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QUASIGROUP HOMOTOPIES, SEMISYMMETRIZATION, AND REVERSIBLE AUTOMATA

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There have been two distinct approaches to quasigroup homotopies, through reversible automata or through semisymmetrization. In the current paper, these two approaches are correlated. Kernel relations of homotopies are characterized combinatorically, and shown to form a modular lattice. Nets or webs are exhibited as purely algebraic constructs, point sets of objects in the category of quasigroup homotopies. A factorization theorem for morphisms in this category is obtained.

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1. Introduction

In the theory of quasigroups, the role played by homotopies (2) is as important as that played by homomorphisms. Historically, there have been two approaches to the study of quasigroup homotopies. The first approach, due to Gvaramiya and Plotkin [6, 7], uses the concept of a reversible automaton, consisting of three mutually interacting state spaces. In terms of universal algebra, reversible automata are modeled as heterogeneous algebras. The second approach uses a semisymmetrization functor, which reduces homotopies between general quasigroups to homomorphisms between semisymmetric quasigroups [16]. In particular, two quasigroups are isotopic if and only if their semisymmetrizations are isomorphic semisymmetric quasigroups. The aim of the present paper is to reconcile these two approaches, to examine the kernels determined by homotopies, and to derive the Factorization theorem that resolves a homotopy as the composite of a regular epimorphism and a monomorphism in the category of quasigroup homotopies. Additionally, the net or web of a quasigroup is identified as the set of points of the quasigroup in the category of quasigroup homotopies.

Quasigroup homotopies are defined in Sec. 2. The process of semisymmetrization is described in Sec. 3. Theorem 1 in Sec. 4 shows how the 3-net or 3-web associated with a quasigroup, originally construed as a combinatorial or geometric object, actually arises as the set of points of the quasigroup in the category of quasigroup homotopies. Section 5 recalls the concept of a reversible automaton. Section 6 begins a more careful examination of reversible automata, required when relinquishing the assumption (made implicitly in [6, 7]) that the state spaces are nonempty. A reversible automaton is said to be pure if its three state spaces are isomorphic sets (Definition 3). Theorem 5 shows that pure reversible automata are equivalent to isotopy classes of quasigroups.

A homotopy from a quasigroup involves three functions defined on that quasigroup, with three corresponding kernel relations. These three kernels constitute a homotopy kernel. In Sec. 7, homotopy kernels are characterized combinatorically in terms of the multiplication on the quasigroup. In particular, a set equipped with a binary multiplication forms a quasigroup iff the triple of equality relations forms a homotopy kernel. Section 8 examines the lattice of homotopy kernels on a quasigroup. Theorem 26 identifies which congruences on the semisymmetrization of a quasigroup correspond to kernels of semisymmetrized homotopies. As a corollary, it is shown that the lattice of homotopy kernels is modular.

Together with its left adjoint, the semisymmetrization functor provides an endofunctor on the category of semisymmetric quasigroups. Section 9 considers the Eilenberg–Moore algebras for this functor. In Sec. 10, these algebras are identified as quasigroup semisymmetrizations. The semisymmetrization adjunction is seen to be monadic. The two concluding sections work out the concrete implications of this categorical fact. The category of quasigroup homotopies is shown to be equivalent to the category of semisymmetrized algebras, semisymmetric quasigroups that have been enriched with a ternary operation satisfying the diagonal identity (36), and interacting appropriately with the quasigroup multiplication (Definition 34). It follows that the category of quasigroup homotopies is complete and cocomplete (Corollary 36). The paper culminates with Theorem 45, providing a factorization of each quasigroup homotopy as a product of a regular epimorphism and a monomorphism in the category of quasigroup homotopies.

Algebraic conventions and notation not explicitly described in the paper will follow the usage of [18]. Note that *corestriction* of a function means trimming its codomain to a subset that still contains the image.

2. Quasigroups

A (*combinatorial*) *quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. The quasigroup $(Q, \cdot)$ is a *loop* $(Q, \cdot, e)$ if there is an *identity* element e of Q such that $ex = x = xe$ for all x in Q. For each element y of a quasigroup Q, the *left multiplication* $L_Q(y)$ or $L(y)$ is the permutation

$$L(y) : Q \to Q; x \mapsto yx,$$

of Q, while the *right multiplication* $R_Q(y)$ or $R(y)$ is the permutation

$$R(y) : Q \to Q; x \mapsto xy.$$

In the equivalent equational description, a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{aligned} &(\mathrm{IL})y\backslash(y \cdot x) = x;\\ &(\mathrm{IR})x = (x \cdot y)/y;\\ &(\mathrm{SL})y \cdot (y\backslash x) = x;\\ &(\mathrm{SR})x = (x/y) \cdot y. \end{aligned} \tag{1}$$

In compound expressions, the multiplication binds more strongly than the divisions, so, for example, the right-hand side of (IR) could be written as xy/y or $x \cdot y/y$. The equational definition of quasigroups means that they form a variety $\mathbf{Q}$, and are thus susceptible to study using standard concepts and methods of universal algebra [18]. In particular, one may also consider $\mathbf{Q}$ as a category with quasigroup homomorphisms as the morphisms.

A triple $(f_1, f_2, f_3) : Q \to Q'$ of maps from the underlying set Q of one quasigroup to the underlying set Q' of another is a *homotopy* if

$$xf_1 \cdot yf_2 = (xy)f_3 \tag{2}$$

for all x, y in Q. The class of all quasigroups then forms the object class of a category $\mathbf{Qtp}$ whose morphisms are quasigroup homotopies. The composite of homotopies $(f_1, f_2, f_3) : Q \to Q'$ and $(g_1, g_2, g_3) : Q' \to Q''$ is the homotopy $(f_1g_1, f_2g_2, f_3g_3) : Q \to Q''$. The isomorphisms in $\mathbf{Qtp}$ are *isotopies*.

There is a forgetful functor

$$\Sigma : \mathbf{Q} \to \mathbf{Qtp} \tag{3}$$

preserving objects, sending a quasigroup homomorphism $f : Q \to Q'$ to the homotopy $(f, f, f) : Q \to Q'$. A function $f : Q \to Q'$ connecting the underlying sets of equational quasigroups $(Q, \cdot, /, \backslash)$ and $(Q', \cdot, /, \backslash)$ is a quasigroup homomorphism if it is a homomorphism $f : (Q, \cdot) \to (Q', \cdot)$ for the multiplications. Thus a homotopy (f_1, f_2, f_3) which has equal components $f_1 = f_2 = f_3$ is an element of the image of the morphism part of the forgetful functor (3).

3. Semisymmetrization

A quasigroup is *semisymmetric* if it satisfies the identity $x \cdot yx = y$. (Compare [17, Example 9] for an interpretation of this identity in terms of the semantic triality of quasigroups.) Let $\mathbf{P}$ denote the category of homomorphisms between semisymmetric quasigroups. Each quasigroup Q or $(Q, \cdot, /, \backslash)$ defines a semisymmetric quasigroup structure $Q\Delta$ on the direct cube Q^3 with multiplication as follows:

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_2//y_3, x_3\backslash\backslash y_1, x_1 \cdot y_2) \tag{4}$$

— writing $x//y = y/x$ and $x\backslash\backslash y = y\backslash x$ [16]. If $(f_1, f_2, f_3) : (Q, \cdot) \to (Q', \cdot)$ is a quasigroup homotopy, define

$$(f_1, f_2, f_3)^\Delta : Q\Delta \to Q'\Delta; (x_1, x_2, x_3) \to (x_1 f_1, x_2 f_2, x_3 f_3). \tag{5}$$

This map is a quasigroup homomorphism. Indeed, for (x_1, x_2, x_3) and (y_1, y_2, y_3) in $Q\Delta$, one has

$$\begin{aligned} &(x_1 f_1, x_2 f_2, x_3 f_3) \cdot (y_1 f_1, y_2 f_2, y_3 f_3) \\ &\quad = (x_2 f_2 // y_3 f_3, x_3 f_3 \backslash\backslash y_1 f_1, x_1 f_1 \cdot y_2 f_2) \\ &\quad = ((x_2 // y_3) f_1, (x_3 \backslash\backslash y_1) f_2, (x_1 \cdot y_2) f_3) \\ &\quad = ((x_1, x_2, x_3) \cdot (y_1, y_2, y_3))(f_1, f_2, f_3)^\Delta. \end{aligned}$$

Thus there is a functor

$$\Delta : \mathbf{Qtp} \to \mathbf{P}, \tag{6}$$

known as the *semisymmetrization functor*, with object Part (4) and morphism Part (5). This functor has a left adjoint, namely the restriction $\Sigma : \mathbf{P} \to \mathbf{Qtp}$ of the forgetful functor (3) [16, Theorem 5.2]. The unit of the adjunction at a semisymmetric quasigroup P is the homomorphism

$$\eta_P : P \to P\Sigma\Delta; x \mapsto (x, x, x) \tag{7}$$

[16, (5.3)]. The counit ε_Q at a quasigroup Q is the homotopy

$$(\pi_1, \pi_2, \pi_3) : Q\Delta\Sigma \to Q \tag{8}$$

with $(x_1, x_2, x_3)\pi_i = x_i$ for $1 \leq i \leq 3$ [16, (5.4)].

4. Webs

A (*three*)-*web* or (*three*)-*net* is a set N carrying equivalence relations α_i for $1 \leq i \leq 3$ such that N is isomorphic to the direct product $N^{\alpha_i} \times N^{\alpha_j}$ for each 2-element subset $\{i, j\}$ of $\{1, 2, 3\}$. Thus $\alpha_i \cap \alpha_j$ is the diagonal or equality relation $\widehat{N}$, while the relation product $\alpha_i \circ \alpha_j$ is the universal relation N^2. For $|N| > 1$, the relations α_i are distinct [18, Proposition I.4.3(d)]. For $1 \leq i \leq 3$, the quotients N^{α_i} are all isomorphic [18, Proposition I.4.3(c)].

Let Q be a set with isomorphisms $\lambda_i : Q \to N^{\alpha_i}$ for $1 \leq i \leq 3$. For q in Q, the α_i-class $q\lambda_i$ is called the *i-line labeled* q. A quasigroup multiplication is defined on Q by setting the product of elements x and y to be the label of the 3-line containing the unique point of intersection of the 1-line labeled x with the 2-line labeled y. The quasigroup Q is said to *coordinatize* the web N. With different labelings, the web is coordinatized by different isotopes of Q.

Conversely, consider a quasigroup Q. A three-web (the *web of the quasigroup* Q) is defined on the direct square Q^2 of Q by taking the respective equivalence relations to be the kernels of the projections $\pi_1 : (x, y) \mapsto x$, $\pi_2 : (x, y) \mapsto y$ and the multiplication $\pi_3 : (x, y) \mapsto xy$. If each i-line $(x, y)^{\ker \pi_i}$ is labeled by $(x, y)\pi_i$ for each index i, then the web of Q is coordinatized by Q.

Theorem 1. *Let Q be a quasigroup. Let T be a singleton quasigroup, the terminal object of* $\mathbf{Qtp}$.

(a) *The web of Q is the set $\mathbf{Qtp}(T, Q)$ of points of Q in* $\mathbf{Qtp}$.
(b) *For a homotopy $p : T \to Q$ and $1 \le i \le 3$, the following are equivalent*:
  (i) *For q in Q, the point p lies on the i-line labeled q.*
  (ii) *The image of $T\Delta p^\Delta$ under the i-th component of the homotopy ε_Q is q.*

5. Reversible Automata

Semisymmetrization reduces homotopies to homomorphisms of semisymmetric quasigroups. The earlier approach due to Gvaramiya and Plotkin [6, 7] reduces homotopies to homomorphisms of heterogeneous algebras. A *reversible automaton* (*of quasigroup type*) is a triple (S_1, S_2, S_3) of sets or state spaces S_i, equipped with operations

$$\mu : S_1 \times S_2 \to S_3; (x_1, x_2) \mapsto x_1 \cdot x_2,$$
$$\rho : S_3 \times S_2 \to S_1; (x_3, x_2) \mapsto x_3/x_2,$$
$$\lambda : S_1 \times S_3 \to S_2; (x_1, x_3) \mapsto x_1 \backslash x_3$$

(known respectively as *multiplication*, *right division* and *left division*) satisfying the identities:

$$\begin{aligned} &(\text{ILA}) x_1 \backslash (x_1 \cdot x_2) = x_2;\\ &(\text{IRA}) x_1 = (x_1 \cdot x_2)/x_2;\\ &(\text{SLA}) x_1 \cdot (x_1 \backslash x_3) = x_3;\\ &(\text{SRA}) x_3 = (x_3/x_2) \cdot x_2; \end{aligned} \tag{9}$$

analogous to (1). Given two reversible automata (S_1, S_2, S_3) and (S_1', S_2', S_3'), a homomorphism (in the sense of heterogeneous algebras [3, 5, 9, 11]) is a triple (f_1, f_2, f_3) of maps $f_i : S_i \to S_i'$ (for $1 \le i \le 3$) such that

$$x_1^{f_1} \cdot x_2^{f_2} = (x_1 \cdot x_2)^{f_3}, \quad x_3^{f_3}/x_2^{f_2} = (x_3/x_2)^{f_1}, \quad x_1^{f_1} \backslash x_3^{f_3} = (x_1 \backslash x_3)^{f_2} \tag{10}$$

for x_i in S_i. With homomorphisms as morphisms, the class of reversible automata forms a category $\mathbf{RAt}$, a variety of heterogeneous algebras.

Each quasigroup Q yields a reversible automaton Q^{at} or (Q, Q, Q) with equal state spaces. From (10), it is then apparent that a quasigroup homotopy $f = (f_1, f_2, f_3) : Q \to Q'$ yields a corresponding homomorphism $f^{\mathsf{at}} : Q^{\mathsf{at}} \to Q'^{\mathsf{at}}$ of reversible automata. Thus a functor $\mathsf{at} : \mathbf{Qtp} \to \mathbf{RAt}$ is defined, corestricting to an equivalence $\mathsf{at} : \mathbf{Qtp} \to \mathbf{QAt}$ of the homotopy category $\mathbf{Qtp}$ with a category $\mathbf{QAt}$ of homomorphisms between reversible automata.

6. Pure Reversible Automata

For any set S (empty or not), there are reversible automata

$$(\varnothing, \varnothing, S), \quad (S, \varnothing, \varnothing), \quad (\varnothing, S, \varnothing) \tag{11}$$

of quasigroup type in which the multiplication, right division, and left division respectively embed the empty set in S, while the other two operations in each are the identity $1_\varnothing$ on the empty set. The remaining possibilities are described by the following.

Proposition 2. *Let (S_1, S_2, S_3) be a reversible automaton of quasigroup type, with at most one empty state space. Then the state spaces S_1, S_2 and S_3 are isomorphic, nonempty sets.*

Proof. If S_2 were empty, the existence of the left division (as a map from $S_1 \times S_3$ to S_2) would imply the emptiness of at least one of S_1 and S_3. Thus S_2 is nonempty. Fix an element s_2 of S_2. Then by (IRA) and (SRA), the maps

$$S_1 \to S_3; x_1 \mapsto x_1 \cdot s_2 \quad \text{and} \quad S_3 \to S_1; x_3 \mapsto x_3/s_2$$

are mutually inverse, showing that S_1 and S_3 are isomorphic. In particular, they are both nonempty. Then for an element s_1 of S_1, the maps

$$S_2 \to S_3; x_2 \mapsto s_1 \cdot x_2 \quad \text{and} \quad S_3 \to S_2; x_3 \mapsto s_1 \backslash x_3$$

are mutually inverse by (ILA) and (SLA), showing that S_3 and S_2 are isomorphic. □

Definition 3. A reversible automaton of quasigroup type is said to be *pure* if its three state spaces are isomorphic sets.

Remark 4. By Proposition 2, the impure reversible automata are given by (11) with non-empty S. Thus Definition 3 agrees with the general definition of Barr [1, p. 367].

Now consider a pure reversible automaton (S_1, S_2, S_3). By the purity, there is a set Q with isomorphisms

$$l_i : Q \to S_i \tag{12}$$

for $1 \leq i \leq 3$. Define a respective multiplication, right division and left division on Q by

$$x \cdot y = (x^{l_1} \cdot y^{l_2}) l_3^{-1}, \tag{13}$$

$$x/y = (x^{l_1} / y^{l_2}) l_3^{-1}, \tag{14}$$

$$x \backslash y = (x^{l_1} \backslash y^{l_2}) l_3^{-1}. \tag{15}$$

The identities (9) on (S_1, S_2, S_3) yield the identities (1) on Q, making Q a quasigroup. The definition of the operations on Q shows that

$$(l_1, l_2, l_3) : Q^{\mathsf{at}} \to (S_1, S_2, S_3)$$

is an isomorphism in **RAt**. If (S_1, S_2, S_3) is isomorphic to Q'^{at} for a quasigroup Q', then the isomorphism between Q^{at} and Q'^{at} shows that Q and Q' are isotopic. Summarizing, one has the following result, originally stated without the explicit purity hypothesis as [6, Theorem 1(1)] [7, Theorem 1].[a]

Theorem 5. *Let R be a pure reversible automaton of quasigroup type.*

(a) *Within the category* **RAt**, *the automaton R is isomorphic to an automaton of the form Q^{at} for a quasigroup Q.*
(b) *The quasigroup Q of* (a) *is unique up to isotopy.*

7. Homotopy Kernels

Definition 6. Let Q be a quasigroup. Consider a triple $(\theta_1, \theta_2, \theta_3)$ of equivalence relations on the set Q. Then $(\theta_1, \theta_2, \theta_3)$ is said to be a *homotopy kernel* if for $xy = z$ and $x'y' = z'$ in Q, any two of the following statements implies the third:

(a) $(x, x') \in \theta_1$;
(b) $(y, y') \in \theta_2$;
(c) $(z, z') \in \theta_3$.

Remark 7. The very definition of quasigroups implies that the triple of equality relations $\big(\widehat{Q}, \widehat{Q}, \widehat{Q}\big)$ on a quasigroup Q forms a homotopy kernel.

Example 8. Let $(f_1, f_2, f_3) : Q \to Q'$ be a homotopy. Then the triple

$$(\ker f_1, \ker f_2, \ker f_3)$$

of kernels forms a homotopy kernel.

Proposition 9. *Let $(\theta_1, \theta_2, \theta_3)$ be a homotopy kernel on a quasigroup Q. Then $(Q^{\theta_1}, Q^{\theta_2}, Q^{\theta_3})$ is a pure reversible automaton.*

Proof. Consider the maps

$$\begin{aligned}
\mu &: Q^{\theta_1} \times Q^{\theta_2} \to Q^{\theta_3}; (x_1^{\theta_1}, x_2^{\theta_2}) \mapsto (x_1 \cdot x_2)^{\theta_3},\\
\rho &: Q^{\theta_3} \times Q^{\theta_2} \to Q^{\theta_1}; (x_3^{\theta_3}, x_2^{\theta_2}) \mapsto (x_3 / x_2)^{\theta_1},\\
\lambda &: Q^{\theta_1} \times Q^{\theta_3} \to Q^{\theta_2}; (x_1^{\theta_1}, x_3^{\theta_3}) \mapsto (x_1 \backslash x_3)^{\theta_2}.
\end{aligned}$$

By Definition 6, these maps are well-defined. To verify (ILA), note that

$$x_1^{\theta_1} \backslash (x_1^{\theta_1} \cdot x_2^{\theta_2}) = x_1^{\theta_1} \backslash (x_1 \cdot x_2)^{\theta_3} = \big(x_1 \cdot (x_1 \backslash x_2)\big)^{\theta_2} = x_2^{\theta_2}$$

by (IL). Verification of the remaining identities (9) is similar. Thus $(Q^{\theta_1}, Q^{\theta_2}, Q^{\theta_3})$ forms a reversible automaton.

If Q is empty, all three quotients Q^{θ_i} for $1 \le i \le 3$ are empty, forming the pure reversible automaton $(\varnothing, \varnothing, \varnothing)$. Otherwise, all three quotients are non-empty. Thus the reversible automaton $(Q^{\theta_1}, Q^{\theta_2}, Q^{\theta_3})$ is pure in all cases. □

[a]These papers used an implicit assumption of nonemptiness.

Corollary 10. *Let $(\theta_1, \theta_2, \theta_3)$ be a homotopy kernel on a quasigroup Q. Then the three quotient sets Q^{θ_1}, Q^{θ_2} and Q^{θ_3} are isomorphic.*

Proposition 11. *Let $(\theta_1, \theta_2, \theta_3)$ be a homotopy kernel on a quasigroup Q. If Q is empty, take $(Q^{\theta_3}, +, -, \sim)$ as the empty quasigroup. Otherwise, for each pair (u, v) of elements of Q, there is a well-defined quasigroup structure $(Q^{\theta_3}, +, -, \sim)$ given by*

$$x^{\theta_3} + y^{\theta_3} = (x/v \cdot u\backslash y)^{\theta_3}, \tag{16}$$

$$x^{\theta_3} - y^{\theta_3} = ((x/(u\backslash y)) \cdot v)^{\theta_3}, \tag{17}$$

$$x^{\theta_3} \sim y^{\theta_3} = (u \cdot ((x/v)\backslash y))^{\theta_3}. \tag{18}$$

Proof. Consider the instances

$$l_1 : Q^{\theta_3} \to Q^{\theta_1}; x^{\theta_3} \mapsto (x/v)^{\theta_1}, \tag{19}$$

$$l_2 : Q^{\theta_3} \to Q^{\theta_2}; y^{\theta_3} \mapsto (u\backslash y)^{\theta_2} \tag{20}$$

$$l_3 : Q^{\theta_3} \to Q^{\theta_1}; x^{\theta_3} \mapsto x^{\theta_3} \tag{21}$$

of (12) for the pure reversible automaton of Proposition 9. Then (13) produces the quasigroup multiplication

$$(x^{\theta_3}, y^{\theta_3}) \mapsto (x/v)^{\theta_1} \cdot (u\backslash y)^{\theta_2} = (x/v \cdot u\backslash y)^{\theta_3}, \tag{22}$$

yielding (16). The right division (17) and left division (18) follow from (14) and (15) in similar fashion. □

Remark 12. If the homotopy kernel is the triple $\left(\widehat{Q}, \widehat{Q}, \widehat{Q}\right)$ of equality relations on a nonempty quasigroup Q, in particular on a loop, then (16) gives the so-called *u,v-isotope* of Q [12, Definitions II.2.5 and III.2.5].

Corollary 13. *If Q is nonempty, with $u, v \in Q$, then $(Q^{\theta_3}, +, (uv)^{\theta_3})$ is a loop. For different choices of u and v, the corresponding loops are isotopic.*

Proof. For x in Q, one has

$$\begin{aligned}(uv)^{\theta_3} + x^{\theta_3} &= ((uv)^{\theta_3}/v^{\theta_2}) \cdot (u^{\theta_1}\backslash x^{\theta_3}) \\ &= (uv/v)^{\theta_1} \cdot (u^{\theta_1}\backslash x^{\theta_3}) = u^{\theta_1} \cdot (u^{\theta_1}\backslash x^{\theta_3}) = x^{\theta_3}\end{aligned}$$

and

$$\begin{aligned}x^{\theta_3} + (uv)^{\theta_3} &= (x^{\theta_3}/v^{\theta_2}) \cdot (u^{\theta_1}\backslash (uv)^{\theta_3}) \\ &= (x^{\theta_3}/v^{\theta_2}) \cdot (u\backslash uv)^{\theta_2} = (x^{\theta_3}/v^{\theta_2}) \cdot v^{\theta_2} = x^{\theta_3}.\end{aligned}$$

The final statement follows from Theorem 5(b). □

Corollary 14. *If $(\theta_1, \theta_2, \theta_3)$ is a homotopy kernel on a quasigroup Q, there is a homotopy $Q \to (Q^{\theta_3}, +)$. If Q is non-empty, with $u, v \in Q$, then the homotopy is given by*

$$(R(v)\mathrm{nat}\,\theta_3, L(u)\mathrm{nat}\,\theta_3, \mathrm{nat}\,\theta_3). \tag{23}$$

Proof. For empty Q, the result is immediate. Otherwise, note that

$$(xv)^{\theta_3} + (uy)^{\theta_3} = (xy)^{\theta_3}$$

by (16) for x, y in Q. □

Proposition 15. *A triple $(\theta_1, \theta_2, \theta_3)$ of equivalence relations on a quasigroup Q is a homotopy kernel if and only if it is of the form*

$$(\ker f_1, \ker f_2, \ker f_3)$$

for a homotopy (f_1, f_2, f_3) with domain Q.

Proof. The "if" direction is Example 8. For the "only if" direction on non-empty Q, note that $(u, u) \in \theta_1$ and $(v, v) \in \theta_2$ imply

$$(x, x') \in \ker(R(v)\mathrm{nat}\,\theta_3) \quad \text{and} \quad (y, y') \in \ker\left(L(u)\mathrm{nat}\,\theta_3\right)$$

in (23) iff $(x, x') \in \theta_1$ and $(y, y') \in \theta_2$. □

Theorem 16. *Let $(\theta_1, \theta_2, \theta_3)$ be a homotopy kernel on a quasigroup Q. Then a semisymmetric quasigroup structure on $Q^{\theta_1} \times Q^{\theta_2} \times Q^{\theta_3}$ is well defined by*

$$(x_1^{\theta_1}, x_2^{\theta_2}, x_3^{\theta_3}) \cdot (y_1^{\theta_1}, y_2^{\theta_2}, y_3^{\theta_3}) = ((y_3/x_2)^{\theta_1}, (y_1\backslash x_3)^{\theta_2}, (x_1 \cdot y_2)^{\theta_3}). \tag{24}$$

This semisymmetric quasigroup is isomorphic to the semisymmetrization of the quasigroup $(Q^{\theta_3}, +, -, \sim)$ from Proposition 11.

Proof. Consider the set isomorphism

$$\begin{aligned}(l_1, l_2, l_3) : (Q^{\theta_3}, +, -, \sim)\Delta &\to Q^{\theta_1} \times Q^{\theta_2} \times Q^{\theta_3};\\ (x_1^{\theta_3}, x_2^{\theta_3}, x_3^{\theta_3}) &\mapsto ((x_1/v)^{\theta_1}, (u\backslash x_2)^{\theta_2}, x_3^{\theta_3})\end{aligned} \tag{25}$$

defined using the set isomorphisms (19)–(21). On the domain side, the product of elements $(x_1^{\theta_3}, x_2^{\theta_3}, x_3^{\theta_3})$ and $(y_1^{\theta_3}, y_2^{\theta_3}, y_3^{\theta_3})$ is

$$(y_3^{\theta_3} - x_1^{\theta_3}, y_1^{\theta_3} \sim x_2^{\theta_3}, x_1^{\theta_3} + y_2^{\theta_3})$$

or

$$\left(((y_3/(u\backslash x_2)) \cdot v)^{\theta_3}, \left(u \cdot ((y_1/v)\backslash x_3)\right)^{\theta_3}, \left(x_1/v \cdot u\backslash y_2\right)^{\theta_3}\right). \tag{26}$$

The corresponding product of the images $((x_1/v)^{\theta_1}, (u\backslash x_2)^{\theta_2}, x_3^{\theta_3})$ and $((y_1/v)^{\theta_1}, (u\backslash y_2)^{\theta_2}, y_3^{\theta_3})$ under the multiplication (24) is

$$\left((y_3/(u\backslash x_2))^{\theta_1}, ((y_1/v)\backslash x_3)^{\theta_2}, (x_1/v \cdot u\backslash y_2)^{\theta_3}\right),$$

which is the image of (26) under the map (25). This map becomes an isomorphism of magmas, so (24) defines a semisymmetric quasigroup on $Q^{\theta_1} \times Q^{\theta_2} \times Q^{\theta_3}$. □

Remark 17. If $(\widehat{Q}, \widehat{Q}, \widehat{Q})$ is the triple of equality relations on a quasigroup Q, then (24) recovers the original definition (4) of the semisymmetrization of Q from [16].

8. The Kernel Lattice

For a quasigroup Q, let $\mathrm{HK}(Q)$ denote the set of homotopy kernels on Q.

Lemma 18. *Consider homotopy kernels $(\theta_1, \theta_2, \theta_3)$ and $(\theta'_1, \theta'_2, \theta'_3)$ on a quasigroup Q. Then in the partition lattice of Q, the following order relations are equivalent:*

(a) $\theta_1 \subseteq \theta'_1$;

(b) $\theta_2 \subseteq \theta'_2$;

(c) $\theta_3 \subseteq \theta'_3$.

Proof. If Q is empty, the lemma is trivial. Otherwise, suppose that (a) holds. Consider a pair (x, y) in θ_3. Now $(v, v) \in \theta_2$ for v in Q. Since $(\theta_1, \theta_2, \theta_3)$ is a homotopy kernel, one has $(x/v, x'/v) \in \theta_1$. Then $(x/v, x'/v) \in \theta'_1$ by (a), while $(v, v) \in \theta'_2$. Since $(\theta'_1, \theta'_2, \theta'_3)$ is a homotopy kernel, it follows that $(x, y) \in \theta'_3$. Thus (c) holds. The other implications are similar. $\square$

Define an order relation $\leq$ on $\mathrm{HK}(Q)$ by setting

$$(\theta_1, \theta_2, \theta_3) \leq (\theta'_1, \theta'_2, \theta'_3)$$

whenever the equivalent statements (a)–(c) of Lemma 18 hold. If $((\theta^i_1, \theta^i_2, \theta^i_3) \mid i \in I)$ is a family of homotopy kernels (for some index set I), then

$$\left(\bigcap_{i \in I} \theta^i_1, \bigcap_{i \in I} \theta^i_2, \bigcap_{i \in I} \theta^i_3 \right)$$

is again a homotopy kernel. Thus $\mathrm{HK}(Q)$ forms a complete lattice under $\leq$, the *kernel lattice* of the quasigroup Q.

For a congruence θ on the semisymmetrization $Q\Delta$ of a quasigroup Q, define relations θ_i on Q for $1 \leq i \leq 3$ as follows:

$$(x, x') \in \theta_1 \Leftrightarrow \forall\ s, t \in Q,\ (x, s, t)\, \theta\, (x', s, t); \tag{27}$$

$$(y, y') \in \theta_2 \Leftrightarrow \forall\ s, t \in Q,\ (s, y, t)\, \theta\, (s, y', t); \tag{28}$$

$$(z, z') \in \theta_3 \Leftrightarrow \forall\ s, t \in Q,\ (s, t, z)\, \theta\, (s, t, z'). \tag{29}$$

Lemma 19. *The triple $(\theta_1, \theta_2, \theta_3)$ is a homotopy kernel on Q.*

Proof. If Q is empty, the lemma is trivial. Otherwise, the θ_i are certainly equivalence relations. Suppose $(y, y') \in \theta_2$ and $(z, z') \in \theta_3$. Then for arbitrary s, t in Q and fixed s_0, t_0 in Q, one has

$$(s, y, t)\theta(s, y', t) \quad \text{and} \quad (s_0, t_0, z)\theta(s_0, t_0, z'),$$

whence

$$\begin{aligned}(z/y, s_0\backslash t, s \cdot t_0) &= (s, y, t) \cdot (s_0, t_0, z)\theta\, (s, y', t) \cdot (s_0, t_0, z') \\ &= (z_0/y_0, s_0\backslash t, s \cdot t_0).\end{aligned}$$

Now as s and t range over Q, the elements $s_0\backslash t$ and $s \cdot t_0$ range over Q. It follows that $(z/y, z'/y') \in \theta_1$. The other implications required by Definition 6 are verified in similar fashion. □

The specifications (27)–(29) thus yield an order-preserving map

$$k : \mathrm{Cg}(Q\Delta) \to \mathrm{HK}(Q); \theta \mapsto (\theta_1, \theta_2, \theta_3) \tag{30}$$

from the congruence lattice of $Q\Delta$ to the kernel lattice of Q.

Conversely, consider a homotopy kernel $(\theta_1, \theta_2, \theta_3)$ on Q. Define a relation $(\theta_1, \theta_2, \theta_3)^c$ on $Q\Delta$ by

$$(x_1, x_2, x_3)\ (\theta_1, \theta_2, \theta_3)^c\ (y_1, y_2, y_3) \Leftrightarrow \forall\ 1 \leq i \leq 3,\ x_i\, \theta_i\, y_i. \tag{31}$$

Lemma 20. *The relation $(\theta_1, \theta_2, \theta_3)^c$ is a congruence on $Q\Delta$.*

Proof. Suppose that (x_i, y_i) and (x'_i, y'_i) lie in θ_i for $1 \leq i \leq 3$. Then $(x'_3/x_2, y'_3/y_2) \in \theta_1$, $(x'_1\backslash x_3, y'_1\backslash y_3) \in \theta_2$, and $(x_1 \cdot x'_2, y_1 \cdot y'_2) \in \theta_3$ since $(\theta_1, \theta_2, \theta_3)$ is a homotopy kernel. □

Remark 21. According to Corollary 14, a homotopy kernel $(\theta_1, \theta_2, \theta_3)$ yields a homotopy $Q \to (Q^{\theta_3}, +)$, semisymmetrizing to a homomorphism $Q\Delta \to (Q^{\theta_3}, +)\Delta$. The relation $(\theta_1, \theta_2, \theta_3)^c$ is the kernel congruence of this homomorphism.

By Lemma 20 or Remark 21, the definition (31) yields an order-preserving map

$$c : \mathrm{HK}(Q) \to \mathrm{Cg}(Q\Delta); (\theta_1, \theta_2, \theta_3) \mapsto (\theta_1, \theta_2, \theta_3)^c \tag{32}$$

from the kernel lattice of Q to the congruence lattice of $Q\Delta$.

Proposition 22. *Let θ be a congruence on $Q\Delta$. Then $\theta \geq \theta^{kc}$.*

Proof. Suppose that $(x, y, z)\, \theta^{kc}\, (x', y', z')$, so one has $(x, x') \in \theta_1$, $(y, y') \in \theta_2$, and $(z, z') \in \theta_3$. Then $(x, y, z)\, \theta\, (x', y, z)\, \theta\, (x', y', z)\, \theta\, (x', y', z')$. □

The inclusion in Proposition 22 may be proper.

Example 23. Let $(A, +)$ be a non-trivial abelian group. Now the *characteristic congruence* ν on the semisymmetrization $A\Delta$ is defined by

$$(x_1, x_2, x_3)\, \nu\, (y_1, y_2, y_3) \Leftrightarrow y_1 - x_1 = y_2 - x_2 = x_3 - y_3$$

[10]. Then ν is a non-trivial congruence. However, $\nu_1 = \nu_2 = \nu_3 = \widehat{A}$ by (27)–(29), so that $\widehat{A\Delta} = \nu^{kc} < \nu$.

Definition 24. A congruence θ on the semisymmetrization $Q\Delta$ of a quasigroup Q is said to be *homotopical* if $\theta = \theta^{kc}$.

In the reverse direction, matters are more straightforward.

Proposition 25. *Let* $(\theta_1, \theta_2, \theta_3)$ *be a homotopy kernel on a quasigroup* Q. *Then* $(\theta_1, \theta_2, \theta_3) = (\theta_1, \theta_2, \theta_3)^{ck}$.

Proof. For $x, x' \in Q$, the reflexivity of θ_2 and θ_3 yields

$$\begin{aligned} x\theta_1 x' &\Leftrightarrow \forall\ s, t \in Q,\ x\,\theta_1\,x',\ s\,\theta_2\,s,\ t\,\theta_3\,t \\ &\Leftrightarrow \forall\ s, t \in Q,\ (x, s, t)\,(\theta_1, \theta_2, \theta_3)^c\,(x', s, t) \\ &\Leftrightarrow x\,(\theta_1, \theta_2, \theta_3)^c_1\,x'. \end{aligned}$$

Similarly, $\theta_2 = (\theta_1, \theta_2, \theta_3)^c_2$ and $\theta_3 = (\theta_1, \theta_2, \theta_3)^c_3$. □

Theorem 26. *The maps* c *of* (32) *and* k *of* (30) *form the respective left and right adjoints in a Galois connection. Moreover, a congruence* θ *on a semisymmetrization* $Q\Delta$ *is homotopical if and only if it is of the form* $(\theta_1, \theta_2, \theta_3)^c$ *for a homotopy kernel* $(\theta_1, \theta_2, \theta_3)$ *on* Q.

Proof. By Propositions 22 and 25, [18, Proposition III.3.3.1] applies. The second statement follows from [18, III(3.3.3)]. □

Corollary 27. *The kernel lattice* $\mathrm{HK}(Q)$ *of a quasigroup* Q *is modular.*

Proof. If Q is empty, the result is immediate. Otherwise, consider two homotopy kernels $\theta_* = (\theta_1, \theta_2, \theta_3)$ and $\varphi_* = (\varphi_1, \varphi_2, \varphi_3)$ on a non-empty quasigroup Q. In the congruence lattice $\mathrm{Cg}(Q\Delta)$, the join of the congruences θ_*^c and φ_*^c is their relation product $\theta_*^c \circ \varphi_*^c$. By Proposition 22, $\theta_*^c \circ \varphi_*^c \geq (\theta_*^c \circ \varphi_*^c)^{kc}$. Conversely, suppose that elements (x_1, x_2, x_3) and (y_1, y_2, y_3) of $Q\Delta$ are related by $(\theta_*^c \circ \varphi_*^c)^{kc}$. Since $x_1\,(\theta_*^c \circ \varphi_*^c)^k_1\,y_1$, one has $(x_1, s, t)\,\theta_*^c \circ \varphi_*^c\,(y_1, s, t)$ for certain elements s, t of Q. Thus there is an element (u, v, w) of $Q\Delta$ such that

$$(x_1, s, t)\,\theta_*^c\,(u, v, w)\,\varphi_*^c\,(y_1, s, t).$$

In particular, $x_1\,\theta_1\,u\,\varphi_1\,y_1$, i.e. $x_1\,\theta_1 \circ \varphi_1\,y_1$. Similarly, $x_i\,\theta_i \circ \varphi_i\,y_i$ for $i = 2$ and 3, so that (x_1, x_2, x_3) and (y_1, y_2, y_3) are related by $\theta_*^c \circ \varphi_*^c$. It follows that $\theta_*^c \circ \varphi_*^c = (\theta_*^c \circ \varphi_*^c)^{kc}$: a join of homotopical congruences is homotopical; and the mutually inverse maps c and k between the sets of closed elements in the Galois connection of Theorem 26 are lattice isomorphisms. In particular, the kernel lattice $\mathrm{HK}(Q)$ is isomorphic to a sublattice of the modular congruence lattice $\mathrm{Cg}(Q\Delta)$. □

9. Eilenberg–Moore Algebras

The semisymmetrization functor $\Delta : \mathbf{Qtp} \to \mathbf{P}$ is right adjoint to the forgetful functor $\Sigma : \mathbf{P} \to \mathbf{Qtp}$. Consider the endofunctor $\Sigma\Delta$ on the category $\mathbf{P}$ of semisymmetric quasigroups. Recall that a $\Sigma\Delta$-*algebra* is a semisymmetric quasigroup P equipped with a homomorphism α_P or

$$\alpha : P\Sigma\Delta \to P \tag{33}$$

known as the *action* or *structure map*. A $\Sigma\varepsilon\Delta$-*algebra*, or *Eilenberg–Moore algebra* for the adjunction $(\Sigma, \Delta, \eta, \varepsilon)$, is a $\Sigma\Delta$-algebra P satisfying the *associative law*

$$\begin{array}{ccc} P\Sigma\Delta\Sigma\Delta & \xrightarrow{\Sigma\varepsilon\Delta_P} & P\Sigma\Delta \\ {\scriptstyle \alpha\Sigma\Delta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ P\Sigma\Delta & \xrightarrow[\alpha]{} & P \end{array} \tag{34}$$

and the *unit law*

$$\begin{array}{ccc} P & \xrightarrow{\eta_P} & P\Sigma\Delta \\ \| & & \downarrow{\scriptstyle \alpha} \\ P & = & P \end{array} \tag{35}$$

[18, IV, Sec. 4.1]. The associative law reduces to the *diagonal identity*

$$((x_{11}, x_{12}, x_{13})^\alpha, (x_{21}, x_{22}, x_{23})^\alpha, (x_{31}, x_{32}, x_{33})^\alpha)\alpha = (x_{11}, x_{22}, x_{33})\alpha \tag{36}$$

for x_{ij} in P with $1 \le i, j \le 3$, while the unit law reduces to the idempotence

$$(x, x, x)\alpha = x \tag{37}$$

for x in P. Thus (P, α) forms a *diagonal algebra* in the sense of Płonka [13, 14] [15, Example 5.2.2].

For a quasigroup Q, the semisymmetrization $Q\Delta$ is an Eilenberg–Moore algebra [18, p. 350]. Its structure map is the image

$$\varepsilon_Q^\Delta : Q\Delta\Sigma\Delta \to Q\Delta; \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \mapsto (x_{11}, x_{22}, x_{33}) \tag{38}$$

of the counit, written using matrix notation for triples of triples.

10. Monadicity

In this section, it will be shown that the adjunction $(\Sigma, \Delta, \eta, \varepsilon)$ is monadic. Indeed, those semisymmetric quasigroups that are (isomorphic to) semisymmetrizations will be characterized as the Eilenberg–Moore algebras of the adjunction.

Let P be an Eilenberg–Moore algebra, with structure map α as in (33). Let θ be the kernel of α, a congruence on $P\Delta$. Define corresponding relations θ_i on P using (27)–(29). By Lemma 19, the triple $(\theta_1, \theta_2, \theta_3)$ of equivalence relations on P forms a homotopy kernel.

Remark 28. On a semisymmetrization $Q\Delta$, (38) shows that the relations θ_i reduce to the respective kernels of the projections π_i making up the counit homotopy (8).

Lemma 29. *The intersection* $\theta_1 \cap \theta_2 \cap \theta_3$ *of the relations* θ_i *is the equality relation* $\widehat{P}$ *on* P.

Proof. Suppose $(x, x') \in \theta_1 \cap \theta_2 \cap \theta_3$. Then

$$x = (x, x, x)\alpha = (x', x, x)\alpha = (x', x', x)\alpha = (x', x', x')\alpha = x'$$

by idempotence (37) and successive use of the relations θ_1, θ_2, θ_3. □

Lemma 30. *For elements x, y, z of P, one has*

$$x \;\theta_1\; (x, y, z)\alpha, \tag{39}$$

$$y \;\theta_2\; (x, y, z)\alpha, \tag{40}$$

$$z \;\theta_3\; (x, y, z)\alpha. \tag{41}$$

Proof. For arbitrary elements s, t of P, the idempotence (37) and diagonal identity (36) yield

$$((x, y, z)^\alpha, s, t)\alpha = ((x, y, z)^\alpha, (s, s, s)^\alpha, (t, t, t)^\alpha)\alpha = (x, s, t)\alpha,$$

exhibiting the relation (39). The other two relations are proved similarly. □

Lemma 31. *The relations θ_1 and θ_2 permute, with common relation product P^2.*

Proof. Let x and y be elements of P. Lemma 30 yields

$$x\theta_1(x, y, x)\alpha\theta_2 y$$

and

$$x\theta_2(y, x, x)\alpha\theta_1 y,$$

as required. □

Lemma 32. *The relations $\theta_1 \cap \theta_2$ and θ_3 permute. Indeed, their common relation product is P^2.*

Proof. Let x and y be elements of P. Lemma 30 yields

$$x\theta_1 \cap \theta_2(x, x, y)\alpha\theta_3 y$$

and

$$x\theta_3(y, y, x)\alpha\theta_1 \cap \theta_2 y,$$

as required. □

Theorem 33. *If a semisymmetric quasigroup P is an Eilenberg–Moore algebra for the adjunction $(\Sigma, \Delta, \eta, \varepsilon)$, then it is isomorphic to the semisymmetrization of a quasigroup Q.*

Proof. Consider the map

$$\Theta : P \to P^{\theta_1} \times P^{\theta_2} \times P^{\theta_3}; x \mapsto (x^{\theta_1}, x^{\theta_2}, x^{\theta_3}).$$

By Lemmas 29, 31 and 32, Θ is a set isomorphism (compare [2, Theorem VII.5]). Since $(\theta_1, \theta_2, \theta_3)$ is a homotopy kernel, Theorem 16 shows that the codomain of Θ carries a semisymmetric quasigroup structure isomorphic to the semisymmetrization of a quasigroup $(P^{\theta_3}, +)$. Then for elements x, y of the semisymmetric quasigroup P, one has

$$\begin{aligned} x\Theta \cdot y\Theta &= (x^{\theta_1}, x^{\theta_2}, x^{\theta_3}) \cdot (y^{\theta_1}, y^{\theta_2}, y^{\theta_3}) \\ &= ((y/x)^{\theta_1}, (y\backslash x)^{\theta_2}, (x \cdot y)^{\theta_3}) \\ &= ((x \cdot y)^{\theta_1}, (x \cdot y)^{\theta_2}, (x \cdot y)^{\theta_3}) = (x \cdot y)\Theta, \end{aligned}$$

so that Θ is a quasigroup isomorphism. □

11. Semisymmetrized Algebras

The next two sections give direct and concrete formulations of the implications of the preceding section.

Definition 34. An algebra $(P, \cdot, \alpha)$ equipped with a binary multiplication denoted by $\cdot$ or juxtaposition, and an idempotent ternary operation α, is a *semisymmetrized algebra* if:

(a) $y \cdot (x \cdot y) = x$;
(b) α satisfies the diagonal identity (36);
(c) for all x_i and y_j in P,

$$(x_1y_1, x_2y_2, x_3y_3)\alpha = (x_3, x_1, x_2)\alpha \cdot (y_2, y_3, y_1)\alpha.$$

Theorem 35. *The category* **Qtp** *of quasigroup homotopies is equivalent to the variety of semisymmetrized algebras.*

Proof. By Definition 34, semisymmetrized algebras form a variety of universal algebras of type $\{(\cdot, 2), (\alpha, 3)\}$, construed as a category with homomorphisms as morphisms. By Definition 34(a) and [16, Corollary 2.2], the reduct $(P, \cdot)$ of a semisymmetrized algebra forms a semisymmetric quasigroup. The remaining parts of Definition 34 then show that P forms an Eilenberg–Moore algebra for the adjunction $(\Delta, \Sigma, \eta, \varepsilon)$. In particular, Definition 34(c) expresses the fact that $\alpha : P\Sigma\Delta \to P$ is a homomorphism. Since the adjunction is monadic by Theorem 33, the Eilenberg–Moore comparison gives the required equivalence. □

Corollary 36. *The homotopy category* **Qtp** *is bicomplete.*

Proof. Each variety of universal algebras is bicomplete [18, IV, Sec. 2.2]. □

Remark 37. By Theorem 5, the category of quasigroup homotopies is equivalent to the category of pure reversible automata. By a result of Barr [1, Theorem 5], reformulated by Goguen and Meseguer [5, p. 331], the category of pure reversible automata is in turn equivalent to some variety of (single-sorted) algebras. Theorem 35 identifies these algebras explicitly as the semisymmetrized algebras.

12. The Factorization Theorem

Throughout this section, consider a homotopy

$$f = (f_1, f_2, f_3) : Q \to Q'. \tag{42}$$

The main result of the section, Theorem 45, presents a factorization of (42). The factorization may be derived indirectly from Theorem 35 and the First Isomorphism Theorem in universal algebra, but it is nevertheless convenient to have a direct and explicit treatment.

Proposition 38. *The following are equivalent*:

(a) *f is a monomorphism in* **Qtp**;
(b) $\exists\, 1 \le i \le 3$. *f_i is a monomorphism in* **Set**;
(c) $\forall\, 1 \le i \le 3$. *f_i is a monomorphism in* **Set**;
(d) (f_1, f_2, f_3) *is a monomorphism in* $\mathbf{Set}^3$;
(e) *$f\Delta$ is a monomorphism in* **Set**;
(f) *$f\Delta$ is a monomorphism in* **Q**;
(g) *$f\Delta$ is a monomorphism in* **P**.

Proof. The equivalence of (c)–(e) is immediate, as is the implication (c)⇒(b), while the equivalence of (e)–(g) follows since **P** and **Q** are categories of homomorphisms of algebras.

(a)⇒(g): The right adjoint Δ : **Qtp** $\to$ **P** preserves monomorphisms [8, Proposition 24.5 and Theorem 27.7].

(c)⇒(a): Suppose $gf = g'f$ in **Qtp**. Then for $1 \le i \le 3$, one has $g_i f_i = g'_i f_i$ in **Set**. By (c), $g_i = g'_i$, so that $g = g'$.

(b)⇒(c): Suppose $yf_2 = y'f_2$ for distinct elements y, y' of Q. Then for an element x of Q, the elements xy and xy' are distinct. But

$$(xy)f_3 = xf_1 \cdot yf_2 = xf_1 \cdot y'f_2 = (xy')f_3,$$

so f_2 not monomorphic implies f_3 not monomorphic. The other cases follow in similar fashion. □

Proposition 39. *Let u and v be elements of Q. Define an operation $+$ on Qf_3 by*

$$xf_3 + yf_3 = (x/v \cdot u\backslash y)f_3 \tag{43}$$

for x, y in Q. Then $(Qf_3, +, (uv)f_3)$ is a well-defined loop.

Proof. Let $(\theta_1, \theta_2, \theta_3)$ be the homotopy kernel on Q given by Example 8. For x in Q, consider the well-defined instances

$$l_1 : Qf_3 \to Q^{\theta_1}; xf_3 \mapsto (x/v)^{\theta_1},$$
$$l_2 : Qf_3 \to Q^{\theta_2}; xf_3 \mapsto (u\backslash x)^{\theta_2},$$
$$l_3 : Qf_3 \to Q^{\theta_1}; xf_3 \mapsto x^{\theta_3},$$

of (12). Then (43) reduces to (16). □

Corollary 40. *The operation* (43) *may be written in the form*

$$X + Y = (X/vf_2) \cdot (uf_1 \backslash Y) \tag{44}$$

for X, Y *in* Qf_3.

Proof. The form (44) of (43) follows from (22). □

For $1 \le i \le 3$, let $j_i : Qf_i \hookrightarrow Q'$ be the embedding of the subset Qf_i into Q'. It is apparent from Corollary 40 that

$$m = (R_{Q'}(vf_2)^{-1} j_1, L_{Q'}(uf_1)^{-1} j_2, j_3) : (Qf_3, +) \to (Q', \cdot) \tag{45}$$

is a well-defined homotopy. In particular, $XR(vf_2)^{-1} \in Qf_1$ and $YL(uf_1)^{-1} \in Qf_2$ for X, Y in Qf_3. By Proposition 38, the homotopy m is a monomorphism in **Qtp**. Thus (45) presents the quasigroup $(Qf_3, +)$ as a subobject of Q' in the category **Qtp**.

Definition 41. If Q is nonempty, then the subobject (45) of Q' is known as the *image* of the homotopy (42). Otherwise, the *image* of (42) is defined to be the empty subquasigroup of Q' (or more precisely the image of the embedding **Q**-morphism $\varnothing \hookrightarrow Q'$ under the forgetful functor Σ).

Proposition 42. *Considered as a subobject of* Q' *in the category* **Qtp**, *the image of the homotopy* (42) *for nonempty* Q *is independent of the choice of the elements* u *and* v *of* Q.

Proposition 43. *Let* m *be the image of the homotopy* f *of* (42). *If* Q *is nonempty,* m *being chosen as in* (45), *there is a homotopy*

$$e = (f_1 R_{Q'}(vf_2), f_2 L_{Q'}(uf_1), f_3) : (Q, \cdot) \to (Qf_3, +) \tag{46}$$

such that $em = f$. *If* f *has empty domain, take* e *to be the identity isotopy on the empty quasigroup, so again* $em = f$.

Proof. For x, y in Q, one has

$$xf_1 R(vf_2) + yf_2 L(uf_1) = xf_1 \cdot yf_2 = (x \cdot y) f_3,$$

showing that e is a homotopy. Then

$$\begin{aligned} em &= (f_1 R(vf_2), f_2 L(uf_1), f_3)(R(vf_2)^{-1} j_1, (uf_1)^{-1} j_2, j_3) \\ &= (f_1 j_1, f_2 j_2, f_3 j_3) = (f_1, f_2, f_3) = f, \end{aligned}$$

as required. □

Proposition 44. *In the category* **Qtp**, *the homotopy* e *of Proposition* 43 *is a regular epimorphism.*

Proof. Consider the kernel congruence

$$\ker f\Delta = \{((x_1,x_2,x_3),(x_1',x_2',x_3')) \in Q\Delta^2 \mid \forall\ 1 \le i \le 3,\ x_i f_i = x_i' f_i\}$$

of the **P**-morphism $f\Delta : Q\Delta \to Q'\Delta$. For $1 \le i \le 3$, define projections

$$\pi_i : \ker f\Delta \to Q; ((x_1,x_2,x_3),(x_1',x_2',x_3')) \mapsto x_i$$

and

$$\pi_i' : \ker f\Delta \to Q; ((x_1,x_2,x_3),(x_1',x_2',x_3')) \mapsto x_i'.$$

Then

$$\pi = (\pi_1,\pi_2,\pi_3) : \ker f\Delta \to Q$$

and

$$\pi' = (\pi_1',\pi_2',\pi_3') : \ker f\Delta \to Q$$

are homotopies (compare the verification in [16] that the counit (8) of the adjunction between Σ and Δ is a homotopy). The homotopy $e = 1_\varnothing$ (for $Q = \varnothing$) or

$$e = (f_1 R(vf_2), f_2 L(uf_1), f_3) : (Q,\cdot) \to (Qf_3,+)$$

(otherwise) is the coequalizer in **Qtp** of the pair (π,π'). In particular, note that

$$xf_1R(vf_2) = x'f_1R(vf_2) \Leftrightarrow xf_1 = x'f_1$$

and

$$xf_2L(uf_1) = x'f_2L(uf_1) \Leftrightarrow xf_2 = x'f_2$$

for x, x' in Q. □

Theorem 45. *Each quasigroup homotopy*

$$f = (f_1,f_2,f_3) : (Q,\cdot) \to (Q',\cdot)$$

factorizes as the product $f = em$ of a regular epimorphism

$$e : (Q,\cdot) \to (Qf_3,+)$$

to its image and a monomorphism

$$m : (Qf_3,+) \to (Q',\cdot)$$

in the category **Qtp** *of quasigroup homotopies. If Q is nonempty, then the image quasigroup $(Qf_3,+)$ is a loop.*

Remark 46. Application of Theorem 45 to the identity homotopy on a nonempty quasigroup Q gives a "natural" confirmation of the well-known fact that each such quasigroup is isotopic to a loop (compare Remark 12).

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