)$ is a unital magma (A, ∇, η) and counital comagma (A, Δ, ε) such that (A, ∇, Δ) is a bimagma, and the biunital diagram (2.5) commutes.

(c.2) A *biunital bimagma homomorphism* $f\colon A \to B$ is a unital magma and counital comagma homomorphism between biunital bimagmas A and B.

Remark 2.6. Joint commuting of the bimagma diagram (2.4) and biunital diagram (2.5) in a biunital bimagma $(A, \nabla, \Delta, \eta, \varepsilon)$ means that $\Delta\colon A \to A \otimes A$ is a unital magma homomorphism, or equivalently, that $\nabla\colon A \otimes A \to A$ is a counital comagma homomorphism.

Definition 2.7. Let $\mathbf{V}$ be a symmetric monoidal category.

(a.1) A *monoid* in $\mathbf{V}$ is an associative unital magma in $\mathbf{V}$.

(a.2) Let A and B be monoids in $\mathbf{V}$. Then a unital magma homomorphism $f\colon A \to B$ is a *monoid homomorphism*.

(b.1) A *comonoid* in $\mathbf{V}$ is a coassociative counital comagma in $\mathbf{V}$.

(b.2) Let A and B be comonoids in $\mathbf{V}$. Then a counital comagma homomorphism $f\colon A \to B$ is a *comonoid homomorphism*.

(c.1) A *bimonoid* in $\mathbf{V}$ is an associative, coassociative, and biunital bimagma.

(c.2) Let A and B be bimonoids in $\mathbf{V}$. Then a biunital bimagma homomorphism $f\colon A \to B$ is a *bimonoid homomorphism*.

(d) A *Hopf algebra* in $\mathbf{V}$ is a bimonoid A in $\mathbf{V}$ equipped with a $\mathbf{V}$-morphism $S\colon A \to A$ known as the *antipode*, such that the antipode diagram (2.6) commutes.

Remark 2.8. In the Sweedler notation of Remark 2.2, the commuting of (2.6) becomes

$$a^{LS}a^{R} = a\varepsilon\eta = a^{L}a^{RS} \tag{2.8}$$

for $a \in A$.

3. Quantum quasigroups and loops

3.1. **The main definitions.**

Definition 3.1. Consider a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. Then a *quantum quasigroup* (A, ∇, Δ) in $\mathbf{V}$ is a bimagma (A, ∇, Δ) in $\mathbf{V}$ for which the *left composite* morphism

$$A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A \tag{3.1}$$

and its dual *right composite*

$$A \otimes A \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\nabla \otimes 1_A} A \otimes A \tag{3.2}$$

are both invertible.

Definition 3.2. Suppose that $(A, \nabla, \Delta, \eta, \varepsilon)$ is a biunital bimagma in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$. If (A, ∇, Δ) is a quantum quasigroup in $\mathbf{V}$, then $(A, \nabla, \Delta, \eta, \varepsilon)$ is said to be a *quantum loop*.

Remark 3.3. (a) The concepts of quantum quasigroup and quantum loop are self-dual.

(b) Quantum loops as presented in Definition 3.2 may be regarded as answers to the challenge ("However, concrete . . . formulation deserve[s] further research") issued by Pérez-Izquierdo in [6].

(c) In the bimonoid context, the left and right composites are often described as *fusion operators* or *Galois operators*.

Since quantum quasigroups and loops are formulated entirely in the language of symmetric monoidal categories, one immediately has the following result. (Compare [16, p.86] for the concept of a symmetric monoidal functor.)

Proposition 3.4. *Suppose that* $(\mathbf{V}, \otimes, \mathbf{1_V})$ *and* $(\mathbf{W}, \otimes, \mathbf{1_W})$ *are symmetric monoidal categories. Let* $F\colon \mathbf{V} \to \mathbf{W}$ *be a symmetric monoidal functor.*

(a) *If* (A, ∇, Δ) *is a quantum quasigroup in* $\mathbf{V}$*, then* (AF, ∇^F, Δ^F) *is a quantum quasigroup in* $\mathbf{W}$*.*
(b) *Suppose that* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a quantum loop in* $\mathbf{V}$*. Then* $(AF, \nabla^F, \Delta^F, \eta^F, \varepsilon^F)$ *is a quantum loop in* $\mathbf{W}$*.*

Noting that the conditions of (co)commutativity and (co)associativity are also formulated entirely in the language of symmetric monoidal categories, one obtains the following.

Corollary 3.5. *In the context of Proposition 3.4, validity of any one of the commutativity, cocommutativity, associativity, or coassociativity conditions for the quantum quasigroup* (A, ∇, Δ) *implies validity of the corresponding condition for the quantum quasigroup* (AF, ∇^F, Δ^F)*.*

3.2. **Trivial quasigroups.** Each symmetric monoidal category contains at least one (isomorphism class of) quantum quasigroup or loop.

Lemma 3.6. *Let* $(\mathbf{V}, \otimes, \mathbf{1})$ *be a symmetric monoidal category. Then* $(\mathbf{1}, 1_{\mathbf{1}}, 1_{\mathbf{1}}, 1_{\mathbf{1}}, 1_{\mathbf{1}})$ *is a quantum loop in* $\mathbf{V}$*.*

Definition 3.7. The quantum loop of Lemma 3.6 is called the *trivial quantum loop* in $\mathbf{V}$. Its reduct $(\mathbf{1}, 1_{\mathbf{1}}, 1_{\mathbf{1}})$ is called the *trivial quantum quasigroup* in $\mathbf{V}$.

It may happen that a nontrivial symmetric monoidal category admits no nontrivial quantum quasigroups. Define **FinSet** to be the category of finite sets and functions between them.

Proposition 3.8. *Let* $(\mathbf{FinSet}, +, \varnothing)$ *be the symmetric monoidal category of finite sets under disjoint union. If* (A, ∇, Δ) *is a quantum quasigroup in* $(\mathbf{FinSet}, +, \varnothing)$*, then it is trivial.*

Proof. The comultiplication $\Delta\colon A \to A + A$ partitions A into two disjoint subsets: the respective preimages A_L and A_R of the first and second summands of $A + A$. Consider the left composite morphism

$$A + A \xrightarrow{\Delta + 1_A} A + A + A \xrightarrow{1_A + \nabla} A + A \tag{3.3}$$

of (3.1) on (A, ∇, Δ). Since (3.3) is invertible in **FinSet**, it is surjective. In particular, the entire first summand A of the codomain of (3.3) lies in the image of (3.3). This forces the image of the restriction $\Delta\colon A_L \to A + A$ to include the entire first summand of its codomain, which in turn forces $A_L = A$. Dual consideration of the right composite (3.2) on (A, ∇, Δ) forces $A_R = A$. Thus $|A| = |A_L| + |A_R| = |A| + |A|$ and $|A| = 0$. $\square$

3.3. **Combinatorial examples.** In this section, the basic symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is taken to be $(\mathbf{Set}, \times, \top)$, the category of sets with the Cartesian product $\times$ and singleton set $\top = \{1\}$ (a terminal object of $\mathbf{Set}$), with the twist symmetry

$$\tau \colon A \times B \to B \times A; (a, b) \mapsto (b, a)$$

and identifications such as $\rho_A \colon A \times \top \to A; (a, 1) \mapsto a$. In order to facilitate reference to the diagrams of §2.3, the direct product of two sets A and B will be written in monoidal category notation as $A \otimes B$, while an ordered pair $(a, b) \in A \times B$ will be written as an element $a \otimes b$ of $A \otimes B$, of tensor rank 1. In this case, the Sweedler notation $\Delta \colon A \to A \otimes A; a \mapsto a^L \otimes a^R$ of Remark 2.2(b) corresponds directly with a pair of functions $L \colon A \to A; a \mapsto a^L$ and $R \colon A \to A; a \mapsto a^R$.

Lemma 3.9. *If (A, Δ, ε) is a counital comagma in* $\mathbf{Set}$*, then the comultiplication is the diagonal embedding $\Delta \colon a \mapsto a \otimes a$. Conversely, the diagonal embedding on each set A yields a cocommutative, coassociative counital comagma (A, Δ, ε) in* $\mathbf{Set}$.

Corollary 3.10. *Quantum loops and counital quantum quasigroups in* $(\mathbf{Set}, \times, \top)$ *are cocommutative and coassociative.*

The lemma leads to a direct identification of quantum loops and counital quantum quasigroups in $\mathbf{Set}$.

Proposition 3.11. *Consider the category* $\mathbf{Set}$ *of sets and functions, with the symmetric monoidal category structure* $(\mathbf{Set}, \times, \top)$.

(a) *Counital quantum quasigroups in* $\mathbf{Set}$ *are equivalent to quasigroups.*
(b) *Quantum loops in* $\mathbf{Set}$ *are equivalent to loops.*

Proof. (a): Let $(A, \nabla, \Delta, \varepsilon)$ be a counital quantum quasigroup in $\mathbf{Set}$. By Lemma 3.9, the left composite function (3.1) takes the form

$$a \otimes b \xmapsto{\Delta \otimes 1_A} a \otimes a \otimes b \xmapsto{1_A \otimes \nabla} a \otimes (a \cdot b) \tag{3.4}$$

for $a, b \in A$. Thus the inverse function may be written as

$$c \otimes (c \backslash d) \longleftarrow\!\!\!\!\longleftarrow\!\!\!\!\dashv c \otimes d \tag{3.5}$$

for $c, d \in A$ and a binary operation $(c, d) \mapsto c \backslash d$ on A. The mutual inverse relationship between (3.4) and (3.5) yields the identities (SL) and (IL) of (2.1) on A. In dual fashion, inversion of the right composite (3.2) yields a binary operation $(c, d) \mapsto c/d$ on A satisfying the identities (SR) and (IR) of (2.1). Thus $(A, \cdot, /, \backslash)$ is a quasigroup.

Conversely, given a quasigroup $(A, \cdot, /, \backslash)$, one may define

$$\nabla \colon a \otimes b \mapsto a \cdot b\,.$$

The quasigroup identities (2.1) yield an inverse (3.5) to (3.1), and a dual inverse to (3.2). The remaining structure is provided by Lemma 3.9, and verification of the bimagma condition (2.4) is immediate.

(b): If $(A, \nabla, \Delta, \eta, \varepsilon)$ is a quantum loop in **Set**, the counital quantum quasigroup reduct $(A, \nabla, \Delta, \varepsilon)$ yields a quasigroup $(A, \cdot, /, \backslash)$ by (a). The unit $\eta \colon \top \to A$ selects an element e of A, which then becomes an identity element for $(A, \cdot, /, \backslash)$ by virtue of the unitality.

Conversely, given a loop $(A, \cdot, /, \backslash, e)$, the quasigroup reduct $(A, \cdot, /, \backslash)$ specifies a counital quantum quasigroup $(A, \nabla, \Delta, \varepsilon)$ by (a). Defining $\eta \colon 1 \mapsto e$ with the identity element e then makes $(A, \nabla, \Delta, \eta, \varepsilon)$ a biunital bimagma. □

General quantum quasigroups in **Set** are subtler. In the finite case, they correspond to certain quasigroups with operators. Treatment of the infinite case is deferred to a later work.

Theorem 3.12. *Quantum quasigroups in* $(\mathbf{FinSet}, \times, \top)$ *are equivalent to triples* (A, L, R) *consisting of a quasigroup* A *with an ordered pair* (L, R) *of automorphisms of* A.

Proof. Let (A, ∇, Δ) be a quantum quasigroup in **FinSet**, with comagma $\Delta \colon a \mapsto a^L \otimes a^R$. Commuting of the bimagma diagram (2.4) shows that the functions $L \colon A \to A$ and $R \colon A \to A$ are endomorphisms of the magma (A, ∇). The left composite function (3.1) takes the form

$$a \otimes b \xmapsto{\Delta \otimes 1_A} a^L \otimes a^R \otimes b \xmapsto{1_A \otimes \nabla} a^L \otimes (a^R \cdot b) \tag{3.6}$$

for $a, b \in A$. Its invertibility implies that L is surjective. Dually, R is surjective. Since A is finite, it follows that both L and R are invertible.

The inverse function to the left composite (3.6) may now be written as

$$c^{L^{-1}} \otimes (c^{L^{-1}R} \backslash d) \longleftarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\longleftarrow\!\!\!\!\dashv c \otimes d \tag{3.7}$$

for $c, d \in A$ and a binary operation $(x, y) \mapsto x \backslash y$ on A. The mutual inverse relationship between (3.6) and (3.7) yields the identities (SL) and (IL) of (2.1) on A. Similarly, inversion of the right composite (3.2) provides a binary operation $(x, y) \mapsto x/y$ on A satisfying the identities (SR) and (IR) of (2.1). Thus $(A, \cdot, /, \backslash)$ is a quasigroup equipped with automorphisms L and R.

Conversely, given a quasigroup $(A, \cdot, /, \backslash)$ with automorphisms L and R, define a multiplication $\nabla\colon A \otimes A \to A; a \otimes b \mapsto ab$ and comultiplication $\Delta\colon A \to A \otimes A; a \mapsto a^L \otimes a^R$. It is then straightforward to verify that (A, ∇, Δ) is a quantum quasigroup in **Set**. $\square$

Since finiteness of the underlying set A was not assumed in the concluding paragraph of the proof of Theorem 3.12, one may immediately observe the following.

Corollary 3.13. *Given a quasigroup $(A, \cdot, /, \backslash)$ with automorphisms L and R, define a multiplication $\nabla\colon A \otimes A \to A; a \otimes b \mapsto ab$ and comultiplication $\Delta\colon A \to A \otimes A; a \mapsto a^L \otimes a^R$. Then (A, ∇, Δ) is a quantum quasigroup in* **Set**.

Corollary 3.14. *Let (A, ∇, Δ) be a quantum quasigroup in* **FinSet**, *with corresponding triple (A, L, R).*

(a) *The quantum quasigroup (A, ∇, Δ) is commutative if and only if the quasigroup A is commutative.*
(b) *The quantum quasigroup (A, ∇, Δ) is associative if and only if the quasigroup A is associative, i.e., is empty or a group.*
(c) *The quantum quasigroup (A, ∇, Δ) is cocommutative if and only if the automorphisms L and R coincide.*

Proposition 3.15. *Let (A, ∇, Δ) be a coassociative quantum quasigroup in* **FinSet**, *with corresponding triple (A, L, R). Then (A, ∇, Δ) is cocommutative, with $L = R = 1_A$.*

Proof. By Definition 2.3(d), one has $L^2 = L$ and $R^2 = R$. Since L and R are automorphisms, it follows that $L = 1_A = R$ and (A, ∇, Δ) is cocommutative. $\square$

Example 3.16. Consider the symmetric group S_3 of degree 3, generated by transpositions λ and ρ. Let L and R be the respective conjugations of S_3 by λ and ρ. Then the quantum quasigroup in **Set** corresponding to the triple (S_3, L, R) is associative, but neither commutative, nor cocommutative, nor coassociative. On the other hand, the quantum quasigroup in **Set** corresponding to the triple (S_3, L, L) is associative and cocommutative, but neither commutative, nor coassociative.

Example 3.17. Consider the quasigroup $(\mathbb{Z}/_3, -)$ of integers modulo 3 under subtraction. Take $L\colon \mathbb{Z}/_3 \to \mathbb{Z}/_3; r \mapsto -r$ and $R = 1_{\mathbb{Z}/_3}$. Then the quantum quasigroup in **Set** corresponding to the triple $\big((\mathbb{Z}/_3, -), L, R\big)$ is neither commutative, nor associative, nor cocommutative, nor coassociative.

3.4. **Quasigroup and loop algebras.** For simplicity, the results of this section are presented within the category $\underline{\underline{S}}$ of modules over a commutative, unital ring S, construed as a symmetric tensor category $(\underline{\underline{S}}, \otimes, S)$ under the tensor product of modules. Discussion of extensions to more general entropic varieties is confined to Remark 3.20.

Proposition 3.18. *Let Q be a quasigroup. Suppose that QS is the free S-module over Q. Define a magma (QS, ∇) by the free extension of the quasigroup multiplication $\nabla\colon Q \otimes Q \to Q; x \otimes y \mapsto xy$. Define a comagma (QS, Δ) by the free extension of the diagonal $\Delta\colon q \mapsto q \otimes q$ for q in Q. Then (QS, ∇, Δ) is a cocommutative, coassociative quantum quasigroup in $\underline{\underline{S}}$.*

Proof. The left composite (3.1) takes the form

$$x \otimes y \xmapsto{\Delta \otimes 1_{QS}} x \otimes x \otimes y \xmapsto{1_{QS} \otimes \nabla} x \otimes (x \cdot y)$$

for $x, y \in Q$. The inverse is given by

$$u \otimes (u \backslash v) \longleftarrow\!\!\!\!\longleftarrow\!\!\!\dashv u \otimes v$$

for $u, v \in Q$. Invertibility of the right composite (3.2) is dual. Commuting of the bimagma diagram (2.4) is straightforward. $\square$

Definition 3.19. The quantum quasigroup QS of Proposition 3.18 is known as the *quasigroup algebra* of Q over the ring S.

Remark 3.20. If $\mathbf{V}$ is an entropic variety, with free algebra functor $V\colon \mathbf{Set} \to \mathbf{V}$, then an analogous quasigroup algebra in $\mathbf{V}$ may be constructed on the free $\mathbf{V}$-algebra QV over the set Q. It is obtained by applying Proposition 3.4, with the free $\mathbf{V}$-algebra functor V, to the quantum quasigroup in $\mathbf{Set}$ corresponding under Theorem 3.12 to the triple $(Q, 1_Q, 1_Q)$. Note that Proposition 3.18 actually represents the special case where $\mathbf{V} = \underline{\underline{S}}$. More generally, taking a triple (Q, L, R) with automorphisms L and R of Q yields a *twisted quasigroup algebra* in $\mathbf{V}$.

Corollary 3.21. *If $(Q, \cdot, e)$ is a loop, then the quasigroup algebra (QS, ∇, Δ) of Q over the ring S admits an augmentation to a quantum loop $(QS, \nabla, \Delta, \eta, \varepsilon)$ in $\underline{\underline{S}}$.*

Proof. The counit $\varepsilon\colon QS \to S$ is the free extension of $\varepsilon\colon Q \to S; x \mapsto 1$. The unit $\eta\colon S \to QS$ is the free extension of $\eta\colon \{1\} \to QS; 1 \mapsto e$. Verification of the unitality, counitality, and biunitality conditions is straightforward. $\square$

Definition 3.22. The quantum loop QS of Corollary 3.21 is known as the *loop algebra* of Q over the ring S.

Example 3.23. If Q is a group, then the loop algebra of Q over a field K is (a reduct of) the usual group Hopf algebra (compare [5, Ex. 1.6]).

Example 3.24. If Q is a Moufang loop, then the loop algebra of Q over a field K, in the sense of Definition 3.22, is (a reduct of) the *loop algebra* of Q in the sense of [1, p.1004]. This example provides instances of Corollaries 4.2 and 4.3 below.

3.5. **Dual quasigroup and loop algebras.** The results of this section are again presented within the category $\underline{\underline{S}}$ of modules over a commutative, unital ring S, taken as a symmetric tensor category under the tensor product of modules. For a finite set Q, recall that the free S-module over Q is modeled by the set S^Q of functions from Q to S, under the pointwise module structure. A basis is provided by the *delta functions* $\delta_q \colon Q \to S$ with

$$x\delta_q = \begin{cases} 1 & \text{if } x = q; \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

for elements x, q of Q. With the *Kronecker delta* $\delta_{x,q}$ to denote the element 1 of S if $x = q$, and 0 otherwise, one may rewrite (3.8) as $x\delta_q = \delta_{x,q}$. The Kronecker delta notation is used below, where its comma will become essential to distinguish it from instances of a delta function δ_{xq} for the product xq of elements x and q of a quasigroup Q.

Proposition 3.25. *Let Q be a finite quasigroup. Define a magma (S^Q, ∇) by pointwise multiplication of S-valued functions. Define a comagma (S^Q, Δ) by the free extension of the factorization*

$$\Delta \colon \delta_q \mapsto \sum_{q^L q^R = q} \delta_{q^L} \otimes \delta_{q^R}$$

for an element q of Q. Then (S^Q, ∇, Δ) is a commutative, associative quantum quasigroup in $\underline{\underline{S}}$.

Proof. The left composite (3.1) takes the form

$$\delta_x \otimes \delta_y \overset{\Delta\otimes 1_{S^Q}}{\longmapsto} \textstyle\sum_{x^L x^R = x} \delta_{x^L} \otimes \delta_{x^R} \otimes \delta_y \overset{1_{S^Q}\otimes\nabla}{\longmapsto} \delta_{x/y} \otimes \delta_y$$

for $x, y \in Q$. Note that $x^R = y$ and $x^L x^R = x$ imply $x^L = x/x^R = x/y$. The inverse is given by

$$\delta_{uv} \otimes \delta_v \longleftarrow\!\!\!\longmapsfrom \delta_u \otimes \delta_v$$

for $u, v \in Q$, since $v = y$ and $u = x/y$ imply $uv = x$. Invertibility of the right composite (3.2) is dual. Commuting of

$$\begin{tikzcd}
\delta_p \otimes \delta_q \arrow[r, mapsto, "\Delta\otimes\Delta"] \arrow[d, mapsto, "\nabla"'] & \sum_{p^Lp^R=p}\sum_{q^Lq^R=q}\delta_{p^L}\otimes\delta_{p^R}\otimes\delta_{q^L}\otimes\delta_{q^R} \arrow[dd, mapsto, "1_{S^Q}\otimes\tau\otimes 1_{S^Q}"] \\
\delta_{p,q}\delta_q \arrow[d, mapsto, "\Delta"'] & \\
\delta_{p,q}\sum_{q^Lq^R=q}\delta_{q^L}\otimes\delta_{q^R} & \sum_{p^Lp^R=p}\sum_{q^Lq^R=q}\delta_{p^L}\otimes\delta_{q^L}\otimes\delta_{p^R}\otimes\delta_{q^R} \arrow[l, mapsto, "\nabla\otimes\nabla"]
\end{tikzcd}$$

for $p, q \in Q$ confirms the bimagma condition (2.4). For the bottom line of the diagram, note that $p^L = q^L$ and $p^R = q^R$ are equivalent to $p = q$ under $p^Lp^R = p$ and $q^Lq^R = q$. □

Definition 3.26. The quantum quasigroup S^Q of Proposition 3.25 is known as the *dual quasigroup algebra* of Q over the ring S.

Remark 3.27. If $\mathbf{V}$ is an entropic Jónsson-Tarski variety (compare [10]), with free algebra functor $V : \mathbf{Set} \to \mathbf{V}$, then an analogous dual quasigroup algebra in $\mathbf{V}$ may be constructed on the free $\mathbf{V}$-algebra QV over a finite quasigroup Q.

Corollary 3.28. *If $(Q, \cdot, e)$ is a finite loop, then the dual quasigroup algebra (S^Q, ∇, Δ) of Q over the ring S admits an augmentation to a quantum loop $(S^Q, \nabla, \Delta, \eta, \varepsilon)$ in $\underline{\underline{S}}$.*

Proof. The counit $\varepsilon : S^Q \to S$ is the delta function δ_e, while the unit $\eta : S \to S^Q$ maps a scalar s to the constant function $Q \to \{s\}$. Verification of the unitality, counitality, and biunitality conditions is straightforward. □

Definition 3.29. The quantum loop S^Q of Corollary 3.28 is known as the *dual loop algebra* of Q over the ring S.

Example 3.30. If Q is a finite group, then the dual loop algebra of Q over a field K is (a reduct of) the usual dual group Hopf algebra.

3.6. **The quantum couple.** As in the previous two sections, this section again works within the category $\underline{\underline{S}}$ of modules over a commutative, unital ring S, regarded as a symmetric tensor category under the tensor product of modules.

Theorem 3.31. *Let G be a (not necessarily finite) group with a (not necessarily faithful) automorphic right action on a finite quasigroup Q. Let GQ be the tensor product of the free S-module GS over G with* *the free S-module QS over Q. For g in G and q in Q, write the basic element $g \otimes q$ of GQ as $g|q$. Define a magma (GQ, ∇) by the free extension of the map*

$$(f|p \otimes g|q)\nabla = \begin{cases} fg|q & \textit{if } pg = q; \\ 0 & \textit{otherwise} \end{cases} \tag{3.9}$$

for f, g in G and p, q in Q. Define a comagma (GQ, Δ) by the free extension of the factorization

$$\Delta \colon g|q \mapsto \sum_{q^L q^R = q} g|q^L \otimes g|q^R$$

for g in G and q in Q. Then (GQ, ∇, Δ) is an associative quantum quasigroup in $\underline{\underline{S}}$.

Proof. The multiplication is associative, since both $(\nabla \otimes 1_{GQ})\nabla$ and $(1_{GQ} \otimes \nabla)\nabla$ map $g_1|q_1 \otimes g_2|q_2 \otimes g_3|q_3$ to $g_1g_2g_3|q_3$ if $q_1g_2 = q_2$ and $q_2g_3 = q_3$, and to 0 otherwise. (Note that $q_1g_2g_3 = q_3$ if and only if $q_1g_2 = q_2$, when $q_2g_3 = q_3$.)

To show that (GQ, ∇, Δ) is a bimagma, it is convenient to abbreviate the multiplication definition (3.9) as $(f|p \otimes g|q)\nabla = \delta_{pg,q} fg|q$ using the Kronecker delta. Now consider the bimagma diagram (2.4). Using the abbreviated version of Sweedler notation, one has

$$\begin{tikzcd}
f|p \otimes g|q \arrow[r, mapsto, "\Delta\otimes\Delta"] \arrow[d, mapsto, "\nabla"'] & f|p^L \otimes f|p^R \otimes g|q^L \otimes g|q^R \arrow[dd, mapsto, "1_{GQ}\otimes\tau\otimes 1_{GQ}"] \\
\delta_{pg,q} fg|q \arrow[d, mapsto, "\Delta"'] & \\
\delta_{pg,q} fg|q^L \otimes fg|q^R & f|p^L \otimes g|q^L \otimes f|p^R \otimes g|q^R \arrow[l, mapsto, "\nabla\otimes\nabla"]
\end{tikzcd}$$

for $g, h \in G$, as required. Note that $pg = q$ and $p^L p^R = p$ imply $p^L g \cdot p^R g = pg$, so $p^L g = q^L$ and $p^R g = q^R$ with $q^L q^R = q$.

The left composite (3.1) acts as

$$f|p \otimes g|q \xmapsto{\Delta\otimes 1_{GQ}} \textstyle\sum_{p^L p^R = p} f|p^L \otimes f|p^R \otimes g|q \xmapsto{1_{GQ}\otimes\nabla} f|\big(p/(qg^{-1})\big) \otimes fg|q$$

for f, g in G and p, q in Q, since $(f|p^R \otimes g|q)\nabla$ is nonzero only for $p^R g = q$ or $p^R = qg^{-1}$, in which case $p^L = p/(qg^{-1})$. Then the basic action of the inverse function is given as

$$h|r \cdot sk^{-1}h \otimes h^{-1}k|s \longleftarrow\!\!\shortmid h|r \otimes k|s$$

for h,k in G and r,s in Q — note that

$$p/(s(h^{-1}k)^{-1})=r \quad\Leftrightarrow\quad p=r\cdot sk^{-1}h\,.$$

Dually, the right composite (3.2) is invertible. Thus (GQ,∇,Δ) is an associative quantum quasigroup. $\square$

Definition 3.32. The quantum quasigroup GQ of Theorem 3.31 is known as the *quantum couple* of G and Q over the ring S.

Example 3.33. If G is the trivial group $\top$, then the quantum couple $\top Q$ reduces to the dual quasigroup algebra S^Q.

Example 3.34. If Q is the trivial singleton quasigroup $\top$, then the quantum couple $G\top$ is the quasigroup algebra GS of the group G.

Example 3.35. If Q is a finite group, acting on itself by conjugation, then the quantum couple (QQ,∇,Δ) is essentially (the bimagma reduct of) the group quantum double studied in [17].

3.7. **Linear quasigroups, trees, and Conway algebras.** In this section, the category $\underline{\underline{\mathbb{Z}}}$ of abelian groups is taken as a symmetric tensor category $(\underline{\underline{\mathbb{Z}}},\oplus,\{0\})$ under the direct sum (biproduct) $\oplus$ of abelian groups.

Lemma 3.36. *Let* (A,∇) *be a magma in* $(\underline{\underline{\mathbb{Z}}},\oplus,\{0\})$*. Then*

$$\nabla\colon A\oplus A\to A; x\oplus y\mapsto x^\rho+y^\lambda \tag{3.10}$$

for endomorphisms ρ,λ *of the abelian group* A*.*

Proof. Note that $A\oplus A$ is the coproduct of two copies of A in $\underline{\underline{\mathbb{Z}}}$. $\square$

Lemma 3.37. *Let* (A,Δ) *be a comagma in* $(\underline{\underline{\mathbb{Z}}},\oplus,\{0\})$*. Then*

$$\Delta\colon A\to A\oplus A; x\mapsto x^L\oplus x^R \tag{3.11}$$

for endomorphisms L,R *of the abelian group* A*.*

Proof. Note that $A\oplus A$ is the product of two copies of A in $\underline{\underline{\mathbb{Z}}}$. $\square$

Remark 3.38. The expression (3.11) serves as a model for the general abbreviated version of Sweedler notation established in Remark 2.2(b).

Proposition 3.39. *Suppose that an abelian group* A *carries a magma structure* (3.10) *and a comagma structure* (3.11)*. Then* (A,∇,Δ) *is a bimagma in* $(\underline{\underline{\mathbb{Z}}},\oplus,\{0\})$ *if and only if the commutation relations*

$$\lambda L=L\lambda\,,\quad \rho L=L\rho\,,\quad \lambda R=R\lambda\,,\quad \rho R=R\rho \tag{3.12}$$

are satisfied by the endomorphisms λ,L,ρ,R *of the abelian group* A*.*

Proof. The bimagma diagram (2.4) takes the form

$$\begin{tikzcd}
x \oplus y \arrow[r, mapsto, "\Delta\otimes\Delta"] \arrow[d, mapsto, "\nabla"'] & x^L \oplus x^R \oplus y^L \oplus y^R \arrow[ddd, mapsto, "1_A\oplus\tau\oplus 1_A"] \\
x^\rho + y^\lambda \arrow[d, mapsto, "\Delta"'] & \\
(x^\rho + y^\lambda)^L \oplus (x^\rho + y^\lambda)^R \arrow[d, equal] & \\
x^{L\rho} + y^{L\lambda} \oplus x^{R\rho} + y^{R\lambda} & x^L \oplus y^L \oplus x^R \oplus y^R \arrow[l, mapsto, "\nabla\oplus\nabla"]
\end{tikzcd}$$

for $x, y \in A$. Thus the relations (3.12) are equivalent to commutativity of the diagram. □

Definition 3.40. A combinatorial quasigroup $(A, \cdot)$ is *linear* if there is an abelian group structure $(A, +, 0)$ with automorphisms ρ, λ such that

$$x \cdot y = x^\rho + y^\lambda \tag{3.13}$$

for x, y in A.

The following theorem may be regarded as a linear version of the combinatorial Theorem 3.12, in a sense made precise by Corollary 3.42 below.

Theorem 3.41. *Finite quantum quasigroups in $(\underline{\underline{\mathbb{Z}}}, \oplus, \{0\})$ are equivalent to triples (A, L, R) consisting of a finite linear quasigroup A with an ordered pair (L, R) of automorphisms of A.*

Proof. Let (A, ∇, Δ) be a finite quantum quasigroup in $(\underline{\underline{\mathbb{Z}}}, \oplus, \{0\})$, with multiplication as in (3.10) and comultiplication as in (3.11). By Proposition 3.39, the commutation relations (3.12) are satisfied by the endomorphisms λ, L, ρ, R of the abelian group A. The invertible left composite morphism (3.1) takes the form

$$x \oplus y \xmapsto{\Delta\oplus 1_A} x^L \oplus x^R \oplus y \xmapsto{1_A\oplus\nabla} x^L \oplus (x^{R\rho} + y^\lambda) \tag{3.14}$$

for $x, y \in A$. Its invertibility implies that L is surjective. Since A is finite, it follows that L is invertible. Then $(x \oplus y)(\Delta \oplus 1_A)(1_A \oplus \nabla) = 0 \oplus y^\lambda$ implies $x = 0$, so $0 \oplus y^\lambda$ can only be the image of $0 \oplus y$. Thus λ is surjective, and since A is finite, it follows that λ is invertible.

The inverse to the left composite (3.14) is given by

$$u^{L^{-1}} \oplus (v - u^{L^{-1}R\rho})^{\lambda^{-1}} \longleftarrow\!\shortmid\, u \oplus v \tag{3.15}$$

for $u, v \in A$, noting that L and λ are automorphisms. Inversion of the right composite (3.2) is dual, in particular implying that R and ρ are automorphisms. Thus the multiplication (3.10) on A yields a linear quasigroup; the commutation relations (3.12) imply that L and R are quasigroup automorphisms.

Conversely, a linear quasigroup $(A, \cdot)$ with $x \cdot y = x^\rho + y^\lambda$ and automorphisms L, R yields a bimagma (A, ∇, Δ) with multiplication (3.10) and comultiplication (3.11) by Proposition 3.39. Invertibility of the left composite (3.1) follows as illustrated above for (3.14) by means of (3.15). Dually, the right composite (3.2) is invertible. $\square$

Since finiteness of the underlying set A was not assumed in the concluding paragraph of the proof of Theorem 3.41, one may observe the following.

Corollary 3.42. *Consider a triple (A, L, R) comprising a linear quasigroup A and an ordered pair (L, R) of automorphisms of A. The triple yields a quantum quasigroup in $(\mathbf{Set}, \times, \top)$ by Corollary 3.13, and also in $(\underline{\underline{\mathbb{Z}}}, \oplus, \{0\})$. Then the former is obtained by applying Proposition 3.4 to the latter, with the underlying set functor $\underline{\underline{\mathbb{Z}}} \to \mathbf{Set}$.*

Example 3.43 (Binary rooted trees and nonassociative powers)**.** Let $\langle \rho, \lambda \rangle$ be the free group over the doubleton $\{\rho, \lambda\}$. Consider the abelian group reduct A of the integral group algebra $\mathbb{Z}\langle \rho, \lambda \rangle$. Define abelian group automorphisms $\rho \colon A \to A; x \mapsto x\rho$ and $\lambda \colon A \to A; x \mapsto x\lambda$. Under the quasigroup multiplication (3.13), A becomes a linear quasigroup. The triple $(A, 1_A, 1_A)$ yields a quantum quasigroup in $(\mathbf{Set}, \times, \top)$ under Corollary 3.13, and in $(\underline{\underline{\mathbb{Z}}}, \oplus, \{0\})$ under Corollary 3.42. Now the submagma generated by $\{1\}$ is free [2, Th. III.5.4], [13, Th. 11.1]. Thus these quantum quasigroups embrace an algebra of binary rooted trees or nonassociative powers.

Example 3.44 (Entropic quasigroups and Conway algebras)**.** If the automorphisms ρ and λ commute, a linear quasigroup 3.13 is entropic. In particular, consider the polynomial ring $\mathbb{Z}[\rho, \rho^{-1}, \lambda, \lambda^{-1}]$ over a pair of invertible indeterminates ρ and λ, i.e., the integral group algebra of the free abelian group over the doubleton $\{\rho, \lambda\}$. Consider the abelian group reduct A of $\mathbb{Z}[\rho, \rho^{-1}, \lambda, \lambda^{-1}]$. As in Example 3.43, define abelian group automorphisms $\rho \colon A \to A; x \mapsto x\rho$ and $\lambda \colon A \to A; x \mapsto x\lambda$. With the quasigroup multiplication (3.13), A forms an entropic linear quasigroup, and in particular an equational quasigroup $(A, \cdot, /, \backslash)$. Let

% denote the opposite right division, with $x\%y = y/x$ for $x, y \in A$. The structure $(A, \%, \backslash)$ is an entropic *right quasigroup*, satisfying the identities $x\%(x\backslash y) = y = x\backslash(x\%y)$. Now consider the quantum quasigroup determined by the triple $(A, 1_A, 1_A)$ in $(\mathbf{Set}, \times, \top)$ by Corollary 3.13, or in $(\underline{\underline{\mathbb{Z}}}, \oplus, \{0\})$ by Corollary 3.42. Then the right quasigroup structure S in $(A, \%, \backslash)$ generated by the set $\{1(\Delta\nabla)^n \mid 0 < n \in \mathbb{Z}\}$ forms a *Conway algebra* in the sense of knot theory (compare [8, 12]). Here $\{1(\Delta\nabla)^n \mid 0 < n \in \mathbb{Z}\}$ is the set of skein polynomials of unlinks, while the skein polynomial of an arbitrary classical oriented link is an element of S.

Example 3.45 (Representations of groups with two generators)**.** The linear quasigroup constructions of this section work more generally in the symmetric tensor category $(\underline{\underline{S}}, \oplus, \{0\})$ of modules over a commutative, unital ring S, under the biproduct $\oplus$ of S-modules. Let A be a faithful representation of a group G in the category $\underline{\underline{S}}$ of modules. Suppose that the group G is generated by two elements r and l, with corresponding representing S-module automorphisms ρ and λ of A. Then the structure of the representation is encoded in the linear quasigroup given by the multiplication (3.13) on A.

4. Quantum loops and Hopf algebras

4.1. **Hopf algebras as quantum loops.** Ostensibly, Proposition 4.1 below shows that a Hopf algebra in a concrete symmetric monoidal category $\mathbf{V}$ is a quantum loop in $\mathbf{V}$, by means of an elementary proof that lends itself to the Moufang-Hopf case treated in Corollary 4.2 and the Hopf quasigroup or coquasigroup cases treated in Corollary 4.3. A diagrammatic version of the proof of Proposition 4.1 (along the lines of [15, Prop. 1.2]) extends the result to arbitrary symmetric monoidal categories $\mathbf{V}$.

Proposition 4.1. *Suppose that* $\mathbf{V}$ *is a concrete symmetric monoidal category. Then if* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *is a Hopf algebra in* $\mathbf{V}$*, the reduct* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a quantum loop in* $\mathbf{V}$*.*

Proof. The biunital bimagma structure in $(A, \nabla, \Delta, \eta, \varepsilon)$ is taken directly from the Hopf algebra. Thus the main task is to demonstrate the invertibility of the composites (3.1) and (3.2). By the coassociativity of the comultiplication, one has

$$x^{LL} \otimes x^{LR} \otimes x^R = x^L \otimes x^{RL} \otimes x^{RR} \tag{4.1}$$

for x in A — compare Definition 2.3(d). Then for an element y in A, tensoring the equation (4.1) on the right with y and applying the function $(1_A\otimes 1_A\otimes S\otimes 1_A)(1_A\otimes 1_A\otimes\nabla)(1_A\otimes\nabla)$ to each side yields

$$\begin{aligned}x^{LL}\otimes x^{LR}x^{RS}y&=x^L\otimes x^{RL}x^{RRS}y=x^L\otimes x^{R\varepsilon\eta}y\\&=x^Lx^{R\varepsilon\eta}\otimes y=x\otimes y\,,\end{aligned}\tag{4.2}$$

using (2.8) and counitality for the second and final equalities respectively. Thus

$$\big((\Delta\otimes 1_A)(1_A\otimes S\otimes 1_A)(1_A\otimes\nabla)\big)\big((\Delta\otimes 1_A)(1_A\otimes\nabla)\big)=1_{A\otimes A}\,.$$

Again, for y in A, tensoring both sides of (4.1) on the right with y and applying $(1_A\otimes S\otimes 1_A\otimes 1_A)(1_A\otimes 1_A\otimes\nabla)(1_A\otimes\nabla)$ gives

$$\begin{aligned}x^{LL}\otimes x^{LRS}x^{R}y&=x^L\otimes x^{RLS}x^{RR}y=x^L\otimes x^{R\varepsilon\eta}y\\&=x^Lx^{R\varepsilon\eta}\otimes y=x\otimes y\end{aligned}\tag{4.3}$$

as before, so

$$\big((\Delta\otimes 1_A)(1_A\otimes\nabla)\big)\big((\Delta\otimes 1_A)(1_A\otimes S\otimes 1_A)(1_A\otimes\nabla)\big)=1_{A\otimes A}\,.$$

It follows that the left composite $(\Delta\otimes 1_A)(1_A\otimes\nabla)$ is invertible. Verification of the invertibility of the right composite $(1_A\otimes\Delta)(\nabla\otimes 1_A)$ is dual. □

The proof of Proposition 4.1 may be applied, *mutatis mutandis*, to yield the following results, which were foreshadowed by Example 3.24. Compare [1, Defn. 1.2] for the concept of a *Moufang-Hopf algebra*.

Corollary 4.2. *Let $(A,\nabla,\Delta,\eta,\varepsilon,S)$ be a Moufang-Hopf algebra. Then the reduct $(A,\nabla,\Delta,\eta,\varepsilon)$ is a quantum loop.*

Proof. Replace the Hopf algebra computation (4.2) by

$$x^{LL}\otimes x^{LR}(x^{RS}y)=x^L\otimes x^{RL}(x^{RRS}y)=x^L\otimes x^{R\varepsilon\eta}y$$

using brackets for nonassociative multiplications. Similarly, the computation

$$x^{LL}\otimes x^{LRS}(x^{R}y)=x^L\otimes x^{RLS}(x^{RR}y)=x^L\otimes x^{R\varepsilon\eta}y$$

replaces (4.3). In each case, the second equalities are applications of identities involving the antipode S within the definition of a Moufang-Hopf algebra. □

As further extensions of Hopf algebras, see [4, Defn. 4.1] for the concept of a *Hopf quasigroup*, and [4, Defn. 5.1] for the dual concept of a *Hopf coquasigroup*.

Corollary 4.3. *Let $(A,\nabla,\Delta,\eta,\varepsilon,S)$ be a Hopf quasigroup or a Hopf coquasigroup. Then the reduct $(A,\nabla,\Delta,\eta,\varepsilon)$ is a quantum loop.*

Proof. In the first case, the above proof of Proposition 4.1 carries over directly, noting only that the associativity of the Hopf algebra multiplication was not used at all, while (2.8) is available here by virtue of [4, Prop. 4.2(1)]. The second case follows by the self-duality of the quantum loop concept. □

Example 4.4. As a Hopf quasigroup [4, Prop. 4.8], the universal enveloping algebra of a Mal'tsev algebra [7] (over a field of characteristic coprime to 6) is a quantum loop.

4.2. **When quantum loops are Hopf algebras.** Let $\underline{\underline{K}}$ be the category of vector spaces over a field K. Recall that a comonoid in $\underline{\underline{K}}$ is *simple* if it has exactly two subcomonoids (one trivial, the other improper) [9, Defn. 2.1.8]. Then a comonoid in $\underline{\underline{K}}$ is *pointed* if each simple subcomonoid is 1-dimensional [9, Defn. 3.4.4]. A bimonoid in $\underline{\underline{K}}$ is *pointed* if its comonoid reduct is pointed [9, Defn. 5.1.13].

Theorem 4.5. *Consider the category $\underline{\underline{K}}$ of vector spaces over a field K. Suppose that $(A, \nabla, \Delta, \eta, \varepsilon)$ is an associative and coassociative finite-dimensional quantum loop in $\underline{\underline{K}}$, such that the comonoid (A, Δ, ε) is pointed. Then $(A, \nabla, \Delta, \eta, \varepsilon)$ is a Hopf algebra.*

Proof. Consider the set

$$A_1 = \{x \in A \mid a\Delta = a \otimes a\,,\ a\varepsilon = 1\}$$

of *setlike* [16, p.40] (or "grouplike" [9, p.25]) elements of A. By the commuting of the bimagma and biunital diagrams in A, it follows that A_1 forms a monoid under multiplication, with $1\eta = e$ as the identity element. Since

$$x \otimes y \xmapsto{\Delta \otimes 1_A} x \otimes x \otimes y \xmapsto{1_A \otimes \nabla} x \otimes xy$$

for $x, y \in A_1$, the invertible composite $(\Delta \otimes 1_A)(1_A \otimes \nabla) \colon A \otimes A \to A \otimes A$ restricts to an injective map

$$j \colon A_1 \otimes A_1 \to A_1 \otimes A_1\,. \tag{4.4}$$

Since the set A_1 is linearly independent in the finite-dimensional vector space A (compare [9, Lemma 2.1.12]), it follows that the set $A_1 \otimes A_1 = \{x \otimes y \mid x, y \in A_1\}$ is finite. Thus the injective map (4.4) is surjective. In particular, for each setlike element u, there are setlike elements w, v such that $w \otimes wv = (w \otimes v)j = u \otimes e$, whence $w = u$ and $uv = e$. A dual argument exhibits a setlike element v' such that $v'u = e$. Thus each setlike element u of A is invertible. Since the bimonoid $(A, \nabla, \Delta, \eta, \varepsilon)$ is pointed, it then becomes a Hopf algebra [9, Prop. 7.6.3]. □

4.3. **Pérez-Izquierdo algebras.** The topic of this paragraph is based on Pérez-Izquierdo's concept of an "H-bialgebra" [6, Defn. 2]. In the category $\underline{\underline{K}}$ of vector spaces over a field K, a Pérez-Izquierdo algebra as defined below reduces to a unital H-bialgebra.

Definition 4.6. Let $\mathbf{V}$ be a symmetric monoidal category. Then a *Pérez-Izquierdo algebra* $\left(A, \nabla, \Delta, \eta, \varepsilon, \overline{/}, \overline{\backslash}\right)$ in $\mathbf{V}$ is a biunital bimagma $(A, \nabla, \Delta, \eta, \varepsilon)$ equipped with a *right division* $\mathbf{V}$-morphism

$$\overline{/}\colon A \otimes A \to A$$

and a *left division* $\mathbf{V}$-morphism

$$\overline{\backslash}\colon A \otimes A \to A$$

such that the diagrams

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[rrr, "1_A \otimes \nabla"] & & & A \otimes A \arrow[d, "\overline{\backslash}"] \\
A \otimes A \arrow[u, "\Delta \otimes 1_A"] \arrow[d, "\Delta \otimes 1_A"'] \arrow[r, "\varepsilon \otimes 1_A"] & \mathbf{1} \otimes A \arrow[r, "\eta \otimes 1_A"] & A \otimes A \arrow[r, "\nabla"] & A \\
A \otimes A \otimes A \arrow[rrr, "1_A \otimes \overline{\backslash}"'] & & & A \otimes A \arrow[u, "\nabla"']
\end{tikzcd} \tag{4.5}$$

and

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[rrr, "\nabla \otimes 1_A"] & & & A \otimes A \arrow[d, "\overline{/}"] \\
A \otimes A \arrow[u, "1_A \otimes \Delta"] \arrow[d, "1_A \otimes \Delta"'] \arrow[r, "1_A \otimes \varepsilon"] & A \otimes \mathbf{1} \arrow[r, "1_A \otimes \eta"] & A \otimes A \arrow[r, "\nabla"] & A \\
A \otimes A \otimes A \arrow[rrr, "\overline{/} \otimes 1_A"'] & & & A \otimes A \arrow[u, "\nabla"']
\end{tikzcd} \tag{4.6}$$

commute.

Remark 4.7. (a) Note that H-bialgebras and Pérez-Izquierdo algebras are not self-dual concepts.

(b) Recovering Pérez-Izquierdo's original notation, write

$$(x \otimes y)\overline{\backslash} = x \backslash y \quad \text{and} \quad (x \otimes y)\overline{/} = x/y$$

for $x, y \in A$ and concrete $\mathbf{V}$. The commuting diagrams (4.5) and (4.6) are then seen to be modeled on the quasigroup identities (2.1).

As with Proposition 4.1, the following result is formulated and proved for a concrete category $\mathbf{V}$, although it may again be extended to an arbitrary symmetric monoidal category $\mathbf{V}$ with a diagrammatic proof.

Theorem 4.8. *Suppose that* $\mathbf{V}$ *is a concrete symmetric monoidal category. Then if* $\big(A, \nabla, \Delta, \eta, \varepsilon, \overline{/}, \overline{\backslash}\big)$ *is a coassociative Pérez-Izquierdo algebra in* $\mathbf{V}$*, the reduct* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a quantum loop in* $\mathbf{V}$*.*

Proof. The biunital bimagma structure in $(A, \nabla, \Delta, \eta, \varepsilon)$ is inherited directly from the Pérez-Izquierdo algebra. Now for an element y in A, tensoring the coassociativity equation (4.1) on the right with y and applying the function $(1_A \otimes 1_A \otimes \overline{\backslash})(1 \otimes \nabla)$ yields

$$x^{LL} \otimes x^{LR}(x^R \backslash y) = x^L \otimes x^{RL}(x^{RR} \backslash y) = x^L \otimes x^{R\varepsilon\eta} y = x \otimes y$$

by the commuting of the lower half of (4.5), so that

$$\big((\Delta \otimes 1_A)(1_A \otimes \overline{\backslash})\big)\big((\Delta \otimes 1_A)(1_A \otimes \nabla)\big) = 1_{A\otimes A}\,.$$

Again, tensoring the coassociativity equation (4.1) on the right with y and applying the function $(1_A \otimes 1_A \otimes \nabla)(1 \otimes \overline{\backslash})$ yields

$$x^{LL} \otimes x^{LR} \backslash (x^R y) = x^L \otimes x^{RL} \backslash (x^{RR} y) = x^L \otimes x^{R\varepsilon\eta} y = x \otimes y$$

by the commuting of the upper half of (4.5), so that

$$\big((\Delta \otimes 1_A)(1_A \otimes \nabla)\big)\big((\Delta \otimes 1_A)(1_A \otimes \overline{\backslash})\big) = 1_{A\otimes A}\,.$$

It follows that the left composite $(\Delta \otimes 1_A)(1_A \otimes \nabla)$ is invertible. Verification of the invertibility of the right composite $(1_A \otimes \Delta)(\nabla \otimes 1_A)$ is similar, using the commuting of (4.6). $\square$

For the concept of universal enveloping algebra of a Sabinin algebra (or *hyperalgebra* in the sense of [2, Defn. XII.7.9], [11]), see [6, §5].

Corollary 4.9. *The universal enveloping algebra of a Sabinin algebra over a field* K *of characteristic zero is a quantum loop in* $\underline{\underline{K}}$*.*

Proof. By [6, Prop. 24, Cor. 25], the universal enveloping algebra forms a coassociative H-bialgebra. $\square$

The following result shows that if the dual loop algebra of a loop is to be a Pérez-Izquierdo algebra, then elements of the loop have to have equal left and right inverses.

Theorem 4.10. *Consider the category* $\underline{\underline{S}}$ *of modules over a non-trivial commutative unital ring* S*. Then the dual loop algebra* $(A, \nabla, \Delta, \eta, \varepsilon)$ *in* $\underline{\underline{S}}$ *of a loop* $(Q, \cdot, /, \backslash, e)$ *admits an augmentation to a Pérez-Izquierdo algebra* $\big(A, \nabla, \Delta, \eta, \varepsilon, \overline{/}, \overline{\backslash}\big)$ *if and only if the identity* $x\backslash e = e/x$ *is satisfied in* Q*.*

Proof. In A, an instance of the fragment of the upper half of (4.5) defined in $(A, \nabla, \Delta, \eta, \varepsilon)$ is provided by

$$\begin{tikzcd}
\sum_{uv=x} \delta_u \otimes \delta_v \otimes \delta_y \arrow[rrr, mapsto, "1_A \otimes \nabla"] & & & \delta_{x/y} \otimes \delta_y \arrow[d, mapsto, dashed, "\overline{\backslash}"] \\
\delta_x \otimes \delta_y \arrow[u, mapsto, "\Delta \otimes 1_A"] \arrow[r, mapsto, "\varepsilon \otimes 1_A"] & \delta_{x,e} \otimes \delta_y \arrow[r, mapsto, "\eta \otimes 1_A"] & \delta_{x,e} \otimes \delta_y \arrow[r, mapsto, "\nabla"] & \delta_{x,e} \delta_y
\end{tikzcd}$$

for $x, y \in Q$. Commuting of this diagram is equivalent to $\delta_{x/y} \backslash \delta_y = \delta_{x,e} \delta_y$, i.e. $\delta_{zy/y} \backslash \delta_y = \delta_{zy,e} \delta_y$ or

$$\delta_z \backslash \delta_y = \delta_{zy,e} \delta_y$$

for $z \in Q$. Under this specification, the lower half of (4.5) appears as (4.7)

$$\begin{tikzcd}
\delta_x \otimes \delta_y \arrow[d, mapsto, "\Delta \otimes 1_A"'] \arrow[r, mapsto, "\varepsilon \otimes 1_A"] & \delta_{x,e} \otimes \delta_y \arrow[r, mapsto, "\eta \otimes 1_A"] & \delta_{x,e} \otimes \delta_y \arrow[r, mapsto, "\nabla"] & \delta_{x,e} \delta_y \\
\sum_{uv=x} \delta_u \otimes \delta_v \otimes \delta_y \arrow[rrr, mapsto, "1_A \otimes \overline{\backslash}"'] & & & \delta_{x/(e/y)} \otimes \delta_y \arrow[u, mapsto, "\nabla"']
\end{tikzcd}$$

noting that for $uv = x$ in Q, one has $vy = e$ if and only if $v = e/y$, in which case $u(e/y) = x$ or $u = x/(e/y)$. Commuting of (4.7) means

$$x = e \quad \text{iff} \quad x/(e/y) = y$$

in Q, and is thus equivalent to satisfaction of the identity $e/(e/y) = y$ or

$$y \backslash e = e/y \tag{4.8}$$

in Q. Since $\{\delta_x \mid x \in Q\}$ is a free basis for the S-module A, satisfaction of (4.8) is equivalent to the good definition and commutativity of (4.5). The chiral symmetry of (4.8) then implies that its satisfaction is also equivalent to the good definition and commutativity of (4.6). $\square$

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