Product Identities for Theta Functions

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Products of Theta Functions

Defining

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1,$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \qquad |q| < 1.$$

Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty, \qquad |q| < 1.$$

Ramanujan's general theta function

For $|ab| < 1$,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

$$\varphi(q) := f(q;q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \frac{1}{2} f(1,q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) := f(-q,-q^2) = (q;q)_\infty.$$

Ramanujan

For $k \in \mathbb{Z}^+$,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = \sum_{r=0}^{k-1} \sum_{n=-\infty}^{\infty} a^{(kn+r)(kn+r+1)/2} b^{(kn+r)(kn+r-1)/2}.$$

Define

$$U_k = a^{k(k+1)/2} b^{k(k-1)/2}, V_k = a^{k(k-1)/2} b^{k(k+1)/2}.$$

$$f(a,b) = f(U_1,V_1) = \sum_{r=0}^{k-1} U_r f(\frac{U_{k+r}}{U_r}, \frac{V_{k-r}}{U_r}).$$

Products of Two or more theta functions

Problem: Conditions under which the product of two or more theta functions can be written as linear combinations of other products of theta functions.

Motivation

M.D.Hirschhorn's generalization of Winquist's identity, 1987
Ramanujan, 1919

$$p(11n+6) \equiv 0 \pmod{11},$$

where $p(n)$ is the number of partitions of the positive integer n.

Theorem 1 (Winquist, 1969)**.** *For any nonzero complex numbers a, b and for $|q| < 1$,*

$$\begin{aligned}&\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{m+n}q^{\frac{3m^2+3n^2+3m+n}{2}}\\ &\quad\times(a^{-3m}b^{-3n}-a^{-3m}b^{3n+1}-a^{-3n+1}b^{-3m-1}+a^{3n+2}b^{-3m-1})\\ =&(q;q)_\infty^2(a;q)_\infty(a^{-1}q;q)_\infty(b;q)_\infty(b^{-1}q;q)_\infty\\ &\quad\times(ab;q)_\infty(a^{-1}b^{-1}q;q)_\infty(ab^{-1};q)_\infty(a^{-1}bq;q)_\infty.\end{aligned}$$

Main idea

Let $z_i \neq 0$, $l_i \in \mathbb{Z}^+$, $h_i \in \mathbb{Z}$, $(i = 1, 2, \ldots, n)$.

$$S := \prod_{i=1}^{n} (-z_i q^{l_i - h_i}; q^{l_i})_\infty (-z_i{}^{-1} q^{h_i}; q^{l_i})_\infty (q^{l_i}; q^{l_i})_\infty$$
$$= \sum_{x_1=-\infty}^{\infty} \sum_{x_2=-\infty}^{\infty} \cdots \sum_{x_n=-\infty}^{\infty} z_1{}^{x_1} z_2{}^{x_2} \cdots z_n{}^{x_n} q^{\sum_{j=1}^{n} (\frac{1}{2} l_j - h_j) x_j + \frac{1}{2} l_j x_j{}^2}.$$

$$y = Ax, \;\; A \text{ is an integral matrix}, \;\; \det A \neq 0.$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

$x = A^{-1} y = \frac{1}{\det A} A^* y$, $k = |\det A|$, $B = \mathrm{sgn}(\det A) A^*$.

$$x = \frac{1}{k} B y.$$

$$By \equiv 0 \pmod{k}.$$

If $y \equiv c_i \pmod{k}$ $(i = 1, 2, \ldots, m)$ is the solution set of the above system of congruences, we substitute y with $ky + c_i$ in $x = \frac{1}{k}By$. Then we have $x = By + \frac{1}{k}Bc_i$ $(i = 1, 2, \ldots, m)$.

A system of congruences $a_i \pmod{n_i}$ with $1 \le i \le k$ is called a **covering system** if every integer y satisfies $y \equiv a_i \pmod{n_i}$ for at least one value of i.

A covering system in which each integer is covered by just one congruence is called an **exact covering system**.

We can take $\{By + \frac{1}{k}Bc_1, \cdots, By + \frac{1}{k}Bc_m\}$ as an exact covering system of $\mathbb{Z}^n$.

We need the coefficients of all $y_i y_j = 0$ $(i \neq j, i, j = 1, 2, \ldots, n)$ in order to separate $y_1, y_2, \ldots, y_n$.

Generalized Orthogonal Relation

$$\left\{\begin{array}{rcrcrl} l_1 b_{11} b_{12} & + & l_2 b_{21} b_{22} & + \cdots + & l_n b_{n1} b_{n2} & = 0, \\ l_1 b_{11} b_{13} & + & l_2 b_{21} b_{23} & + \cdots + & l_n b_{n1} b_{n3} & = 0, \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ l_1 b_{1(n-1)} b_{1n} & + & l_2 b_{2(n-1)} b_{2n} & + \cdots + & l_n b_{n(n-1)} b_{nn} & = 0. \end{array}\right.$$

Procedure for obtaining series-product identities

1. For fixed positive integers $l_1, l_2, \ldots, l_n$, find $n \times n$ matrices B satisfying

- The "generalized orthogonal" relation,
- $\det B = \pm k^{n-1}$,
- kB^{-1} is an integral matrix,

  where $k \in N$.

2. Solve system of congruences $By \equiv 0 \pmod{k}$.
Suppose we have m solutions to it. By computing the contribution of each solution, we can write each product of n theta functions as a linear combination of m products of n theta functions.

Products of Two Theta Functions

The generalized orthogonal relation is

$$l_1 b_{11} b_{12} + l_2 b_{21} b_{22} = 0.$$

Without lose of generality, we can assume three entries of matrix B are positive, one is negative. We also assume $\det B = k$.
Choose

$$B = \begin{pmatrix} k_1 & k_2 \\ -1 & 1 \end{pmatrix},$$

Corollary 1. *If $|ab| < 1, (cd) = (ab)^{k_1 k_2}$, where both k_1 and k_2 are positive integers, then*

$$f(a,b)f(c,d) = \sum_{i=0}^{k_1+k_2-1} a^{\frac{i^2+i}{2}} b^{\frac{i^2-i}{2}} f(a^{\frac{k_1^2+k_1}{2}+k_1 i} b^{\frac{k_1^2-k_1}{2}+k_1 i} d, a^{\frac{k_1^2-k_1}{2}-k_1 i} b^{\frac{k_1^2+k_1}{2}-k_1 i} c)$$
$$\times f(a^{\frac{k_2^2+k_2}{2}+k_2 i} b^{\frac{k_2^2-k_2}{2}+k_2 i} c, a^{\frac{k_2^2-k_2}{2}-k_2 i} b^{\frac{k_2^2+k_2}{2}-k_2 i} d).$$

Proof.

$$B = \begin{pmatrix} k_1 & k_2 \\ -1 & 1 \end{pmatrix},$$

$\det B = k = k_1 + k_2$.

$$y_1 - y_2 \equiv 0 \pmod{k}.$$

k cases: $y_1 \equiv y_2 \equiv 0 \pmod{k}, \ldots, y_1 \equiv y_2 \equiv k-1 \pmod{k}$.
For $y_1 \equiv y_2 \equiv i \pmod{k}$, replace y_1 by $ky_1 + i$ and y_2 by $ky_2 + i$ in $x = \frac{1}{k}By$.
We find that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} i \\ 0 \end{pmatrix}.$$

Summing up the k parts in the sum S, we obtain Corollary 1. □

Special case
For $ab = cd$,

$$f(a,b)f(c,d) = f(ad,bc)f(ac,bd) + af(\frac{c}{a},\frac{a}{c}abcd)f(\frac{d}{a},\frac{a}{d}abcd).$$

Gasper, "three term relation for sigma functions" .

Corollary 2.

$$\begin{aligned} &b(ad,q/ad,bc,q/bc,d/a,qa/d,c/b,qb/c;q)_\infty \\ +&c(ab,q/ab,cd,q/cd,b/a,qa/b,d/c,qc/d;q)_\infty \\ +&d(ac,q/ac,bd,q/bd,c/a,qa/c,b/d,qd/b;q)_\infty = 0 \end{aligned}$$

Hirschhorn's generalization of the quintuple product identity
For $cd = (ab)^2$,

$$\begin{aligned} f(a,b)f(c,d) =& f(ac,bd)f(a^3bd,ab^3c) + af(\frac{d}{a},\frac{a}{d}abcd)f(\frac{bc}{a},\frac{a}{bc}a^2b^2c^2d^2) \\ &+ bf(\frac{c}{b},\frac{b}{c}abcd)f(\frac{ad}{b},\frac{b}{ad}a^2b^2c^2d^2). \end{aligned}$$

Göllnitz-Gordon functions

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty},$$

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}.$$

$$f(1,q)f(1,q^7) = \sum_{i=0}^{7} q^{\frac{i^2-i}{2}} f(q^{7+i}, q^{1-i}) f(q^{21+7i}, q^{35-7i}),$$

$$\begin{aligned} 2\psi(q)\psi(q^7) =& f(q,q^7)f(q^{21},q^{35}) + q^3 f(q^3,q^5) f(q^{49},q^7) \\ &+ q\psi(q^2)\psi(q^{14}) + \psi(q^8)\varphi(q^{28}) + q^6\varphi(q^4)\psi(q^{56}). \end{aligned}$$

$$f(-1,-q)f(-1,-q^7) = \sum_{i=0}^{m} (-1)^i q^{\frac{i^2-i}{2}} f(q^{m+i}, q^{1-i}) f(q^{\frac{m^2-m}{2}+mi}, q^{\frac{m^2+3m}{2}-mi}) = 0,$$

$$f(q,q^7)f(q^{21},q^{35}) + q^3 f(q^3,q^5) f(q^{49},q^7) = \psi(q^8)\varphi(q^{28}) + q^6\varphi(q^4)\psi(q^{56}) + q\psi(q^2)\psi(q^{14}).$$

$$\psi(q)\psi(q^7) = \psi(q^8)\varphi(q^{28}) + q^6\varphi(q^4)\psi(q^{56}) + q\psi(q^2)\psi(q^{14}),$$

$$\begin{aligned}
\psi(q)\psi(q^7) &= f(q, q^7)f(q^{21}, q^{35}) + q^3 f(q^3, q^5)f(q^{49}, q^7), \\
\psi(q^2)\varphi(-q^3) &= f(-q^3, -q^5)f(-q^9, -q^{15}) + q^2 f(-q, -q^7)f(-q^3, -q^{21}), \\
\psi(q)\varphi(-q^2) &= f^2(-q^3, -q^5) + qf^2(-q, -q^7), \\
\psi(q^6)\varphi(q) &= f(q, q^7)f(q^9, q^{15}) + qf(q^3, q^5)f(q^3, q^{21}).
\end{aligned}$$

Corollary 3 (S.-S. Huang, 2002)**.**

$$\begin{aligned}
S(q^7)T(q) - q^3 S(q)T(q^7) &= 1, \\
S(q^3)S(q) + q^2 T(q^3)T(q) &= \frac{(q^3;q^3)_\infty (q^4;q^4)_\infty}{(q;q)_\infty (q^{12};q^{12})_\infty}, \\
S^2(q) + qT^2(q) &= \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^3 (q^4;q^4)_\infty^3}, \\
S(q^3)T(q) - qS(q)T(q^3) &= \frac{(q;q)_\infty (q^{12};q^{12})_\infty}{(q^3;q^3)_\infty (q^4;q^4)_\infty}.
\end{aligned}$$

Definition 1. *Let $P(q)$ denote any power series in q. Then the t-dissection of P is given by*

$$P(q) =: \sum_{k=0}^{t-1} q^k P_k(q^t).$$

$2m$-dissection of $\varphi(q)\varphi(q^{m^2})$

$$\varphi(q)\varphi(q^{m^2}) = \sum_{i=0}^{2m-1} q^{i^2} f^2(q^{2m^2+2mi}, q^{2m^2-2mi}).$$

m-dissection of $\psi(q)\varphi(2q^{m^2})$

$$\psi(q)\varphi(q^{2m^2}) = \sum_{i=0}^{2m-1} q^{2i^2-i} f^2(q^{4m^2-m+4mi}, q^{4m^2+m-4mi}).$$

4-dissection of $\psi^2(-q^2)f^2(-q)$

$$\psi^2(-q^2)f^2(-q) = \varphi^2(-q^8)f(-q^8) - 2q\psi(q^8)f^2(-q^4).$$

$$B = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Corollary 4. *For $ab = cd$,*

$$\begin{aligned} f(a,b)f(c,d) =& f(a^2bc^2, ab^2d^2)f(ab^3c, a^3bd) + af(a^3b^2c^2, bd^2)f(bc/a, a^5b^3d) \\ &+ cf(a^4b^3c^2, d^2/a)f(b^2c^3d^2, a^2d) \\ &+ acf(a^5b^4c^2, b/c^2)f(b^2c, a^4b^2d) \\ &+ adf(ac^2, a^2b^3d^2)f(c/a^2, a^6b^4d). \end{aligned}$$

Special case The septuple product identity

$$\begin{aligned} &(z;q^2)_\infty(z^{-1}q^2;q^2)_\infty(z^2;q^2)_\infty(z^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty^2 \\ =&(q^4;q^{10})_\infty(q^6;q^{10})_\infty(q^{10};q^{10})_\infty^2\{z^3(z^5q^8;q^{10})_\infty(z^{-5}q^2;q^{10})_\infty \\ &+(z^5q^2;q^{10})_\infty(z^{-5}q^8;q^{10})_\infty\} - (q^2;q^{10})_\infty(q^8;q^{10})_\infty(q^{10};q^{10})_\infty^2 \\ &\times\{z(z^5q^4;q^{10})_\infty(z^{-5}q^6;q^{10})_\infty + z^2(z^5q^6;q^{10})_\infty(z^{-5}q^4;q^{10})_\infty\}. \end{aligned}$$

Products of Three or More Theta Functions

$$B = 6\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

- The set of the 3 columns of B is an orthogonal set.
- $\det B = 1296 = 36^{3-1}$.
- $36 \cdot B^{-1}$ is an integer matrix.

$k = 36$.

$$\begin{cases} y_1 + y_2 + y_3 \equiv 0 \pmod 6, \\ 2y_1 - y_2 \equiv 0 \pmod 6, \\ y_1 + y_2 - y_3 \equiv 0 \pmod 6. \end{cases}$$

$$
\begin{aligned}
S :=& (z_1;q)_\infty ({z_1}^{-1}q;q)_\infty (z_2;q)_\infty ({z_2}^{-1}q;q)_\infty \\
&\times (z_3;q)_\infty ({z_3}^{-1}q;q)_\infty (q;q)_\infty^3 \\
=& (-z_1{z_2}^2z_3q;q^6)_\infty (-{z_1}^{-1}{z_2}^{-2}{z_3}^{-1}q^5;q^6)_\infty (q^6,q^6)_\infty \\
&\times (z_1{z_2}^{-1}z_3q;q^3)_\infty ({z_1}^{-1}z_2{z_3}^{-1}q^2;q^3)_\infty (q^3,q^3)_\infty \\
&\times (-z_1{z_3}^{-1}q;q^2)_\infty (-{z_1}^{-1}z_3q;q^2)_\infty (q^2,q^2)_\infty \\
&+\cdots
\end{aligned}
$$

$$
B = 6\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.
$$

$1^2+2^2+1^2=6$ $\quad$ $1^2+(-1)^2+1^2=3$ $\quad$ $1^2+0^2+(-1)^2=2$

$z_1{z_2}^2z_3$ $\quad$ $z_1{z_2}^{-1}z_3$ $\quad$ $z_1{z_3}^{-1}$

Corollary 5. *For $a, b \neq 0$, and $|q| < 1$,*

$$\begin{aligned}
&(a;q)_\infty(a^{-1}q;q)_\infty(b;q)_\infty(b^{-1}q;q)_\infty(ab;q)_\infty(a^{-1}b^{-1}q;q)_\infty(q;q)_\infty^2 \\
=&(-ab^{-1}q;q^2)_\infty(-a^{-1}bq;q^2)_\infty(q^2,q^2)_\infty \\
&\times \{(-a^3b^3q;q^6)_\infty(-a^{-3}b^{-3}q^5;q^6)_\infty(q^6,q^6)_\infty \\
&- a^2b^2(-a^3b^3q^5;q^6)_\infty(-a^{-3}b^{-3}q;q^6)_\infty(q^6,q^6)_\infty\} \\
&+ (-ab^{-1}q^2;q^2)_\infty(-a^{-1}b;q^2)_\infty(q^2,q^2)_\infty \\
&\times \{a^2b(-a^3b^3q^4;q^6)_\infty(-a^{-3}b^{-3}q^2;q^6)_\infty(q^6,q^6)_\infty \\
&- a(-a^3b^3q^2;q^6)_\infty(-a^{-3}b^{-3}q^4;q^6)_\infty(q^6,q^6)_\infty\}.
\end{aligned}$$

Corollary 6.

$$\begin{aligned}
16(q;q)_\infty^8 = \sum_{m,n=-\infty}^{\infty} &\{(2n-6m+1)(2n+6m-2)^2q^{-2m} \\
&- (2n-6m-3)(2n+6m+2)^2q^{2m} \\
&+ (2n-6m-1)(2n+6m+2)^2q^{n+m} \\
&- (2n-6m+1)(2n+6m)^2q^{n-m}\}q^{n^2+3m^2}.
\end{aligned}$$

$l_1 = l_2 = 1,\ l_3 = 2$

$$B = 4\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix},$$

Corollary 7. *For* $a_1b_1 = a_2b_2 = q,\ a_3b_3 = q^2,\ |q| < 1,$

$$\begin{aligned} & f(a_1, b_1)f(a_2, b_2)f(a_3, b_3) \\ =& f(a_1a_2a_3, b_1b_2b_3)f(a_1b_2, b_1a_2)f(a_1a_2b_3, b_1b_2a_3) \\ & + a_1f(a_1a_2a_3q, b_1b_2b_3/q)f(a_1b_2q, b_1a_2/q)f(a_1a_2b_3q, b_1b_2a_3/q) \\ & + a_1a_2f(a_1a_2a_3q^2, b_1b_2b_3/q^2)f(a_1b_2, b_1a_2)f(a_1a_2b_3q^2, b_1b_2a_3/q^2) \\ & + a_1a_3f(a_1a_2a_3q^3, b_1b_2b_3/q^3)f(a_1b_2q, b_1a_2/q)f(a_1a_2b_3/q, b_1b_2a_3q). \end{aligned}$$

$l_1 = l_2 = l_3 = l_4 = 1$

$$B = 8 \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

For $q = a_1b_1 = a_2b_2 = a_3b_3 = a_4b_4$, $|q| < 1$,

$$f(a_1, b_1)f(a_2, b_2)f(a_3, b_3)f(a_4, b_4)$$

is a linear combination of eight products of four theta functions. Corollary 7 is a special case of the identity generated by this matrix.

Generalized Schröter Formula

$T(x,q)=\sum_{n=-\infty}^{\infty} x^n q^{n^2}$, where $x\neq 0$, $|q|<1$.

Theorem 2 (Schröter's Formula)**.** *For positive integers a, b,*

$$T(x,q^a)T(y,q^b)=\sum_{n=0}^{a+b-1} y^n q^{bn^2} T(xyq^{2bn};q^{a+b})T(x^{-b}y^a q^{2abn},q^{ab^2+a^2b}).$$

Theorem 3. *For positive integers a,b,k_i $(i=1,2)$, $k_2|k_1b$ and $k_2|a$,*

$$T(x,q^a)T(y,q^b)=\sum_{n=0}^{a/k_2+k_1^2b/k_2-1} y^n q^{bn^2} T(xy^{k_1}q^{2k_1bn};q^{a+bk_1{}^2})$$
$$\times T(x^{-k_1b/k_2}y^{a/k_2}q^{2abn/k_2},q^{ab^2k_1{}^2/k_2{}^2+a^2b/k_2{}^2}).$$

Special case: W. Chu and Q. Yan, 2007

Corollary 8. *Let* α, β, γ *be three natural integers with* $gcd(\alpha, \gamma) = 1$ *and* $\lambda = 1 + \alpha\beta^2\gamma$. *For two indeterminate* x *and* y *with* $x \neq 0$ *and* $y \neq 0$, *there holds the algebraic identity*

$$\langle x; q^\alpha \rangle_\infty \langle x^{\beta\gamma} y; q^\gamma \rangle_\infty = \sum_{l=0}^{\alpha\beta^2\gamma} (-1)^l q^{\binom{l}{2}\alpha} x^l \langle (-1)^{\alpha\beta} x^\lambda y^{\alpha\beta} q^{\binom{\alpha\beta}{2}\gamma + l\alpha}; q^{\lambda\alpha} \rangle \times \langle (-1)^{\beta\gamma} y q^{\binom{\beta\gamma+1}{2}\gamma - l\alpha\beta\gamma}; q^{\lambda\gamma} \rangle,$$

where $\langle x; q \rangle_\infty = (q; q)_\infty (x; q)_\infty (q/x; q)_\infty$.

Corollary 9. *For positive integer $a, b,$and k, with $k|a$,*

$$T(x,q^a)T(y,q^b) = \sum_{n=0}^{\frac{a}{k}+bk-1} y^n q^{bn^2} T(xy^k q^{2bkn}, q^{a+bk^2}) T(x^{-b} y^{\frac{a}{k}} q^{\frac{2ab}{k}n}, q^{ab^2+\frac{a^2b}{k^2}}).$$

Corollary 10 (**Blecksmith-Brillhart-Gerst Theorem, 1988**)**.** *Define $f_0(a,b) = f(a,b)$ and $f_1(a,b) = f(-a,-b)$. Let a, b, c and d denote complex numbers such that $|ab|$, $|cd| < 1$, and suppose that there exist positive integer α, β, and m such that*

$$(ab)^\beta = (cd)^{\alpha(m-\alpha\beta)}$$

Let $\varepsilon_1, \varepsilon_2 \in \{0,1\}$. Then

$$\begin{aligned} &f_{\varepsilon_1}(a,b) f_{\varepsilon_2}(c,d) \\ &= \sum_{r\in R} (-1)^{\varepsilon_2 r} c^{r(r+1)/2} d^{r(r-1)/2} f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2n)/2}}{c^\alpha}, \frac{b(cd)^{\alpha(\alpha+1+2n)/2}}{d^\alpha}\right) \\ &\quad \times f_{\delta_2}\left(\frac{(a/b)^{\beta/2}(cd)^{(m-\alpha\beta)(m+1+2n)/2}}{d^{m-\alpha\beta}}, \frac{(b/a)^{\beta/2}(cd)^{(m-\alpha\beta)(m+1-2n)/2}}{c^{m-\alpha\beta}}\right), \end{aligned}$$

where R is a complete residue system $\pmod m$,

$$\delta_1 = \begin{cases} 0, & \textit{if } \varepsilon_1 + \alpha\varepsilon_2 \textit{ is even}, \\ 1, & \textit{if } \varepsilon_1 + \alpha\varepsilon_2 \textit{ is odd}, \end{cases}$$

and

$$\delta_2 = \begin{cases} 0, & \textit{if } \varepsilon_1\beta + \varepsilon_2(m - \alpha\beta) \textit{ is even}, \\ 1, & \textit{if } \varepsilon_1\beta + \varepsilon_2(m - \alpha\beta) \textit{ is odd}. \end{cases}$$