Available online at www.sciencedirect.com

ScienceDirect

Discrete Mathematics 307 (2007) 1 – 11

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Poset extensions, convex sets, and semilattice presentations

G. Bińczak[a], A.B. Romanowska[a], J.D.H. Smith[b]

[a] *Faculty of Mathematics and Information Sciences, Warsaw University of Technology, 00-661 Warsaw, Poland*
[b] *Department of Mathematics, Iowa State University, Ames, IA 50011, USA*

Received 11 December 2001; received in revised form 22 January 2005; accepted 3 April 2006
Available online 7 November 2006

Abstract

The paper is devoted to an algebraic and geometric study of the feasible set of a poset, the set of finite probability distributions on the elements of the poset whose weights satisfy the order relationships specified by the poset. For a general poset, this feasible set is a barycentric algebra. The feasible sets of the order structures on a given finite set are precisely the convex unions of the primary simplices, the facets of the first barycentric subdivision of the simplex spanned by the elements of the set. As another fragment of a potential complete duality theory for barycentric algebras, a duality is established between order-preserving mappings and embeddings of feasible sets. In particular, the primary simplices constituting the feasible set of a given finite poset are the feasible sets of the linear extensions of the poset. A finite poset is connected if and only if its barycentre is an extreme point of its feasible set. The feasible set of a (general) disconnected poset is the join of the feasible sets of its components. The extreme points of the feasible set of a finite poset are specified in terms of the disjointly irreducible elements of the semilattice presented by the poset. Semilattices presented by posets are characterised in terms of various distributivity concepts.

MSC: 06A07; 06A12

Keywords: Poset extension; Linear extension; Szpilrajn theorem; Poset dimension; Barycentric algebra; Duality theory; Convex set; Semilattice; Semilattice presentation; Distributive semilattice; Weakly distributive; Mildly distributive; Irreducible element; Maximal chain; Extreme point

1. Introduction

If R or $(R, \leqslant)$ is a partially ordered set ("poset"), the *feasible set* of R is defined to be the set of probability distributions on the underlying set R satisfying the constraints that if $x < y$ in R, then the probability p_x of x does not exceed the probability p_y of y (Definition 3.1). Feasible sets of partial orders come into play when one has only partial information about the probabilities of certain events, the partial information being represented entirely by order relations stating that the probability of one given event is known not to exceed the probability of another. For example, the occurrence of the first event might necessarily entail the occurrence of the second. A different kind of example arises in ecological experiments counting the number of individuals of various species in a given region, where it might be known, say that marmots are not more likely than deer, nor deer more likely than crows. Similar considerations of relative frequencies of species govern the construction of phylogenetic trees [13,20,21,25].

E-mail addresses: binczak@mini.pw.edu.pl (G. Bińczak), aroman@alpha.mini.pw.edu.pl (A.B. Romanowska), jdhsmith@math.iastate.edu (J.D.H. Smith).

0012-365X/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2006.09.021

The purpose of the current paper is to investigate the feasible sets of partial orders, especially from the standpoint of the theory of barycentric algebras (as axiomatised in Definition 2.1). Barycentric algebras are modes in the sense of [18,19], algebras in which each element forms a singleton subalgebra, and for which each operation is a homomorphism. The set of all finite probability distributions on a given set is characterised algebraically as the free barycentric algebra over the set. If the set is finite, there is an equivalent geometrical characterisation as the simplex spanned by the set (Theorem 2.2). The feasible sets are thus described using both geometric and algebraic tools. The main geometric tool used is the first barycentric subdivision of a simplex, as realised by exact bisections of the simplex. (Readers accustomed to the traditional association of a poset with its simplicial complex of finite chains should note that throughout the paper, consideration is limited to this particular realisation of the first barycentric subdivision. The emphasis is on convex geometry, not topology.) The elementary maximal-dimensional simplices appearing in the specific first barycentric subdivision are called the *primary simplices*. They are the feasible sets of finite linear orders (Proposition 3.6). Theorem 3.5 identifies the feasible sets of finite partial orders as the convex unions of primary simplices. There is a duality between posets and their feasible sets (Proposition 3.8), so the primary simplices constituting the feasible set of a finite partial order R are precisely the feasible sets of the linear orders L that extend R (Theorem 3.11). The original determinations of these linear extensions were in terms of maximal chains in the distributive lattice of possibly empty lower subsets ("ideals") of R (compare [2,4,5,12,23,24], etc.). The duality between feasible sets and finite partial orders may be viewed, along with Pontryagin duality for semilattices [10], as another fragment of a duality theory for barycentric algebras (compare [15,16]; [19, Chapter 9]). Construction of a complete duality theory for barycentric algebras is still a major open problem.

Although the feasible set of a poset R is itself a barycentric algebra, it turns out that much of its geometry in the finite case is describable in terms of a simpler, more combinatorial barycentric algebra, a semilattice. This semilattice is the semilattice S *presented* by the ordered set R. The order relations in R are interpreted as equational relations in the language of semilattices. The semilattice S is then the largest semilattice generated by the elements of the underlying set of R in which the equations expressed by the order structure of R are required to hold (Definition 4.1). The semilattice S is realised by the semilattice of non-empty upper subsets of R under the operation of union (Proposition 4.2). The poset R is recovered from S as its poset of irreducible elements (Theorem 4.5). Semilattices presentable by finite posets are identified as precisely the weakly distributive semilattices (Corollary 5.8) or the mildly distributive semilattices in the sense of Hickman [9] (Theorem 5.3). General presentable semilattices are characterised by Theorem 5.4 and Corollary 5.7.

The concluding section of the paper uses the semilattice S presented by a poset R to study the geometry of the feasible set of R, particularly in the finite case. In Theorem 6.1, a finite poset R is seen to be connected if and only if its barycentre is an extreme point of the feasible set.[1] The feasible set of a (general) disconnected poset is just the join of the feasible sets of its components (Theorem 6.2). In particular, the feasible set of a disconnected finite poset is the convex hull of the feasible sets of its components (Corollary 6.3). The vertices of the primary triangulation of the feasible set are precisely the barycentres of the elements of S (Theorem 6.4). Finally, the extreme points of the feasible set of R are identified combinatorially as the barycentres of the so-called *disjointly irreducible* elements of S (Theorem 6.7). (Harremoës and Topsøe have obtained a similar description of the extreme points of the feasible set, using different methods, as part of their work on optimal predictors and coding strategies [8].)

For algebraic concepts and notational conventions that are not otherwise explained in the paper, readers are referred to [19,22]. In particular, note that mappings are generally written in their natural place on the right of their arguments, except for those mappings which are images of morphisms under contravariant functors. An *upper set* of a poset R is a subset U of R with the property that $U \ni x < y \Rightarrow y \in U$. The *principal upper set* generated by an element x of R is the set $x^{\leqslant} = \{y \in R \mid x \leqslant y\}$.

2. Barycentric algebras

This section briefly outlines the basic facts about barycentric algebras that are used in the paper. For more details, readers may consult [18,19]. Let I° denote the open unit interval $]0, 1[$. For p in I°, define $p' = 1 - p$.

[1] Note that while the order polytope of a finite poset appears as a suspension of the feasible set, the geometry of the order polytope does not appear to admit such a direct relation with the properties of the poset. For example, the order polytope of the connected two-element chain does not have an extreme point corresponding to the barycentre of the chain.

Definition 2.1. A *barycentric algebra* A or (A, I°) is an algebra of type $I^{\circ} \times \{2\}$, equipped with a binary operation

$$\underline{p} : A \times A \to A; (x, y) \mapsto xy\underline{p}$$

for each p in I°, satisfying the identities

$$xx\underline{p} = x \tag{2.1}$$

of *idempotence* for each p in I°, the identities

$$xy\underline{p} = yx\underline{p'} \tag{2.2}$$

of *skew-commutativity* for each p in I°, and the identities

$$xy\underline{p}z\underline{q} = xyz\underline{q/(p'q')'}\,\underline{(p'q')'} \tag{2.3}$$

of *skew-associativity* for each p, q in I°.

A convex set C forms a barycentric algebra (C, I°), with $xy\underline{p} = (1-p)x + py$ for x, y in C and p in I°. A *semilattice* $(S, \cdot)$ is an idempotent, commutative semigroup. The semilattice S becomes a barycentric algebra on setting $xy\underline{p} = x \cdot y$ for x, y in S and p in I°. Products in a semilattice are often expressed simply by juxtaposition of their arguments. Recall that semilattices may equivalently be described as partially ordered sets in which each two-element subset $\{x, y\}$ has a greatest lower bound, namely the product of x and y. Indeed, one then has $x \leqslant y$ if and only if $x = x \cdot y$. Semilattices obtained in this way are described as *meet semilattices*.

For the following result, see [14], Section 2.1 of [18], Section 5.8 of [19]. The equivalence of the final two structures in the theorem corresponds to the identification of the barycentric coordinates in a simplex with the weights in finite probability distributions.

Theorem 2.2. *Let X be a finite set. The following structures are equivalent*:

(a) *the free barycentric algebra XB on X*;
(b) *the simplex spanned by X*;
(c) *the set of all probability distributions on X.*

Using the identifications given by Theorem 2.2, the *barycentre* of a non-empty set is defined as the uniform distribution over the set. The *barycentre* of a non-empty word is defined as the uniform distribution on its set of letters. For general sets, the following restricted version of Theorem 2.2 applies (compare [19, Section 9.2]).

Corollary 2.3. *Let X be a set. The following structures are equivalent*:

(a) *the free barycentric algebra XB on X*;
(b) *the simplex spanned by X.*

3. Feasible sets

Definition 3.1. Let R or $(R, \leqslant)$ be a poset. Then the *feasible set* $F(R)$ of R is the set of probability distributions on R satisfying the constraints $p_x \leqslant p_y$ for $x < y$ in R.

Proposition 3.2. *The feasible set of a finite poset R is a closed convex subset of the simplex spanned by the elements of R.*

Proof. For each element $x < y$ of the strict order relation on R, let $C_{x<y}$ be the subset of the simplex consisting of all points whose barycentric coordinates satisfy $p_x \leqslant p_y$. Then $C_{x<y}$ is a closed convex subset of the simplex. The feasible set of R, as the intersection of the finite family of closed convex subsets $C_{x<y}$ ranging over all pairs $x < y$ in the poset R, is itself a closed convex subset of the simplex. It is non-empty, since it contains the uniform distribution on R. □

Corollary 3.3. *The feasible set of a general poset R is a subalgebra of the free barycentric algebra RB on R.*

Proof. In the general case, the corresponding subsets $C_{x<y}$ are subalgebras of *RB*. Thus, the feasible set of *R*, as the intersection of the family of subalgebras $C_{x<y}$ ranging over all pairs $x < y$ in the poset *R*, is itself a subalgebra of *RB*. □

The feasible set $F(\mathbb{N}, \leqslant)$ of the natural numbers with their usual ordering is empty. On the other hand, if an element x of a poset $(R, \leqslant)$ has the property that the principal upper set $x^{\leqslant}$ is finite, then the uniform distribution $\underline{x^{\leqslant}}$ on this upper set is an element of the feasible set of *R*.

Theorem 3.4. *The feasible set of the poset* $(\mathbb{N}, \geqslant)$ *is the free barycentric algebra on a countable set of generators.*

Proof. Let p_k denote the probability of a natural number k. Each element of $F(\mathbb{N}, \geqslant)$ has the form $(p_0, p_1, \ldots)$ with

$$p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0 = p_{n+1} = p_{n+2} \ldots \tag{3.1}$$

and $\sum_0^n p_i = 1$ for some natural number n. This element has a unique expression of the form

$$\sum_0^n q_r \cdot \underline{r^{\geqslant}}$$

with $q_r = (r+1)(p_r - p_{r+1})$ for $0 \leqslant r < n$ and $q_n = (n+1)p_n$. Now by (3.1), each q_r is non-negative, while $\sum_0^n q_r = \sum_0^n p_r = 1$. Thus, $F(\mathbb{N}, \geqslant)$ is the free barycentric algebra on the countable set $\{\underline{r^{\geqslant}} \mid r \in \mathbb{N}\}$ of uniform distributions on upper sets. □

If *R* is finite, then the internal bounding hyperplanes $H_{x<y}$ of the feasible set, consisting of the distributions with $p_x = p_y$ for each covering pair $x < y$ of *R*, are part of the first barycentric subdivision of the simplex spanned by *R*. The (elementary) maximal-dimensional simplices of this first barycentric subdivision are called the *primary simplices*. The restriction to the feasible set $F(R)$ of the first barycentric subdivision of the simplex spanned by *R* is called the *primary triangulation* of the feasible set.

Theorem 3.5. *Let Q be a finite, non-empty set. Then the feasible sets of order relations on Q are precisely the convex unions of primary simplices in the first barycentric subdivision of the simplex spanned by Q.*

Proof. It is immediate from the above considerations that the feasible sets are of the required form. Conversely, a convex union *U* of primary simplices is bounded internally (within the simplex over *Q*) by hyperplanes that are describable in either of the two equivalent forms $H_{x<y}$ or $H_{y<x}$. For each such hyperplane, choose between the two alternative potential covering relations $x < y$ or $y < x$ according to which of the choices $p_x < p_y$ or $p_y < p_x$ appears in the barycentric coordinates of points inside *U*. The set *U* then becomes the feasible set for the partial order defined by the collection of all the chosen covering relations. □

Note that the primary simplices are the feasible sets of linear orders:

Proposition 3.6. *Let Q be a finite, non-empty set. Then the primary simplices of the simplex spanned by Q are precisely the feasible sets of the linear orders on Q.*

Proof. The proof is by induction on the order of *Q*. Let *P* be a primary simplex of the simplex *QB* spanned by *Q*. Then *P* is the cone extending from the barycentre of *Q* to a primary simplex P' of a face of *QB*. This face of *QB*, opposite to the vertex q say, is the simplex $Q'B$ spanned by the subset $Q' = Q - \{q\}$ of *Q*. By induction, P' is the feasible set of a linear order L' on Q'. Let *L* be the ordinal sum of $\{q\}$ with L'. Then *P*, as the cone from the barycentre of *Q* to P', is the feasible set of *L*.

Conversely, let $L = \{q_1 < q_2 < \cdots < q_n\}$ be a linear order on *Q*. By induction, the feasible set of $L' = \{q_2 < \cdots < q_n\}$ is a primary simplex P' of the face of *QB* opposite to the vertex q_1. Then the feasible set of *L*, which is the cone to P' from the barycentre of *Q*, is a primary simplex of *QB*. □

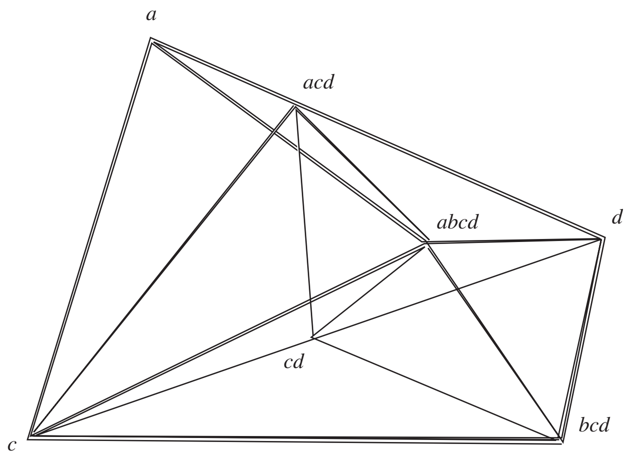


Fig. 1. Feasible set of $\{a < c > b < d\}$.

Example 3.7. Let R be the poset $\{a < c > b < d\}$ (the "standard example" used in [2,24], etc.). The feasible set of R, together with its primary triangulation, is shown in Fig. 1. A word labelling a vertex stands for the barycentre of (the set of letters of) the word. In order to enhance the clarity, the diagram has not been drawn to scale. Note that the primary simplices in the primary triangulation are the respective spans of the following vertex sets, and the feasible sets of the corresponding linear orders:

$\{d, cd, acd, abcd\} - \{d > c > a > b\}$,

$\{c, ac, acd, abcd\} - \{c > a > d > b\}$,

$\{c, cd, acd, abcd\} - \{c > d > a > b\}$,

$\{c, cd, bcd, abcd\} - \{c > d > b > a\}$,

$\{d, cd, bcd, abcd\} - \{d > c > b > a\}$.

Proposition 3.8. *Let $f : (Q, \leqslant_1) \to (Q, \leqslant_2)$ be a bijective order-preserving map between two posets $Q_1 = (Q, \leqslant_1)$ and $Q_2 = (Q, \leqslant_2)$ on the same finite, non-empty underlying set Q. Then the map*

$$F(f) : F(Q_2) \to F(Q_1); \ \sum_{x \in Q} p_x x \mapsto \sum_{x \in Q} p_{xf} x \tag{3.2}$$

is a well-defined barycentric algebra homomorphism.

Proof. For well-definedness, it must be shown that the image of (3.2) lies in the feasible set $F(Q_1)$. But since f is order-preserving and bijective, an order relation $y <_1 z$ in Q_1 implies $yf <_2 zf$ in Q_2, so that $p_{yf} \leqslant p_{zf}$, as required. To see that $F(f)$ is a barycentric algebra homomorphism, note that it is the restriction to $F(Q_2)$ of the unique barycentric algebra automorphism of the free algebra QB specified by the inverse $f^{-1} : Q \to Q$ of the bijection $f : Q \to Q$. □

For a given finite, non-empty set Q, let $\mathscr{O}(Q)$ denote the category whose objects are the various poset structures $(Q, \leqslant)$ on the set Q, and whose morphisms are the order-preserving bijections between these poset structures on Q.

Corollary 3.9. *The assignment F yields a contravariant functor from the category $\mathscr{O}(Q)$ to the category of barycentric algebras.*

Remark 3.10. Let $\mathscr{B}(Q)$ be the image of F. Then Corollary 3.9 provides a duality between the categories $\mathscr{O}(Q)$ and $\mathscr{B}(Q)$. On the other hand, F cannot distinguish between, say, the linear orderings $\{0 < 1 < 2 < 3 < \cdots\}$ and $\{1 < 0 < 2 < 3 < \cdots\}$ of $\mathbb{N}$, assigning the empty barycentric algebra to each.

Theorem 3.11. *A finite poset R extends to a linear order L if and only if the feasible set of L is a primary simplex in the primary triangulation of the feasible set of R.*

Proof. If $f : R \to L$ embeds R in L, then $F(f) : F(L) \to F(R)$ embeds the primary simplex corresponding to L into (the primary triangulation of) the feasible set of R.

Conversely, suppose that the feasible set of L is a primary simplex in the primary triangulation of $F(R)$. Then the order constraints defined by the linear order L are consistent with the constraints defined by R. It follows that the binary order relation of R is a subset of the binary order relation of L, so that R extends to L. □

4. Semilattice presentations

Definition 4.1. Let R be a poset. Then the *semilattice presented* by R is the semilattice S defined to within isomorphism by the following properties:

(a) there is an order-preserving embedding $\eta : R \to S$;
(b) for each order-preserving map $f : R \to H$ into a semilattice H, there is a unique semilattice homomorphism $f' : S \to H$ such that $\eta f' = f$.

A semilattice S is said to be *presentable* if it is presented as above by a poset R.

An upper set U in a poset R is said to be *finitely generated* if it is of the form

$$U = x_1^{\leqslant} \cup \cdots \cup x_r^{\leqslant} \tag{4.1}$$

for a finite subset $\{x_1, \ldots, x_r\}$ of R. The following result, which immediately guarantees the existence of the semilattice presented by a poset, appears as Proposition 4.4 in [17].

Proposition 4.2. *Let R be a poset. Then the semilattice presented by R is the meet semilattice $U(R)$ or $(U(R), \cup)$ of non-empty finitely generated upper sets of R, under the operation of set-theoretical union.*

Note that the meet-semilattice ordering $\leqslant_\cup$ on $(U(R), \cup)$ is given by

$$U_1 \leqslant_\cup U_2 \quad \Leftrightarrow \quad U_1 = U_1 \cup U_2 \quad \Leftrightarrow \quad U_1 \supseteq U_2. \tag{4.2}$$

Corollary 4.3. *Let R be a finite poset. Then the semilattice presented by R is a subsemilattice of the free semilattice over the underlying set of R.*

Proof. The free semilattice over R is the semilattice of finite, non-empty subsets of R under the operation of union. □

Definition 4.4. An element x of a (meet) semilattice S is said to be

- *irreducible* if $x = yz$ in S implies $x = y$ or $x = z$;
- *prome* if $x \geqslant yz$ in S implies $x \geqslant y$ or $x \geqslant z$.

Theorem 4.5. *Let R be a poset. Then the irreducible elements of the meet semilattice S presented by R form a poset isomorphic with R.*

Proof. Realise S according to Proposition 4.2 as the meet semilattice of non-empty finitely generated upper sets of R, under the operation of union. Consider the map

$$R \to S; r \mapsto r^{\leqslant} \tag{4.3}$$

given by the generation of principal upper sets. Each element U of S has a unique expression in form (4.1) with $\{x_1, \ldots, x_m\}$ as an antichain. Thus, the image of (4.3) is precisely the set of irreducible elements of S. Now for x and y

in R, one has $x \leqslant y$ if and only if $x^{\leqslant} \supseteq y^{\leqslant}$. Thus, (4.3) corestricts to an order isomorphism between R and the poset of irreducible elements of the meet semilattice S. □

Corollary 4.6. *If a semilattice is presentable, then it is presented by its poset of irreducible elements.*

5. Distributivity

From the lattice-theoretical point of view, the content of the preceding section, as it applies to finite posets, might be regarded as part of the duality between posets and distributive lattices (compare [3,6]). However, it is noteworthy that the above treatment takes place entirely within modal theory. Moreover, unlike the semilattice presented by a finite, non-empty poset R, the distributive lattice dual of R is not necessarily the correct object for the purposes of Theorem 6.4 below. The current section (which may be skipped by readers interested primarily in feasible sets) examines the relationship between semilattice presentations and distributivity in more detail.

Recall that a meet semilattice S is said to be *distributive* if $x \geqslant y_1 y_2$ in S implies the existence of elements $x_1 \geqslant y_1$ and $x_2 \geqslant y_2$ such that $x = x_1 x_2$ (compare [7, p. 99], working with join semilattices). Now finite distributive meet semilattices are *lattices*, i.e. posets having least upper bounds as well as greatest lower bounds (compare [7, Exercise II.5.27]). However, the semilattice presented by a two-element antichain is a free semilattice which is not a lattice. Thus, in general, presentable semilattices are not distributive. Conversely, Remark 5.5 below gives an example of a distributive semilattice which is not presentable.

Two relevant extensions of the distributivity concept for semilattices have appeared in the literature. Following [11], a meet semilattice is defined to be *weakly distributive* if $x \geqslant y_1 y_2$ in S with $x \ngeqslant y_1$ and $x \ngeqslant y_2$ implies the existence of elements $x_1 \geqslant y_1$ and $x_2 \geqslant y_2$ such that $x = x_1 x_2$ (compare [1, Proposition 3.2]). Note that the semilattice presented by a two-element antichain, while not distributive in the above sense, is weakly distributive. In [9] Hickman introduced another less stringent form of distributivity for semilattices. For a non-empty semilattice S, set

$$T(S) = \{s_1^{\leqslant} \cap \cdots \cap s_r^{\leqslant} \mid r > 0,\ \ s_i \in S\} \tag{5.1}$$

with the order induced from the power set of S. Then S is said to be *mildly distributive* if and only if the poset $T(S)$ is a distributive lattice. (Hickman used a slightly different definition, and then proved the equivalence of his definition with the one given here in Corollary 2.4 of [9].) We begin by relating mild distributivity to presentability in the finite case.

Lemma 5.1. *Let S be a finite meet semilattice. Then each non-empty element of $T(S)$ is a principal upper set of S.*

Proof. Suppose that a non-empty element $s_1^{\leqslant} \cap \cdots \cap s_r^{\leqslant}$ of (5.1) were not a principal upper set of S. Then it would have two incomparable minimal elements, a and b say, with $s_i \leqslant a$ and $s_i \leqslant b$ for $1 \leqslant i \leqslant r$. But then $s_i \leqslant a \cdot b$ for $1 \leqslant i \leqslant r$ would contradict the minimality of a and b. □

Lemma 5.2. *For a finite meet semilattice S, the following two conditions are equivalent*:

(a) *S is upwardly directed*;
(b) *S is a lattice*.

Theorem 5.3. *Let S be a finite, non-empty semilattice. Then S is presentable if and only if it is mildly distributive.*

Proof. There are two cases to consider, according to whether S does or does not satisfy the equivalent conditions of Lemma 5.2. If S is upwardly directed, then each element of (5.1) is non-empty, so by Lemma 5.1 the mapping

$$S \to T(S);\ s \mapsto s^{\leqslant} \tag{5.2}$$

is a dual isomorphism. If S is not a lattice, then $T(S)$ is the ordinal sum of $\{\emptyset\}$ with the image of (5.2). In either case, suppose that S is presented by a poset R. Then $T(S)$ is isomorphic with the distributive lattice of upper sets of R, so that S is mildly distributive. Conversely, suppose that $T(S)$ is distributive. Then S is presented by the poset of non-empty irreducible elements of the dual of $T(S)$. □

For general semilattices, the following theorem is fundamental.

Theorem 5.4. *A semilattice S is presentable if and only if*:

(a) *each element is a product of a finite number of irreducible elements*; *and*
(b) *each irreducible element is prime*.

Proof. If S is presentable, then it may be taken to be of the form $(U(R), \cup)$ given by Proposition 4.2. The irreducible elements are the principal upper sets $x^{\leqslant}$ for $x \in R$, so condition (a) of the theorem is clearly satisfied by virtue of the defining relation (4.1). It remains to be shown that a principal upper set $x^{\leqslant}$ is prime. Recalling the specification (4.2) of the meet-semilattice ordering of $U(R)$, suppose that $x^{\leqslant}$ is contained in the union $U_1 \cup U_2$ of two upper sets U_1 and U_2. Then the element x of R lies in one of U_1 and U_2, so that $x^{\leqslant}$ is contained in one of U_1 and U_2, as required.

Conversely, suppose that a semilattice S satisfies the conditions of the theorem. Let R be the set of irreducible elements of S, equipped with the ordering induced from the meet-semilattice ordering of S. Consider the unique semilattice-homomorphic extension

$$(U(R), \cup) \to S; x_1^{\leqslant} \cup \ldots x_m^{\leqslant} \mapsto x_1 \ldots x_m \tag{5.3}$$

of the order-preserving embedding of R in S that is guaranteed by Proposition 4.2. By condition (a) of the theorem, (5.3) surjects. It remains to be shown that (5.3) injects. Suppose that there are irreducible elements $x_1, x_2, \ldots x_m, y_1, y_2 \ldots y_n$ of S with

$$x_1 x_2 \ldots x_m = y_1 y_2 \ldots y_n. \tag{5.4}$$

For $1 \leqslant i \leqslant m$, condition (b) of the theorem identifies the irreducible element x_i as prime. Since (5.4) says that $x_i \geqslant y_1 y_2 \ldots y_n$, it follows that there is some $1 \leqslant j(i) \leqslant n$ with $x_i \geqslant y_{j(i)}$ or $x_i \in y_{j(i)}^{\leqslant}$, so $x_i^{\leqslant} \subseteq y_1^{\leqslant} \cup \cdots \cup y_n^{\leqslant}$. Thus

$$x_1^{\leqslant} \cup \cdots \cup x_m^{\leqslant} \subseteq y_1^{\leqslant} \cup \cdots \cup y_n^{\leqslant}.$$

The reverse containment follows similarly, and so (5.3) injects as required. □

Remark 5.5. Theorem 5.4 shows that the infinite distributive semilattice $(2^{\mathbb{N}}, \cup)$ is not presentable, since infinite subsets of $\mathbb{N}$ are not expressible as finite unions of the irreducible elements, namely the singletons.

Lemma 5.6. *A presentable semilattice is weakly distributive*.

Proof. Consider a presentable semilattice, without loss of generality having the form $(U(R), \cup)$ for a poset R. Suppose that there are upper sets $V = x_1^{\leqslant} \cup \cdots \cup x_l^{\leqslant}$, $W_1 = y_1^{\leqslant} \cup \cdots \cup y_m^{\leqslant}$, and $W_2 = z_1^{\leqslant} \cup \cdots \cup z_n^{\leqslant}$ with $V \subseteq W_1 \cup W_2$ but $V \not\subseteq W_1$ and $V \not\subseteq W_2$. Now $x_1, \ldots, x_l \in W_1 \cup W_2$, so without loss of generality there is $1 \leqslant r < l$ with $x_1, \ldots, x_r \in W_1$ and $x_{r+1}, \ldots, x_l \in W_2$. Taking $V_1 = x_1^{\leqslant} \cup \cdots \cup x_r^{\leqslant}$ and $V_2 = x_{r+1}^{\leqslant} \cup \cdots \cup x_l^{\leqslant}$, one then has $V_1 \subseteq W_1$, $V_2 \subseteq W_2$, and $V = V_1 \cup V_2$. □

Corollary 5.7. *A semilattice S is presentable if and only if it is weakly distributive*, *and each of its elements is a product of a finite number of irreducible elements*.

Proof. If a semilattice is presentable, then it satisfies the required conditions by Theorem 5.4 and Lemma 5.6. Conversely, consider a semilattice S satisfying the conditions of the corollary. By Theorem 5.4, each irreducible element x of S must then be identified as prime. Suppose $x \geqslant y_1 y_2$ for elements y_1 and y_2 of S. It must be shown that $x \geqslant y_1$ or $x \geqslant y_2$. Assuming otherwise, namely $x \ngeqslant y_1$ and $x \ngeqslant y_2$, weak distributivity would guarantee the existence of elements x_1 and x_2 with $x = x_1 x_2$, $x_1 \geqslant y_1$, and $x_2 \geqslant y_2$. Since x is irreducible, one would then have $x = x_1$ or $x = x_2$, implying the contradiction $x \geqslant y_1$ or $x \geqslant y_2$. □

For a finite semilattice, condition (a) of Theorem 5.4 is immediate. For weak distributivity, one thus obtains the following analogue of Theorem 5.3:

Corollary 5.8. *Let S be a finite, non-empty semilattice. Then S is presentable if and only if it is weakly distributive.*

6. Extreme points of the feasible set

The purpose of this final section is to specify the geometry of the feasible set of a finite partial order, and in particular to determine its extreme points. The first result shows how the connectivity of a finite partial order correlates with the extremality of its barycentre in its feasible set.

Theorem 6.1. *Let R be a finite poset. Then R is connected if and only if the barycentre of the underlying set of R is an extreme point of the feasible set of R.*

Proof. Suppose that R is disconnected. Then the barycentre of R, as a mixture of the barycentres of the components of R, is not an extreme point of the feasible set of R.

Conversely, suppose that the barycentre of R appears as a mixture in the feasible set of R. Now the primary triangulation of the feasible set is a subcomplex of the first barycentric subdivision of the simplex spanned by the underlying set of R. Thus, the only way in which the barycentre of R appears in the feasible set as a mixture is as a mixture of the barycentres of two non-empty complementary subsets X, Y of R. Consider each point x in X and each point y in Y. Then the barycentre of X is a feasible point with each $p_x > 0$ and $p_y = 0$, while the barycentre of Y is a feasible point with each $p_x = 0$ and $p_y > 0$. Since no comparabilities relate the non-empty, complementary subsets X and Y of R, it follows that R is disconnected. □

The following result describes the feasible set of a general disconnected poset in terms of the feasible sets of its components.

Theorem 6.2. *Let R be a poset. Then in the subalgebra poset* $\mathrm{Sb}(RB)$ *of the free barycentric algebra on the underlying set of R, the feasible set of R is the least upper bound of the feasible sets of its components.*

Proof. By the independence of the components, the feasible sets of the components are certainly subalgebras of the feasible set of R, so the least upper bound of the components is again a subset of the feasible set of R.

Conversely, let $\{R_i \mid i \in I\}$ be the set of components of R, with $R_i = \{x_{ij} \mid j \in J_i\}$ for each $i \in I$. Consider an element $p = \sum_{i\in I} \sum_{j\in J_{ip}} p_{ij} x_{ij}$ of the feasible set of R, with finite subsets J_{ip} of J_i such that $\forall i \in I,\ \forall j \in J_{ip},\ \ p_{ij} > 0$. Define

$$I_p = \left\{ i \in I \;\middle|\; \sum_{j\in J_{ip}} p_{ij} > 0 \right\}.$$

Note that I_p is finite. For each $i \in I_p$, define $q_i = \sum_{j\in J_{ip}} p_{ij}$ and $p_i = q_i^{-1} \sum_{j\in J_{ip}} p_{ij} x_{ij}$. Each such distribution p_i lies in the feasible set of the corresponding component R_i. Then the arbitrary element p of the feasible set of R appears as the convex combination $p = \sum_{i\in I_p} q_i p_i$ of elements of the feasible sets of the components. □

Corollary 6.3. *Let R be a finite poset. Then the feasible set of R is the convex hull of the feasible sets of its components.*

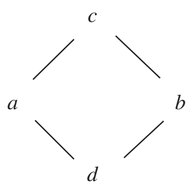
Theorem 6.4. *Let R be a finite, non-empty poset presenting the semilattice S. Then the barycentres of the elements of S are precisely the vertices of the primary triangulation of the feasible set of R.*

Proof. The vertices of the primary triangulation of the feasible set of R are the vertices appearing in the primary simplices of the feasible set of R. By Theorem 3.11, these vertices are the vertices of the feasible sets of the linear orders L into which R embeds. By standard results (compare [4,24], Example 3.7, etc.) and Proposition 4.2, the linear extensions L are given by the maximal chains of the poset S of non-empty upper subsets of R. (Note that [4,24] work with the complementary lower subsets of R.) The elements of the maximal chains are precisely the elements of S, and the corresponding vertices of the feasible sets of the linear extensions L are the barycentres of the elements of these maximal chains. □

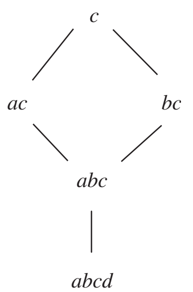
Definition 6.5. Let S be the semilattice presented by a non-empty finite poset R. Consider the realisation of S given by Corollary 4.3 as a subsemilattice of the free semilattice over the underlying set of R. Then an element x of S is said to be *disjointly irreducible* if it cannot be expressed as the product yz of two elements y, z of S having disjoint sets of arguments.

Theorem 4.5 shows that disjoint irreducibility is actually an abstract property of elements of the semilattice S presented by a non-empty poset R, independent of the particular realisation of S used in Definition 6.5. In other words, if there is a semilattice isomorphism $\phi : S \to S'$ between S and another presentable semilattice S', then x is a disjointly irreducible element of S if and only if $x\phi$ is a disjointly irreducible element of S', presented by its poset of irreducible elements according to Theorem 4.5.

Example 6.6. Let R be the poset whose covering relation is displayed by the following Hasse diagram:



Then the semilattice S presented by R has its Hasse diagram as follows:



The element abc of S is disjointly irreducible, even though it is not irreducible in the sense of Definition 4.4.

Theorem 6.7. *Let R be a finite, non-empty poset. Then the extreme points of the feasible set of R are precisely the barycentres of the disjointly irreducible elements of the semilattice S presented by R.*

Proof. By Theorem 6.4, the vertices of the primary triangulation of the feasible set of R are the barycentres of the elements of S. Recall that the primary triangulation of the feasible set of R is a subtriangulation of the first barycentric subdivision of the simplex spanned by R. By Theorem 6.4 again, the vertices of this first barycentric subdivision are the barycentres of the free semilattice over the underlying set of R, which contains S as a subsemilattice according to Corollary 4.3. Now a vertex u of the first barycentric subdivision appears as a mixture of two other vertices v and w if and only if these vertices are the barycentres of disjoint sets with u as the barycentre of their union. Thus, a vertex of the primary triangulation of the feasible set of R is an extreme point of the feasible set if and only if it is the barycentre of a disjointly irreducible element of S. □

Acknowledgements

The paper was written within the framework of COST Action 274. The third author acknowledges the hospitality and financial support received during the preparation of this paper from both the Stefan Banach International Mathematical Centre of the Polish Academy of Sciences and the Faculty of Mathematics and Information Sciences of Warsaw University of Technology.

References

[1] S. Badaev, S. Goncharov, A. Sorbi, Isomorphism types and theories of Rogers semilattices of arithmetical numberings, Computability and Models, University Series in Mathematics, Kluwer Academic Publishers, Plenum Press, Dordrecht, New York, NY, 2003 pp. 79–91.
[2] P. Baldy, M. Morvan, E. Thierry, Extending a partially ordered set: links with its lattice of ideals, Order 16 (1999) 305–312.
[3] G. Birkhoff, Lattice Theory, third ed., American Mathematical Society, Providence, RI, 1967.
[4] R. Bonnet, M. Pouzet, Extensions et stratifications d'ensembles dispersés, C. R. Acad. Sci. Paris Sér. A 268 (1969) 1512–1515.
[5] R. Bonnet, M. Pouzet, Linear extensions of ordered sets, in: I. Rival (Ed.), Ordered Sets, Reidel, Dordrecht, 1982, pp. 125–170.
[6] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, Cambridge, 1990.
[7] G. Grätzer, General Lattice Theory, Akademie-Verlag, Berlin, 1978.
[8] P. Harremoës, F. Topsøe, Instances of exact optimal prediction and universal coding, preprint, 2002.
[9] R. Hickman, Mildly distributive semilattices, J. Austral. Math. Soc. (Ser. A) 36 (1984) 287–315.
[10] K.H. Hofmann, M. Mislove, A. Stralka, Pontryagin Duality of Compact 0-Dimensional Semilattices and its Applications, Springer, Berlin, 1974.
[11] T. Katriňák, Pseudokomplementäre Halbverbände, Mat. Časopis 18 (1968) 121–143.
[12] D. Kelly, W.T. Trotter Jr., Dimension theory for ordered sets, in: I. Rival (Ed.), Ordered Sets, Reidel, Dordrecht, 1982, pp. 171–211.
[13] R.E. Krichevskiĭ, B.Ya. Ryabko, A.Yu. Haritonov, Optimal key for taxons in accordance with their frequencies, Discrete Appl. Math. 3 (1981) 67–72.
[14] W.D. Neumann, On the quasivariety of convex subsets of affine spaces, Arch. Math. 21 (1970) 11–16.
[15] K.J. Pszczoła, A.B. Romanowska, J.D.H. Smith, Duality for some free modes, Discuss. Math. Gen. Algebra and Appl. 23 (2003) 45–61.
[16] K.J. Pszczoła, A.B. Romanowska, J.D.H. Smith, Duality for quadrilaterals, Contrib. to Gen. Algebra 14 (2004) 127–134.
[17] A. Romanowska, From dissemilattices to snack algebras, Fund. Informat. 31 (1997) 65–77.
[18] A.B. Romanowska, J.D.H. Smith, Modal Theory, Heldermann, Berlin, 1985.
[19] A.B. Romanowska, J.D.H. Smith, Modes, World Scientific, Singapore, 2002.
[20] B.Ya. Ryabko, Encoding of a source with unknown but ordered probabilities, Problemy Peredachi Informatsii 14 (1978) 46–52 English translation: Problems Inform. Transmission 15 (1979) 71–77 (in Russian).
[21] B.Ya. Ryabko, F. Topsøe, Universal coding for sources with partially ordered probabilities, in: Proceedings of the 2001 IEEE International Symposium on Information Theory, IEEE, New York, NY, 2001, p. 337.
[22] J.D.H. Smith, A.B. Romanowska, Post-Modern Algebra, Wiley, New York, NY, 1999.
[23] R. Stanley, Ordered structures and partitions, Ph.D. Thesis, Harvard University, 1971.
[24] R. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc. 119 (1972).
[25] F. Topsøe, Exact prediction and universal coding for trees, in: Proceedings of the 2001 IEEE International Symposium on Information Theory, IEEE, New York, NY, 2001, p. 79.