POSET LOOPS

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Abstract. Given a ring and a locally finite poset, an *incidence loop* or *poset loop* is obtained from a new and natural extended convolution product on the set of functions mapping intervals of the poset to elements of the ring. The paper investigates the interplay between properties of the ring, the poset, and the loop. The annihilation structure of the ring and extremal elements of the poset determine commutative and associative properties for loop elements. Nilpotence of the ring and height restrictions on the poset force the loop to become associative, or even commutative. Constraints on the appearance of nilpotent groups of class 2 as poset loops are given. The main result shows that the incidence loop of a poset of finite height is nilpotent, of nilpotence class bounded in terms of the height of the poset.

1. Introduction

Given a commutative, unital ring S, the standard incidence algebra of a locally finite poset P is an S-algebra structure built on the set of functions mapping intervals in P to elements of S. This incidence algebra may be interpreted as an algebra of upper-triangular matrices, under the usual addition and multiplication of matrices. In particular, the full set of all upper-triangular matrices (of given degree d) corresponds to the incidence algebra of a (d-element) chain.

Within the incidence algebras of chains, the sets of upper-triangular matrices taking the value 1 on the diagonal (*unipotent* matrices) form groups under matrix multiplication (compare [2]). However, the representation theory of these groups is wild. It was tamed by a rather drastic fusion of conjugacy classes, leading to so-called *supercharacters* and *superclasses* [1, 6]. In some recent work, a less drastic, intermediate fusion has received attention, based on a modification of matrix multiplication leading to a generally non-associative loop product within the incidence algebra of a chain [8].

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The present paper initiates a general study, for arbitrary locally finite posets, of loops analogous to those based on chains with the modified matrix multiplication of [8]. Thus there are three ingredients:

(1) An arbitrary (not necessarily commutative or unital, but certainly associative) ring S;
(2) A locally finite poset P;
(3) A loop I_S^P, the so-called *incidence loop* of the poset P over the ring S.

The focus of the study falls on the interplay between the respective properties of the ring S, the poset P, and the loop I_S^P. One example of this interplay is worth singling out here, since it is used repeatedly within proofs: the duality between products $x \cdot y$ in rings and loops along with relations $u < v$ in posets on the one hand, and reversed products $y \cdot x$ along with converse relations $u > v$ on the other.

Background on loops (and quasigroups) is presented in Section 2. Incidence loops (or *poset loops*) are then introduced in Section 3, in particular the key definition (3.3) of the *extended convolution product* on I_S^P, the set of functions mapping ordered pairs (x, y) with $x < y$ in P to elements of the ring S. Theorem 3.3 shows that the extended convolution product induces a loop structure on I_S^P. Section 4 relates the annihilation structure of the ring S and extremal elements of the poset P with commutative and associative properties of individual elements of the loop I_S^P. Theorem 4.11 shows that commutants of poset loops are subloops (compare [11]). In Section 5, ring homomorphisms are correlated with loop homomorphisms, disjoint unions (or "parallel compositions") of posets are correlated with products of loops, and poset extensions are correlated with subloops. The respective Sections 6 and 7 investigate how nilpotence of the ring S and height restrictions on P correspond to special cases where the loop I_S^P is an abelian or non-abelian group. Section 8 initiates investigation of nilpotent groups of class 2 appearing as incidence loops of posets of height 3.

The main result of the paper is Theorem 9.6, showing that the incidence loop I_S^P of a poset of height at most $c + 1$ is nilpotent, of class bounded above by c. Since loops of nilpotence class at most c form a variety, the theorem may be used to obtain identities satisfied by the incidence loops of posets of height at most $c+1$. In a slightly different direction, the concluding Problem 10.1 seeks varieties $\mathbf{V}_n$ of loops (for each positive integer n) such that, whenever $|S^n| > 1$ in a ring S, the incidence loop I_S^P of a locally finite poset P lies in $\mathbf{V}_n$ if and only if the height of P is at most n. The paper covers the initial cases where $n < 4$.

2. Quasigroups and loops

Quasigroups may be defined combinatorially or equationally. Combinatorially, a *quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. A *loop* is a quasigroup Q with an *identity* element 1 such that $1 \cdot x = x = x \cdot 1$ for all x in Q.

Equationally, a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{llll}(\mathrm{SL}) & x \cdot (x\backslash z) = z\,; & (\mathrm{SR}) & z = (z/x) \cdot x\,; \\ (\mathrm{IL}) & x\backslash(x \cdot z) = z\,; & (\mathrm{IR}) & z = (z \cdot x)/x\,.\end{array}$$

For each element x of a quasigroup Q, consider the *right multiplication*

$$R(x) \colon Q \to Q; y \mapsto y \cdot x$$

and *left multiplication*

$$L(x) \colon Q \to Q; y \mapsto x \cdot y\,.$$

The right and left multiplications are elements of the group $Q!$ of bijections from the set Q to itself. For example, the identity (SL) says that each $L(x)$ surjects, while (IL) gives the injectivity of $L(x)$. The *multiplication group* $\mathrm{Mlt}\, Q$ of a quasigroup Q is the subgroup of $Q!$ generated by $\{R(q), L(q) \mid q \in Q\}$.

Now consider a loop $(Q, \cdot, 1)$. The *inner multiplication group* $\mathrm{Inn}\, Q$ is the stabilizer $\mathrm{Mlt}\, Q_1$ in $\mathrm{Mlt}\, Q$ of the identity element 1. For example, if Q is a group, then $\mathrm{Inn}\, Q$ is the inner automorphism group of Q. (In a general loop Q, elements of $\mathrm{Inn}\, Q$ are not necessarily automorphisms of Q.) For elements q, r of Q, define the *conjugation*

$$T(q) = R(q)L(q)^{-1}\,, \tag{2.1}$$

the *right inner mapping*

$$R(q, r) = R(q)R(r)R(qr)^{-1}\,, \tag{2.2}$$

and the *left inner mapping*

$$L(q, r) = L(q)L(r)L(rq)^{-1} \tag{2.3}$$

in $\mathrm{Inn}\, Q$. Collectively, (2.1)–(2.3) are known as *inner mappings*. Note that $\mathrm{Inn}\, Q$ is generated by the subset

$$\{T(q), R(q, r), L(q, r) \mid q, r \in Q\}$$

of $\operatorname{Mlt} Q$ [5, Lemma IV.1.2], [13, §2.8]. A subloop P of Q is *normal* if $P \operatorname{Inn} Q \subseteq P$. This condition means that there is a loop homomorphism

$$Q \to Q/P; x \mapsto Px \tag{2.4}$$

from Q to the *quotient loop* $Q/P = \{Px \mid x \in Q\}$ with loop structure given precisely by declaring that (2.4) is a homomorphism (compare [5, §IV.1], [14, §I.2.4]).

3. Poset loops

Recall that a poset $(P, \leq)$ is *locally finite* if the *interval*

$$[x, y] = \{z \in P \mid x \leq z \leq y\}$$

is finite for each pair $x < y$ of elements of P.

Definition 3.1. Let S be a ring, not necessarily commutative or unital, but certainly associative. Let $(P, \leq)$ be a locally finite poset. For $x < y$ in P, let $\Pi(x, y)$ denote the set of (finite) paths

$$x = x_0 < x_1 < x_2 < \cdots < x_{r-1} < x_r = y \tag{3.1}$$

from x to y in P. Write $l(x, y)$ for the length r of the longest path (3.1) from x to y.

(a) The *restricted incidence algebra* I_S^P is the set of functions from

$$<^P = \{(x, y) \in P \times P \mid x < y\} \tag{3.2}$$

to S, equipped with the abelian group structure induced pointwise from S.

(b) If $p = (x_0 < x_1 < x_2 < \cdots < x_{r-1} < x_r)$ is a path in the poset $(P, \leq)$, and $f \in I_S^P$, then the *path extension* f_p or $f_{x_0, x_1, x_2, \ldots, x_{r-1}, x_r}$ of f is the product

$$f(x_0, x_1) f(x_1, x_2) \ldots f(x_{r-1}, x_r)$$

in S.

(c) The *extended convolution product* $\cdot$ on I_S^P is defined by

$$(f \cdot g)(x, y) = f_{x,y} + g_{x,y} + \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} f_p g_q \tag{3.3}$$

for $x < y$ in P.

(d) The *trivial function* e is the zero element of the abelian group I_S^P.

Remark 3.2. In Definition 3.1(c), the local finiteness of P guarantees that the sum in (3.3) is finite.

Theorem 3.3. *Let S be a ring, and let $(P, \leq)$ be a locally finite poset. Then the restricted incidence algebra I_S^P forms a loop $(I_S^P, \cdot, e)$ under the extended convolution product, with the trivial function as identity element.*

Proof. For f in I_S^P, the equations $e \cdot f = f = f \cdot e$ are immediate from (3.3). Now consider solving $h \cdot f = g$ for h in terms of given f, g in I_S^P. Consider $x < y$ in P. It will be shown by induction on the maximum length $l(x, y)$ of a path (3.1) from x to y that the function value $h(x, y)$ is uniquely determined. If $l(x, y) = 1$, so that y covers x, then $g_{x,y} = h_{x,y} + f_{x,y}$ by (3.3), and $h_{x,y}$ is uniquely determined as $g_{x,y} - f_{x,y}$. Now consider a general pair $x < y$ in P. By (3.3), $g = h \cdot f$ implies

$$g_{x,y} = h_{x,y} + f_{x,y} + \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} h_p f_q \,,$$

so

$$h_{x,y} = g_{x,y} - f_{x,y} - \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} h_p f_q \,. \tag{3.4}$$

For $x < z < y$, one has $l(x, z) < l(x, y)$, If

$$x = x_0 < x_1 < \cdots < x_r = z$$

is a path p from x to z, one has $l(x_{i-1}, x_i) \leq l(x, z) < l(x, y)$ for $i = 1, \ldots, r$. By the induction hypothesis, it follows that all the terms $h_p = h_{x_0,x_1} \ldots h_{x_{r-1},r}$ on the right hand side of (3.4) have already been determined uniquely. Hence $h_{x,y}$ itself is determined uniquely. In dual fashion, it may be shown that there is a unique solution h to the equation $f \cdot h = g$ in terms of given f, g in I_S^P. Thus $(I_S^P, \cdot, e)$ is a loop. □

Definition 3.4. In the context of Theorem 3.3, the loop I_S^P is known as the *incidence loop* of the locally finite poset P (over the ring S). More generically, loops I_S^P for locally finite posets P and rings S are known as *poset loops*.

For many purposes, the specific ring S is irrelevant. Thus throughout the paper, the symbol S will denote an arbitrary ring (not necessarily unital), unless more explicit specification is provided (as, for example, in the hypothesis of Theorem 6.8).

4. Centrality

For the following, compare [7, p.53]. Recall that the rings under consideration in this paper are not necessarily unital.

Definition 4.1. Let S be a ring.

(a) The subset

$$\{z \in S \mid \forall\ s \in S\,,\ zs = 0\}$$

is defined as the *left annihilator* $ZL(S)$ of S.

(b) The subset

$$\{z \in S \mid \forall\ s \in S\,,\ sz = 0\}$$

is defined as the *right annihilator* $ZR(S)$ of S.

(c) The ring S is said to be *centerless* if $ZL(S) = ZR(S) = \{0\}$.

Lemma 4.2. *Let S be a ring which satisfies at least one of the following conditions:*

(a) *S is unital;*
(b) *S has no divisors of zero;*
(c) *S has no nilpotent elements.*

Then S is centerless.

Proof. Consider $z \in ZL(S)$. Under each of the conditions, it transpires that $z = 0$:

(a) Since $z \in ZL(S)$, one has $z = z \cdot 1 = 0$.

(b) Since $z \in ZL(S)$, one has $z \cdot z = 0$, so $z = 0$.

(c) Since $z \in ZL(S)$, one has $z^2 = z \cdot z = 0$, so z is nilpotent. Thus $z = 0$.

The proof that $ZR(S) = \{0\}$ is dual. □

Definition 4.3. Let L be a loop.

(a) The subset

$$\{z \in L \mid \forall\ x \in L\,,\ zx = xz\}$$

is defined as the *commutant* $C(L)$ of L (compare [10, 11]).

(b) The set

$$\{z \in C(L) \mid \forall\ x, y \in L\,,\ z(xy) = (zx)y \text{ and } (yx)z = y(xz)\}$$

of elements of L fixed by the inner mappings (2.1) – (2.3) is defined as the *center* $Z(L)$ of L [4], [5, p. 57], [13, (3.31)].

Remark 4.4. For a Mal'tsev algebra A, the *center congruence* $\zeta(A)$ is defined as the largest subalgebra of A^2 containing the diagonal $\widehat{A} = \{(a, a) \mid a \in A\}$ as a normal subalgebra [12].

(a) For a commutative ring S, the coinciding annihilators $ZL(S) = ZR(S)$ form the congruence class $0^{\zeta(S)}$. Indeed for any ring S, one has $ZL(S) \cap ZR(S) = 0^{\zeta(S)}$ (compare [9, Ex. 2.III, p. 25]).

(b) The center of a loop L with identity element e is the congruence class $e^{\zeta(L)}$. In particular, the center of L is always a normal subloop of L. In general, it is not known when the commutant of a loop is a normal subloop.

Definition 4.5. Let $(P, \leq)$ be a locally finite poset.

(a) An element u of P is *minimal* if $x \leq u \in P$ implies $x = u$.
(b) An element v of P is *maximal* if $v \leq y \in P$ implies $v = y$.

The choice of terminology in the following definition is motivated by Remark 4.4(a).

Definition 4.6. Let S be a ring, and let $(P, \leq)$ be a locally finite poset.

(a) Define $ZR_S^P =$

$$\{f \in I_S^P \mid \forall\ u < v \in P\,,\ u \text{ not minimal } \Rightarrow f(u,v) \in ZR(S)\}\,,$$

the *right central subset* of the restricted incidence algebra I_S^P.

(b) Dually, define $ZL_S^P =$

$$\{f \in I_S^P \mid \forall\ u < v \in P\,,\ v \text{ not maximal } \Rightarrow f(u,v) \in ZL(S)\}\,,$$

the *left central subset* of the restricted incidence algebra I_S^P.

(c) The subset $Z_S^P = ZR_S^P \cap ZL_S^P$ is described as the *central subset* of the restricted incidence algebra I_S^P.

Proposition 4.7. *Let S be a ring, and let $(P, \leq)$ be a locally finite poset. Then $Z_S^P \subseteq C(I_S^P, \cdot, e)$.*

Proof. Consider an element d of Z_S^P, and an element f of I_S^P. Then for $x < y$ in P, one has

$$\begin{aligned}(f \cdot d)(x,y) &= f_{x,y} + d_{x,y} + \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} f_p d_q \\ &= f_{x,y} + d_{x,y}\end{aligned}$$

because, for a path $q = (z < \ldots < y)$ in $\Pi(z,y)$, with initial step $z < z'$, one has $d(z,z') \in ZR(S)$ since $d \in ZR_S^P$ and z is not minimal in P. Dually, one has

$$\begin{aligned}(d \cdot f)(x,y) &= d_{x,y} + f_{x,y} + \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} d_p f_q \\ &= f_{x,y} + d_{x,y}\end{aligned}$$

because, for a path $p = (x < \ldots < z)$ in $\Pi(x,z)$, with final step $z'' < z$, one has $d(z'',z) \in ZL(S)$ since $d \in ZL_S^P$ and z is not maximal in P. $\square$

Theorem 4.8. *Let S be a ring, and let $(P, \leq)$ be a locally finite poset. Then $Z_S^P = C(I_S^P, \cdot, e)$.*

Proof. Given Proposition 4.7, it suffices to establish the containment $C(I_S^P, \cdot, e) \subseteq Z_S^P$. Consider an element f of the commutant $C(I_S^P, \cdot, e)$. Then it will be shown, by induction on the length $l(u', v')$ of a maximal path from u' to v', that $f(u', v') \in ZR(S)$ for any $u' < v' \in P$ where u' is not minimal. The dual proof that f lies in the left central subset ZL_S^P is omitted.

For notational purposes, the induction proof will work throughout with an element u of P that is not minimal. It will be shown that $f(u, v) \in ZR(S)$ for $u < v \in P$. Since u is not minimal, and P is locally finite, there is an element x of P that is covered by u. Consider an arbitrary element s of S. Take an element d of I_S^P with $d(x, u) = s$, and all other values on the relation set $<^P$ being 0. Then

$$(f \cdot d)(x, v) = f_{x,v} + d_{x,v} + \sum_{x<z<v} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,v)} f_p d_q = f_{x,v} \tag{4.1}$$

since $d_{x,v} = 0$, and the step $x < u$ cannot appear in a path from z to v with $x < z$.

Induction basis: v covers u. Here

$$\begin{aligned}(d \cdot f)(x, v) &= d_{x,v} + f_{x,v} + \sum_{x<z<v} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,v)} d_p f_q \\ &= f_{x,v} + d(x, u) f(u, v) = f_{x,v} + s f(u, v) \qquad (4.2)\end{aligned}$$

since $d_{x,v} = 0$ as before, and furthermore the only non-zero coefficient d_p is obtained from the one-step path $p = (x < u)$ from x to $z = u$. Since f lies in the commutant $C(I_S^P, \cdot, e)$, the values of $f \cdot d$ and $d \cdot f$ agree on $u < v$. Thus $s f(u, v) = 0$, by comparison between (4.2) and (4.1). Since the element s of S was arbitrary, it follows that $f(u, v) \in ZR(S)$ for this basic case.

Induction step: Suppose that the function value $f(u', v')$ lies in $ZR(S)$ for all $u' < v'$ in P, with u' not minimal, when the length $l(u', v')$ of a maximal path from u' to v' does not exceed some positive integer k. Suppose that $l(u, v) = k + 1$. Then

$$\begin{aligned}(d \cdot f)(x, v) &= d_{x,v} + f_{x,v} + \sum_{x<z<v} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,v)} d_p f_q \\ &= f_{x,v} + d(x, u) f(u, v) = f_{x,v} + s f(u, v) \qquad (4.3)\end{aligned}$$

since $d_{x,v} = 0$ as before, and furthermore the only non-zero coefficient d_p is obtained from the one-step path $p = (x < u)$ from x to $z = u$. Note that for a path q from $z = u$ to v, the one-step case $u < v$ is accounted for in the computation, namely by the term $f(u, v)$. For a longer path $q = (u < u_1 < \ldots\ v)$, one has $d(x, u) f(u, u_1) = 0$, since the induction hypothesis, along with the fact that $l(u,u_1) \leq k$, implies that $f(u,u_1) \in ZR(S)$. Thus $d(x,u)f_q = d(x,u)f(u,u_1)\cdots = 0$. Finally, one has $sf(u,v) = 0$ by comparison between (4.3) and (4.1), so that $f(u,v)$ lies in $ZR(S)$. □

Corollary 4.9. *Let S be a ring, and let $(P,\leq)$ be a locally finite poset. Then $Z(I_S^P,\cdot,e) \subseteq Z_S^P$.*

Proof. It suffices to note that $Z(I_S^P,\cdot,e) \subseteq C(I_S^P,\cdot,e)$. □

For the following, recall that a loop $(L,\cdot,e)$ is *centerless* if its center is trivial: $Z(L) = \{e\}$.

Corollary 4.10. *Let S be a centerless ring. Let $(P,\leq)$ be an infinite, locally finite chain. Then the loop I_S^P is centerless.*

Proof. Since P is an infinite, locally finite chain, it has no minimal or no maximal elements. By the assumption on S, one has $ZR(S) = ZL(S) = \{0\}$. Thus $ZR_S^P = \{e\}$ or $ZL_S^P = \{e\}$, whence $Z_S^P = \{e\}$, and then $Z(I_S^P) = \{e\}$ by Corollary 4.9. □

In general, the commutant of a loop need not be a subloop (compare [11]). The following result confirms that such behavior is not seen in poset loops.

Theorem 4.11. *Let S be a ring, and let $(P,\leq)$ be a locally finite poset. Then the commutant $C(I_S^P,\cdot,e)$ is a subloop of $(I_S^P,\cdot,e)$.*

Proof. To begin, note that $e \in C(I_S^P)$. For the remainder of the proof, consider elements f,g of $C(I_S^P)$, and $u < v$ in P. By Theorem 4.8, $f,g \in Z_S^P$.

If u is not minimal, one has

$$(f\cdot g)(u,v) = f_{u,v} + g_{u,v} + \sum_{u<z<v}\ \sum_{p\in\Pi(u,z)}\ \sum_{q\in\Pi(z,v)} f_p g_q \in ZR(S)$$

since $f_{u,v}, g_{u,v} \in ZR(S)$, and for each path $p = u < u_1 \ldots z$ from u to z with $u < z < v$, one has $f_{u,u_1} \in ZR(S)$. Thus $f\cdot g \in ZR_S^P$. Dually, $f\cdot g \in ZL_S^P$, so $f\cdot g \in ZR_S^P \cap ZL_S^P = Z_S^P = C(I_S^P)$ by Theorem 4.8: the commutant is closed under products.

Now $\big(f\cdot(f\backslash g)\big)(u,v) =$

$$f_{u,v} + (f\backslash g)_{u,v} + \sum_{u<z<v}\ \sum_{p\in\Pi(u,z)}\ \sum_{q\in\Pi(z,v)} f_p (f\backslash g)_q = g_{u,v}$$

implies

$$(f\backslash g)(u,v) = g_{u,v} - f_{u,v} - \sum_{u<z<v}\ \sum_{p\in\Pi(u,z)}\ \sum_{q\in\Pi(z,v)} f_p (f\backslash g)_q\,. \tag{4.4}$$

If u is not minimal, it follows that $(f\backslash g)(u,v) \in ZR(S)$, since for each path $p = (u < u_1 \dots z)$ from u to z with $u < z < v$, one has $f_{u,u_1} \in ZR(S)$. If v is not maximal, it will be now shown by induction on the length $l(u,v)$ of the interval $u < v$ that $(f\backslash g)(u,v) \in ZL(S)$. If v covers u, (4.4) yields

$$(f\backslash g)(u,v) = g_{u,v} - f_{u,v} \in ZL(S)\,.$$

For the induction step, assume that $(f\backslash g)(v',v) \in ZL(S)$ for intervals $v' < v$ of length less than $l(u,v)$. Then (4.4) again exhibits $(f\backslash g)(u,v)$ as a member of $ZL(S)$, since for each path $q = (z \dots v_1 < v)$ from z to v with $u < z < v$, one has $(f\backslash g)_{v_1,v} \in ZL(S)$ by the induction hypothesis. Thus $f\backslash g \in ZR_S^P \cap ZL_S^P = Z_S^P = C(I_S^P)$ by Theorem 4.8: the commutant is closed under left division. The closure under right division is dual. □

5. Structural properties

This section considers a number of relations between structural properties of rings, posets, and loops. First, an algebraic lemma:

Lemma 5.1. *Suppose that $(M,\cdot,/,\backslash,1_M)$ and $(N,\cdot,/,\backslash,1_N)$ are loops. If a function $\Theta\colon M \to N; m \mapsto m^\Theta$ preserves multiplication, then it also preserves the left and right divisions, as well as the multiplicative identities (and thus constitutes a loop homomorphism).*

Proof. Consider $x = y\backslash z$ in M. Then $x \cdot y = z$, so $x^\Theta \cdot y^\Theta = z^\Theta$. Thus $(y\backslash z)^\Theta = x^\Theta = y^\Theta\backslash z^\Theta$. Preservation of right division is similar. Finally, 1_M^Θ is an idempotent element of the loop N, and thus coincides with 1_N. □

Theorem 5.2. *Suppose that P is a locally finite poset. Then a ring homomorphism $\theta\colon S \to T$ induces a loop homomorphism $\Theta\colon I_S^P \to I_T^P$ with*

$$f^\Theta(x,y) = f(x,y)^\theta$$

for $x < y \in P$ and $f \in I_S^P$.

Proof. For $g \in I_S^P$, one has

$$\begin{aligned}
(f \cdot g)^\Theta(x,y) &= \big(f_{x,y} + g_{x,y} + \sum_{x<z<y} \sum_{p\in\Pi(x,z)} \sum_{q\in\Pi(z,y)} f_p g_q\big)^\theta \\
&= (f_{x,y})^\theta + (g_{x,y})^\theta + \sum_{x<z<y} \sum_{p\in\Pi(x,z)} \sum_{q\in\Pi(z,y)} (f_p)^\theta (g_q)^\theta \\
&= f^\Theta_{x,y} + g^\Theta_{x,y} + \sum_{x<z<y} \sum_{p\in\Pi(x,z)} \sum_{q\in\Pi(z,y)} \big(f^\Theta\big)_p \big(g^\Theta\big)_q \\
&= \big(f^\Theta \cdot g^\Theta\big)(x,y) \qquad (5.1)
\end{aligned}$$

for all $x < y \in P$. To justify the penultimate equality in (5.1), note that

$$\begin{aligned}
(f_p)^\theta &= (f_{x_0,x_1} \cdots f_{x_{r-1},x_r})^\theta = \big(f_{x_0,x_1}\big)^\theta \cdots \big(f_{x_{r-1},x_r}\big)^\theta \\
&= f^\Theta_{x_0,x_1} \cdots f^\Theta_{x_{r-1},x_r} = \big(f^\Theta\big)_p
\end{aligned}$$

for a path $p = (x_0 < x_1 \dots x_r)$ in P. Now from (5.1) it is apparent that $(f \cdot g)^\Theta = f^\Theta \cdot g^\Theta$, i.e., $\Theta \colon I_S^P \to I_T^P$ preserves the multiplications of its domain and codomain. By Lemma 5.1, it follows that Θ is a loop homomorphism. □

For the remainder of this section, the ring S is relegated to a purely subsidiary role. Recall that the disjoint union (or "parallel composition") $P_1 + P_2$ of two component posets P_1 and P_2 just consists of disjoint induced copies of P_1 and P_2, with no order relations between elements of the disjoint components [15].

Theorem 5.3. *Suppose that a locally finite poset P is the disjoint union*

$$P = P_1 + P_2$$

of components P_1, P_2. Then the loop I_S^P factorizes as the direct product

$$I_S^P \cong I_S^{P_1} \times I_S^{P_2}$$

of the incidence loops of the components.

Proof. Define a function $\Theta \colon I_S^{P_1} \times I_S^{P_2} \to I_S^P$ by setting

$$(f_1, f_2)^\Theta(x,y) = \begin{cases} f_1(x,y) & \text{if } \{x,y\} \subseteq P_1 \\ f_2(x,y) & \text{if } \{x,y\} \subseteq P_2 \end{cases}$$

for $x < y$ in P and $(f_1, f_2) \in I_S^{P_1} \times I_S^{P_2}$. Note that Θ is well-defined, since $x < y$ in P implies just one of $\{x,y\} \subseteq P_1$ or $\{x,y\} \subseteq P_2$. The restriction function

$$\Phi \colon I_S^P \to I_S^{P_1} \times I_S^{P_2}; f \mapsto \left(f|_{<^{P_1}}, f|_{<^{P_2}}\right),$$

defined using the notation of (3.2), forms a two-sided inverse for the function $\Theta\colon I_S^{P_1} \times I_S^{P_2} \to I_S^P$. Now consider f, g in I_S^P and $x < y$ in P, say $\{x, y\} \subseteq P_1$ without loss of generality. Then $(f \cdot g)(x, y) = (f|_{<^{P_1}} \cdot g|_{<^{P_1}})(x, y)$, since (3.3) only involves paths in P_1. Applying Lemma 5.1, it follows that Φ is a loop homomorphism. □

Corollary 5.4. *Non-isomorphic locally finite posets may have isomorphic incidence loops.*

Proof. For a given locally finite poset P, let P' be the disjoint union of P with an isolated singleton $\top$. Since $I_S^\top = \{e\}$, Theorem 5.3 shows that $I_S^P \cong I_S^P \times I_S^\top \cong I_S^{P'}$. □

For the following result, recall that a poset $P' = (X, \leq')$ is said to be an *extension* of a poset $P = (X, \leq)$ if $x \leq y$ in P implies $x \leq' y$.

Proposition 5.5. *Suppose that a locally finite poset $P' = (X, \leq')$ is an extension of a poset $P = (X, \leq)$. Then I_S^P is a subloop of $I_S^{P'}$.*

Proof. Define

$$T = \{(x, y) \in X \times X \mid x \leq' y \text{ and } x \nleq y\}\,.$$

Note that

$$I_S^P = \{f \in I_S^{P'} \mid (x, y) \in T \ \Rightarrow f_{x,y} = 0\} \tag{5.2}$$

at the set level. It remains to be shown that the loop structure of $I_S^{P'}$ induces the incidence loop structure of I_S^P on the subset (5.2) of $I_S^{P'}$. Extending the notation of Definition 3.1, for $x <' y$ in P', write $\Pi'(x, y)$ for the set of paths from x to y in P'. Then for elements f, g of I_S^P, and $x < y$ in P, one has

$$\begin{aligned}(f \cdot g)(x, y) &= f_{x,y} + g_{x,y} + \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} f_p g_q \\ &= f_{x,y} + g_{x,y} + \sum_{x<'z<'y} \sum_{p' \in \Pi'(x,z)} \sum_{q' \in \Pi'(z,y)} f_{p'} g_{q'}\end{aligned}$$

since $f_{p'} = 0$ if any step of the path p' lies in T, and similarly $g_{q'} = 0$ if any step of the path q' lies in T. □

6. Abelian groups

Recall that the *height* of a locally finite poset is the maximum cardinality of a chain that it contains. Further, a loop is *abelian* if it forms an abelian group (and is thus both commutative and associative, or equivalently, has an improper center).

Proposition 6.1. *Let P be a locally finite poset of height less than 3. Suppose that the set of covering pairs in P has cardinality κ. Then the incidence loop I_S^P is abelian, isomorphic to the power $(S,+,0)^\kappa$.*

Proof. Consider f in I_S^P. Suppose $x < y$ in P. Since the height of P is less than 3, it follows that y covers x. Thus if K is the set of covering pairs of P, the set S^K of functions from K to S coincides with the full set I_S^P.

Now consider an additional element g in I_S^P. Then by (3.3), one has

$$(f \cdot g)(x,y) = f_{x,y} + g_{x,y}\,. \tag{6.1}$$

Thus $f \cdot g = f + g$: the extended convolution product on I_S^P agrees with the pointwise addition on I_S^P. □

Corollary 6.2. *Let P be a locally finite poset. Suppose S is a zero ring, so that $|S \cdot S| = 1$. Suppose that λ is the cardinality of the set $\{(x,y) \in P^2 \mid x < y\}$. Then the incidence loop I_S^P is abelian, isomorphic to the power $(S,+,0)^\lambda$.*

Proof. By the condition on S, each extended convolution product (3.3) in I_S^P reduces to the form (6.1). □

Corollary 6.3. *Each abelian group may be represented as the incidence loop of a locally finite poset.*

Proof. Let $(A,+,0)$ be an abelian group, written additively. Define a constant product $a \circ b = 0$ on A, so that A forms a ring $(A,+,\circ)$. Consider the poset $P = \{\bot < \top\}$, with the unique covering pair $\bot < \top$. Then $I_A^P \cong (A,+,0)$ by Proposition 6.1 or Corollary 6.2. □

In Corollary 6.3, the incidence loop is taken over a zero ring. The following result provides incidence loop representations over rings of integers (possibly to a certain modulus).

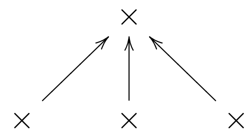
Corollary 6.4. *For $d = \infty$ or $1 < d \in \mathbb{Z}$, let A be the free abelian group of exponent d over a set X. Then A may be represented as the incidence loop of a locally finite poset P over the ring $\mathbb{Z}$ for $d = \infty$, and over the ring $\mathbb{Z}/_d$ of integers modulo d for finite d.*

Proof. Take P to be the poset $X \times \{\bot < \top\}$, the disjoint union of $|X|$ copies of $\{\bot < \top\}$. Then apply Proposition 6.1, or Theorem 5.3. □

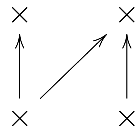
Proposition 6.1 yields a stronger version of Corollary 5.4.

Proposition 6.5. *Connected locally finite posets may have isomorphic incidence loops, even if they are neither isomorphic nor anti-isomorphic.*

Proof. Let P be the poset with Hasse diagram



Let P' be the poset with Hasse diagram



Then $I_S^P \cong I_S^{P'} \cong (S, +, 0)^3$ by Proposition 6.1. □

In both Corollary 5.4 and Proposition 6.1, the pairs of non-isomorphic posets with isomorphic incidence loops include one poset which is not directed.

Problem 6.6. Let S be a ring.

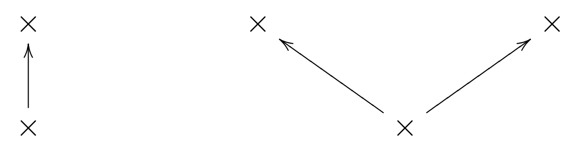
(a) Let P be a poset. Is there always a directed poset D such that $I_S^P \cong I_S^D$?
(b) Are there non-isomorphic directed posets D and D' such that $I_{\mathbb{Z}}^D$ is isomorphic to $I_{\mathbb{Z}}^{D'}$?

Theorem 5.3 showed how the incidence loop of a disjoint union (or "parallel composition") of locally finite posets may be determined from the incidence loops of the uniands. On the other hand, the following result, which again makes use of Proposition 6.1, shows that the incidence loop of an ordinal sum (or "series composition") $P_1 \oplus P_2$ of posets P_1 and P_2 cannot be determined by the incidence loops of the summands. Recall that the ordinal sum $P_1 \oplus P_2$ consists of induced copies of P_1 and P_2, where each element of P_1 is less than each element of P_2. (This is the notation of [15], while the same notation is used in for the disjoint union in [3].)

Proposition 6.7. *Let S be a nontrivial ring. Then there are locally finite posets P_1 and P_2, with isomorphic incidence loops $I_{P_1}^S \cong I_{P_2}^S$, such that $I_{P_1 \oplus P_1}^S$ is not isomorphic to $I_{P_1 \oplus P_2}^S$.*

Proof. Let P_1 be a singleton. Take $P_2 = P_1 + P_1$. Since $I_{P_1}^S$ is trivial, Theorem 5.3 shows that $I_{P_2}^S$ is trivial, so $I_{P_1}^S \cong I_{P_2}^S$. The respective ordinal sums are as follows:

$\underline{P_1 \oplus P_1}$ $\underline{P_1 \oplus P_2}$



By Proposition 6.1, one has $I^S_{P_1 \oplus P_1} \cong (S, +, 0)$, while $I^S_{P_1 \oplus P_2} \cong (S, +, 0)^2$. Since S is nontrivial, it follows that $I^S_{P_1 \oplus P_1} \ncong I^S_{P_1 \oplus P_2}$. □

Theorem 6.8. *Let P be a locally finite poset. Suppose that $|S \cdot S| > 1$ in the ring S. Then the incidence loop I^P_S is abelian if and only if the height of P is less than* 3.

Proof. The "if" statement of the theorem is covered by Proposition 6.1. Now suppose that P has a chain $x < y < z$. Consider elements f, g of I^P_S, supported on $\{(x, y), (y, z), (x, z)\}$, with

$$f_{x,y} = a, \quad f_{y,z} = f_{x,z} = 0$$

and

$$g_{y,z} = b, \quad g_{x,y} = g_{x,z} = 0$$

for $ab \neq 0$ in S. Then by (3.3), one has

$$\begin{aligned}(f \cdot g)(x, z) &= f_{x,z} + g_{x,z} + f_{x,y}g_{y,z} = ab \\ &\neq 0 = g_{x,z} + f_{x,z} + g_{x,y}f_{y,z} = (g \cdot f)(x, z)\,,\end{aligned}$$

showing that the loop I^P_S is not abelian in this case. □

Remark 6.9. Note that Corollary 6.2 demonstrates the need for the condition on S in the formulation of Theorem 6.8.

7. Groups

Recall that a *central series* in a loop (or group) $(L, \cdot, 1)$ is a nested sequence

$$L = L_0 \geq L_1 \geq \cdots \geq L_{c-1} \geq L_c = \{1\} \tag{7.1}$$

of normal subloops of L such that $L_i/L_{i+1} \subseteq Z(L/L_{i+1})$ for $0 \leq i < c$. If such a central series exists, then the loop L is said to be *nilpotent*, of *class* not exceeding c. Thus the *nilpotence class* c of a nilpotent loop L is the smallest natural number c for which a central series of the form (7.1) exists in L.

Proposition 7.1. *Let P be a locally finite poset of height less than 4. Then the incidence loop I_S^P is a nilpotent group of class at most 2.*

Proof. Consider elements f, g, h of I_S^P. Suppose $x < y$ in P. Since the height of P is less than 4, it follows that $l(x, y)$ is 1 or 2. Then

$$\begin{aligned}
&[(f \cdot g) \cdot h](x, y) \\
&\quad = \Big(f_{x,y} + g_{x,y} + \sum_{x<z<y} f_{x,z} g_{z,y}\Big) + h_{x,y} + \sum_{x<z<y} (f \cdot g)(x, z) h_{z,y} \\
&\quad = f_{x,y} + g_{x,y} + h_{x,y} + \sum_{x<z<y} (f_{x,z} g_{z,y} + f_{x,z} h_{z,y} + g_{x,z} h_{z,y}) \\
&\quad = f_{x,y} + \Big(g_{x,y} + h_{x,y} + \sum_{x<z<y} g_{x,z} h_{z,y}\Big) + \sum_{x<z<y} f_{x,z} (g \cdot h)(z, y) \\
&\quad = [f \cdot (g \cdot h)](x, y)
\end{aligned}$$

by (3.3). Thus $(f \cdot g) \cdot h = f \cdot (g \cdot h)$, so I_S^P is a group.

In groups, centers and commutants coincide. Theorem 4.8 thus implies that $Z_S^P = C(I_S^P) = Z(I_S^P)$. Now consider two elements f, g of I_S^P. By the product definition (3.3), $(f \cdot g)(x, y)$ and $(g \cdot f)(x, y)$ both equal $f_{x,y} + g_{x,y}$ if y covers x in P, so the group commutator $[f, g]$ can only take non-zero values on pairs $u < v$ for which there is a path $u < z < v$ of length 2. Since the height of P is less than 4, it follows that u is minimal and v is maximal in this case. Thus $[f, g] \in Z_S^P = Z(I_S^P)$, and I_S^P is nilpotent of class at most 2. □

Corollary 7.2. *Let P be a locally finite poset of height 3. Suppose that $|S \cdot S| > 1$ in the ring S. Then the incidence loop I_S^P of P over S is a nilpotent group of class 2.*

Proof. By Theorem 6.8, the loop I_S^P is not abelian. Then by Proposition 7.1, the incidence loop I_S^P is a nilpotent group of class 2. □

Proposition 7.1 exhibited incidence loops I_S^P that are associative because of a restriction on the poset P. By contrast, the following result exhibits incidence loops I_S^P that are associative by virtue of a restriction on the ring S.

Proposition 7.3. *Let P be a locally finite poset. Let S be a ring with $|S \cdot S \cdot S| = 1$. Then the incidence loop I_S^P is a group.*

Proof. Consider $f, g, h \in I_S^P$. Note that (3.3) reduces to

$$(f \cdot g)(x, y) = f_{x,y} + g_{x,y} + \sum_{x<z<y} f_{x,z} g_{z,y}$$

since any terms f_p or g_q with respective paths p or q having more than one step will lead to a zero product $f_p g_q$. Then

$$
\begin{aligned}
&[(f \cdot g) \cdot h](x, y) \\
&\quad = \Big(f_{x,y} + g_{x,y} + \sum_{x<z<y} f_{x,z} g_{z,y} \Big) + h_{x,y} + \sum_{x<z<y} (f \cdot g)(x, z) h_{z,y} \\
&\quad = f_{x,y} + g_{x,y} + h_{x,y} + \sum_{x<z<y} (f_{x,z} g_{z,y} + f_{x,z} h_{z,y} + g_{x,z} h_{z,y}) \\
&\quad = f_{x,y} + \Big(g_{x,y} + h_{x,y} + \sum_{x<z<y} g_{x,z} h_{z,y} \Big) + \sum_{x<z<y} f_{x,z} (g \cdot h)(z, y) \\
&\quad = [f \cdot (g \cdot h)](x, y)
\end{aligned}
$$

as in the proof of Proposition 7.1. □

Theorem 7.4. *Let P be a locally finite poset. Suppose that $|S{\cdot}S{\cdot}S| > 1$ in the ring S. Then the incidence loop I_S^P is a group if and only if the height of P is less than* 4.

Proof. Proposition 7.1 covers the "if" direction. Now suppose that $a \cdot b \cdot c \neq 0$ in S and $t < u < v < w$ in P. Suppose:

$$
\begin{aligned}
&f_{u,v} = b, \text{ and } f_{x,y} = 0 \text{ for } (x, y) \neq (u, v) \text{ and } x < y \text{ in } P\,; \\
&g_{t,u} = a, \text{ and } g_{x,y} = 0 \text{ for } (x, y) \neq (t, u) \text{ and } x < y \text{ in } P\,; \\
&h_{v,w} = c, \text{ and } h_{x,y} = 0 \text{ for } (x, y) \neq (v, w) \text{ and } x < y \text{ in } P\,.
\end{aligned}
$$

The non-zero values of $f \cdot g$ are $(f \cdot g)_{t,u} = a$ and $(f \cdot g)_{u,v} = b$, while the non-zero values of $g \cdot h$ are $(g \cdot h)_{t,u} = a$ and $(g \cdot h)_{v,w} = c$. One then has

$$
[(f \cdot g) \cdot h]_{t,w} = abc \neq 0 = [f \cdot (g \cdot h)]_{t,w} \tag{7.2}
$$

showing that I_S^P is not a group in this case. □

Remark 7.5. Note that Proposition 7.3 demonstrates the need for the condition on S in the formulation of Theorem 7.4.

8. Representing nilpotent groups

For a given abstract nilpotent group G of class 1 (i.e., an abelian group), the results of Section 6 show that G is isomorphic to the incidence loop I_S^P of a locally finite poset P over a ring S. For nilpotence class 2, the situation is more complicated. To begin, here is the general problem.

Problem 8.1. Let S be a ring with $|S \cdot S \cdot S| > 1$. Which nilpotent groups of class 2 appear as incidence loops of locally finite posets P of height 3 over S?

Some partial answers to Problem 8.1 are provided in this section, particularly for the case where S is the ring of integers. In fact, the focus on $\mathbb{Z}$ is not as restrictive as it may appear, since by Theorem 5.2, the integral incidence loop $I_{\mathbb{Z}}^P$ of a locally finite poset P enjoys a universality property over the class of all incidence loops I_S^P for which the ring S is unital. A simple application of this universality is illustrated in Corollary 8.4 below.

For a unital ring S and $u < v$ in a poset P, define an element E^{uv} of I_S^P by

$$E^{uv}(x,y) = \begin{cases} 1 & \text{if } (u,v) = (x,y); \\ 0 & \text{otherwise} \end{cases}$$

(compare [8, Defn. 2.5]). Recall that the *derived group* or *commutator subgroup* G' of a group G is the normal subgroup generated by all the commutators $[x,y] = x^{-1}y^{-1}xy$ in G. In a nilpotent group $(G,\cdot,1)$ of class 2, the derived group is contained in the center, so $G > G' > \{1\}$ is a central series in G. The following observation provides an initial counterpart to Corollary 6.4.

Proposition 8.2. *Let P be the 3-element chain. Then $I_{\mathbb{Z}}^P$ is the free nilpotent group on two generators.*

Proof. The group $G = (I_{\mathbb{Z}}^P, \cdot, e)$ is (isomorphic to) the group of unipotent upper-triangular matrices over $\mathbb{Z}$. This group, in turn, is a free nilpotent group of class 2 on the two generating matrices $[x_{ij}]$ with (x_{12}, x_{13}, x_{23}) respectively equal to $(1,0,0)$ and $(0,0,1)$. □

Corollary 8.3. *Let P be a locally finite poset of height 3. Then for each 3-element chain $a < b < c$ in P, the group $I_{\mathbb{Z}}^P$ contains a subgroup isomorphic to the free nilpotent group on the two generators E^{ab} and E^{bc}, with $E^{ac} = [E^{ab}, E^{bc}]$.*

Corollary 8.4. *Consider a modulus $1 < d \in \mathbb{Z}$. Suppose that P is a 3-element chain. Then $I_{\mathbb{Z}/_d}^P$ is the free nilpotent group of exponent d on two generators.*

Proof. Starting with the free nilpotent group $I_{\mathbb{Z}}^P$ from Proposition 8.2, apply the group homomorphism induced, according to Theorem 5.2, from the ring homomorphism $\mathbb{Z} \to \mathbb{Z}/_d$ that is given by reduction modulo d. □

The following result provides general structural information about incidence loops of (possibly infinite) locally finite posets of height 3 over the ring of integers.

Proposition 8.5. *Let P be a locally finite poset of height 3. Let K be the set of covering pairs in P. Let L be the set of intervals of height 3 in P. Set $G = I_{\mathbb{Z}}^P$. Then the group G/G' is free abelian of rank $|K|$, while the group G' is free abelian of rank $|L|$.*

Proof. By Corollary 8.3, the group G is generated by

$$\{E^{xy} \mid y \text{ covers } x \text{ in } P\}.$$

Then the group G/G' appears as

$$\bigoplus_{y \text{ covers } x \text{ in } P} \mathbb{Z}E^{xy},$$

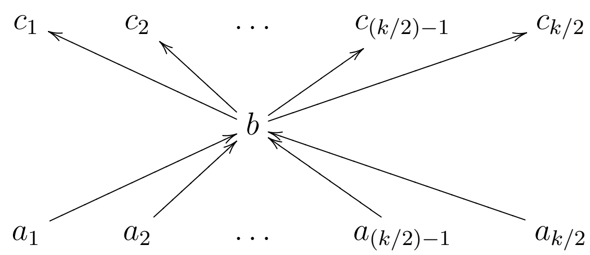
a free abelian group of rank $|K|$, while the group G' appears as

$$\bigoplus_{\{(x,y) \mid \exists\ z \in P\,.\ x<z<y\}} \mathbb{Z}E^{xy},$$

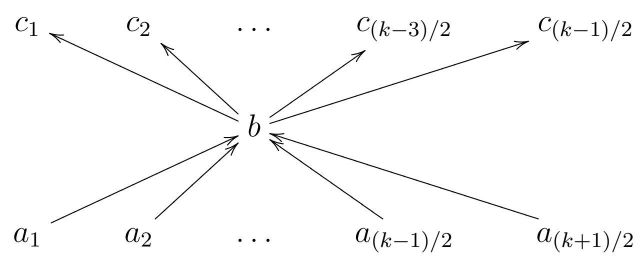
a free abelian group of rank $|L|$. □

Proposition 8.6. *Let P be a finite poset of height 3, with k covering pairs and l intervals of height 3. Then $4l \leq k^2$.*

Proof. Among all the posets with the given number k of covering pairs, the number of intervals of height 3 is maximized by posets of the form



for k even, and by



(or its dual) for k odd. There are $k^2/4$ intervals of height 3 in the first case, namely $[a_i, c_j]$ with $1 \leq i, j \leq k/2$. In the second case as illustrated, there are $(k^2 - 1)/4$ intervals of height 3, namely $[a_i, c_j]$ with $1 \leq i \leq (k+1)/2$ and $1 \leq j < k/2$. Thus $l \leq k^2/4$. □

Theorem 8.7. *Consider a nilpotent group G of class 2 for which G/G' is a free abelian group of finite rank k, while G' is a free abelian group of finite rank l. Then if $4l > k^2$, the group G does not appear as the incidence loop $I_{\mathbb{Z}}^P$ of a locally finite poset P over $\mathbb{Z}$.*

Proof. Suppose that $G = I_{\mathbb{Z}}^P$ for a locally finite poset P. By Theorem 7.4, the height of P is less than 4. But since G' is nontrivial, Theorem 6.8 shows that the height of P is not less than 3. Thus the height of P is 3. By Proposition 8.5, P has k covering pairs, and l intervals of height 3. Now $4l \leq k^2$ by Proposition 8.6. This contradicts the hypothesis $4l > k^2$. □

Corollary 8.8. *Let G be a free nilpotent group of class 2 and finite rank k. Then if $k > 2$, the group G does not appear as the incidence loop $I_{\mathbb{Z}}^P$ of a locally finite poset P over $\mathbb{Z}$.*

Proof. In G, while G/G' is free abelian of rank k, the derived group G' is free abelian of rank $l = k(k-1)/2$. Then $4l - k^2 = k^2 - 2k \geq 0$, with equality only if $k = 2$ (for $0 < k \in \mathbb{Z}$). The result thus follows from Theorem 8.7. □

9. Nilpotent incidence loops

Proposition 7.1 showed that if P is a poset of height at most 3, then the incidence loop I_S^P is nilpotent, of class at most 2. The main result of this section, Theorem 9.6, extends that aspect of Proposition 7.1 by showing that if P is a poset of height at most $c+1$, for some positive integer c, then the incidence loop I_S^P is nilpotent, of class at most c.

Definition 9.1. Let P be a locally finite poset. For each natural number i, define the subset

$$\Gamma_i^P = \{f \in I_S^P \mid f(x,y) = 0 \text{ if } l(x,y) \leq i\}$$

of the incidence loop I_S^P over a ring S.

Note that $\Gamma_0^P = I_S^P$, while Γ_1^P is the set of functions in I_S^P that vanish on covering pairs.

Lemma 9.2. *For a positive integer c, let P be a poset of height at most $c+1$. Then there is a filtration*

$$I_S^P = \Gamma_0^P \geq \Gamma_1^P \geq \ldots \geq \Gamma_{c-1}^P \geq \Gamma_c^P = \{e\} \tag{9.1}$$

of the set I_S^P.

Proposition 9.3. *For* $0 \le i \le c$*, suppose* $f \in \Gamma_i^P$ *and* $g, h \in I_S^P$*. Then on all ordered pairs* $u < v \in P$ *with* $l(u, v) \le i$*, the following triples of elements of* I_S^P *agree:*

(a) *The product* $f \cdot g$*, the dual product* $g \cdot f$*, and* g*;*
(b) *The product* $(f \cdot g) \cdot h$*, the re-associated product* $f \cdot (g \cdot h)$*, and* $g \cdot h$*;*
(c) *The product* $(h \cdot g) \cdot f$*, the re-associated product* $h \cdot (g \cdot f)$*, and* $h \cdot g$*.*

Proof. (a): On all ordered pairs $x < y \in P$ with $l(x, y) \le i$, one has

$$(f \cdot g)(x, y) = f_{x,y} + g_{x,y} + \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} f_p g_q = g_{x,y} \tag{9.2}$$

since the interval $x < y$, as well as each step in any path p, all have lengths not exceeding i. Dually,

$$(g \cdot f)(x, y) = g_{x,y} + f_{x,y} + \sum_{x<z<y} \sum_{p \in \Pi(x,z)} \sum_{q \in \Pi(z,y)} g_p f_q = g_{x,y}$$

since the interval $x < y$, as well as each step in any path q, all have lengths not exceeding i.

(b): On all ordered pairs $u < v \in P$ with $l(u, v) \le i$, one has

$$\big((f \cdot g) \cdot h\big)(u, v) = (g \cdot h)(u, v) = \big(f \cdot (g \cdot h)\big)(u, v)$$

by (a).

(c): is dual to (b). □

Proposition 9.4. *For* $0 \le i \le c$*, the set* Γ_i^P *is a subloop of* I_S^P*.*

Proof. For elements f, g of Γ_i^P, Proposition 9.3(a) shows that $f \cdot g$ and g agree, and therefore vanish, on intervals $x < y \in P$ of length not exceeding i. This suffices to show that $f \cdot g \in \Gamma_i^P$.

Now for any elements f, g of I_S^P, one has

$$f \cdot (f \backslash g) = g \tag{9.3}$$

in I_S^P. Suppose that Γ_i^P is not closed under $\backslash$, so there are elements f, g of Γ_i^P for which $f \backslash g \notin \Gamma_i^P$. This means that there is at least one interval $u < v \in P$, of length not exceeding i, such that $(f \backslash g)(u, v) \neq 0$. Then by Proposition 9.3(a),

$$\big(f \cdot (f \backslash g)\big)(u, v) = (f \backslash g)(u, v) \neq 0 = g(u, v)$$

in contradiction to the equality (9.3). Thus Γ_i^P is closed under left division. The proof of closure under right division is similar. Finally, note that $e \in \Gamma_i^P$. $\square$

Proposition 9.5. *For $0 \le i \le c$, the set Γ_i^P is a normal subloop of I_S^P.*

Proof. Consider an element f of Γ_i^P, and elements g, h of I_S^P. It will first be shown that

$$fT(g) = fR(g)L(g)^{-1} = g\backslash(f \cdot g) \in \Gamma_i^P \,. \tag{9.4}$$

If (9.4) fails, then there is an interval $u < v \in P$, of minimal length $l(u,v) \le i$, such that $\big(g\backslash(f\cdot g)\big)(u,v) \neq 0$. Thus

$$\begin{aligned}
&\Big(g \cdot \big(g\backslash(f\cdot g)\big)\Big)(u,v) \\
&\qquad = g_{u,v} + \big(g\backslash(f\cdot g)\big)(u,v) + \sum_{u<z<v}\ \sum_{p\in\Pi(u,z)}\ \sum_{q\in\Pi(z,v)} g_p\big(g\backslash(f\cdot g)\big)_q \\
&\qquad = g_{u,v} + \big(g\backslash(f\cdot g)\big)(u,v) \\
&\qquad \neq g(u,v)\,,
\end{aligned}$$

the second equality holding since $g\backslash(f\cdot g)$ vanishes on each step of each path q from z to v, by the choice of the interval $u<v$. Then

$$(f\cdot g)(u,v) = \Big(g \cdot \big(g\backslash(f\cdot g)\big)\Big)(u,v) \neq g(u,v)$$

with $l(u,v) \le i$, contradicting Proposition 9.3(a).

Next, it will be shown that

$$fR(g,h) = fR(g)R(h)R(g\cdot h)^{-1} = \big((f\cdot g)\cdot h\big)/(g\cdot h) \in \Gamma_i^P \,. \tag{9.5}$$

If (9.5) fails, then there is an interval $u < v \in P$, of minimal length $l(u,v) \le i$, such that $\Big(\big((f\cdot g)\cdot h\big)/(g\cdot h)\Big)(u,v) \neq 0$. Thus

$$\begin{aligned}
&\bigg(\Big(\big((f\cdot g)\cdot h\big)/(g\cdot h)\Big)\cdot(g\cdot h)\bigg)(u,v) \\
&\qquad = \Big(\big((f\cdot g)\cdot h\big)/(g\cdot h)\Big)_{u,v} + (g\cdot h)(u,v) \\
&\qquad\quad + \sum_{u<z<v}\ \sum_{p\in\Pi(u,z)}\ \sum_{q\in\Pi(z,v)} \Big(\big((f\cdot g)\cdot h\big)/(g\cdot h)\Big)_p (g\cdot h)_q \\
&\qquad = \Big(\big((f\cdot g)\cdot h\big)/(g\cdot h)\Big)(u,v) + (g\cdot h)(u,v) \\
&\qquad \neq (g\cdot h)(u,v)\,,
\end{aligned}$$

the second equality holding since $\big((f\cdot g)\cdot h\big)/(g\cdot h)$ vanishes on each step of each path p from u to z, by the choice of the interval $u<v$. Then

$$\begin{aligned}\big((f\cdot g)\cdot h\big)(u,v) &= \Big(\Big(\big((f\cdot g)\cdot h\big)/(g\cdot h)\Big)\cdot(g\cdot h)\Big)(u,v)\\ &\neq (g\cdot h)(u,v)\end{aligned}$$

with $l(u,v)\leq i$, contradicting Proposition 9.3(b).

The final step, dual to the preceding, shows that

$$fL(g,h)=fL(g)L(h)L(h\cdot g)^{-1}=(h\cdot g)\backslash\big(h\cdot(g\cdot f)\big)\in\Gamma_i^P \tag{9.6}$$

by obtaining a contradiction to Proposition 9.3(c) if (9.6) fails. □

Theorem 9.6. *For a positive integer c, let P be a locally finite poset of height at most $c+1$. Then the incidence loop I_S^P is nilpotent, of class at most c.*

Proof. The filtration (9.1) will be shown to be a central series in the loop I_S^P. By Proposition 9.5, each term Γ_i^P is a normal subloop of I_S^P. Thus the proof reduces to establishing the containment

$$\Gamma_i^P/\Gamma_{i+1}^P\subseteq Z(I_S^P/\Gamma_{i+1}^P)$$

for $0\leq i<c$. Consider $f\in\Gamma_i^P$ and $g,h\in I_S^P$. Suppose $u<v\in P$ with $l(u,v)=i+1$. It will be shown that:

(a) $(f\cdot g)(u,v)=(g\cdot f)(u,v)$;
(b) $\big((f\cdot g)\cdot h\big)(u,v)=\big(f\cdot(g\cdot h)\big)(u,v)$;
(c) $\big(h\cdot(g\cdot f)\big)(u,v)=\big((h\cdot g)\cdot f\big)(u,v)$.

For (a), one has

$$\begin{aligned}(f\cdot g)(u,v) &= f_{u,v}+g_{u,v}+\sum_{u<z<v}\sum_{p\in\Pi(u,z)}\sum_{q\in\Pi(z,v)}f_pg_q\\ &= f_{u,v}+g_{u,v}\end{aligned} \tag{9.7}$$

since f_p vanishes on each path p from u to z, and

$$(g\cdot f)(u,v)=g_{u,v}+f_{u,v}+\sum_{u<z<v}\sum_{p\in\Pi(u,z)}\sum_{q\in\Pi(z,v)}g_pf_q=g_{u,v}+f_{u,v}$$

since f_q vanishes on each path q from z to v. For (b), one has

$$\begin{aligned}\big((f\cdot g)\cdot h\big)(u,v) &= (f\cdot g)_{u,v}+h_{u,v}+\sum_{u<z<v}\sum_{p\in\Pi(u,z)}\sum_{q\in\Pi(z,v)}(f\cdot g)_ph_q\\ &= f_{u,v}+g_{u,v}+h_{u,v}+\sum_{u<z<v}\sum_{p\in\Pi(u,z)}\sum_{q\in\Pi(z,v)}g_ph_q\end{aligned}$$

by (9.7) and Proposition 9.3(a), while

$$\begin{aligned}\big(f\cdot(g\cdot h)\big)(u,v) &= f_{u,v}+(g\cdot h)_{u,v}+\sum_{u<z<v}\sum_{p\in\Pi(u,z)}\sum_{q\in\Pi(z,v)} f_p(g\cdot h)_q\\ &= f_{u,v}+(g\cdot h)_{u,v}\\ &= f_{u,v}+g_{u,v}+h_{u,v}+\sum_{u<z<v}\sum_{p\in\Pi(u,z)}\sum_{q\in\Pi(z,v)} g_p h_q\end{aligned}$$

since f_p vanishes on each path p from u to z for the second equality. Verification of (c) is dual to that for (b). □

Remark 9.7. Corollary 4.10 shows that the incidence loop of a locally finite poset of infinite height need not be nilpotent.

10. Conclusion

Consider the respective consequences of Theorems 5.3, 6.8 and 7.4, as they apply to a locally finite poset P and a ring S:

- Suppose that $|S|>1$. Then the loop I_S^P is trivial if and only if the height of P is at most 1.
- Suppose that $|S^2|>1$. Then the loop I_S^P is abelian if and only if the height of P is at most 2.
- Suppose that $|S^3|>1$. Then the loop I_S^P is a group if and only if the height of P is at most 3.

This list suggests the following problem within the same context.

Problem 10.1. For each positive integer n, find a variety $\mathbf{V}_n$ of loops such that when $|S^n|>1$, the incidence loop I_S^P lies in $\mathbf{V}_n$ if and only if the height of P is at most n.

Problem 10.1 is initialized by taking $\mathbf{V}_1$ to be the variety of trivial loops, $\mathbf{V}_2$ to be the variety of loops that are commutative and associative, and $\mathbf{V}_3$ as the variety of loops that are associative. In general, the larger the variety $\mathbf{V}_n$, the better. In this spirit, the full variety of all associative loops is a better candidate for $\mathbf{V}_3$ than the variety of nilpotent associative loops of class at most 2.

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