Plane Cubic Curves, Three-Webs and the Dimensionality of Space-Time

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Abstract

Within the theory of pencil-generated space-times, two "cubic-web" models are presented.

In the first approach, addressing perceptional aspects of time and space, the arrow of time is modelled by a planar pencil of singular proper curves of order *three* (the so-called point-cubics). If the characteristic of the ground field is not two, this pencil features three distinct types of singular point-cubics, viz. crunodal (representing past events), acnodal (standing for future events), and a single cuspidal cubic (the present moment). The spatial dimensions are regarded as the foliations of a hexagonal m-web associated with each curve of the *reciprocal* pencil. It is shown that the generic m-web-space associated with the past/future event ($m = 4$) is different from that coupled to the moment of the present ($m = 3$, i.e. the *observed* dimensionality of space). The situation changes profoundly as we switch to fields of characteristic two, where two cases must be distinguished according to whether the completely "skew" term in the equation of a cubic is zero or not. In the former case, the time dimension consists solely of the present events and the corresponding web spaces are one dimensional; in the latter case, the arrow features the past and future, but lacks the present, and the web spaces are endowed with two dimensions.

The second approach, more akin to a physicist's view of space-time, enjoys the property that both time and space are, at the beginning, *one*-dimensional and mutually dual geometrical structures. Time is defined as a pencil of cuspidal point-cubics, and space as the dual line-pencil. It is after this original symmetry is broken, by viewing both the pencils in *one and the same* representation, that of points, where the difference in dimensionality of time and space appears: whereas time retains its single dimension, space acquires, in general, another two dimensions.

Both approaches exhibit an intimate connection between the intrinsic structure of time and the multiplicity of spatial dimensions.

Keywords: arrow of time, dimensionality of space, three-webs, cubic curves

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1.0 Introduction

The theory of pencil-generated (P-G) space-times [1–8] is a fruitful source of models, being able not only to represent our "ordinary" sense of space and time [1, 3], but also to shed substantial light on the character of the most pronounced "defective" space-time constructs reported by people suffering severe mental disorders or experiencing so-called altered states of consciousness [4–5, 7–8]. Its weakness is a consequence of its strength: the very generality has made it almost impossible to draw any quantitative conclusion from it yet. So, there is a need to develop the theory into a form that can be investigated quantitatively. The aim of the present paper is to outline possible directions for such development.

The version of the theory discussed below, characterized by a more intricate connection between time and space than that embodied in the original version [1–8], is based on two important observations. The first is the fact that the three planar collinear pencils of lines, which stand in our original theory [1, 3] for the three observed dimensions of psycho-physical space, represent nothing but a particular example of a geometric configuration called a hexagonal m-web [9, 10]. The second observation is the following: if the curves forming a hexagonal m-web are lines, they envelop (i.e., they are the tangent lines of) an algebraic curve of *class* m (henceforth referred to as a *line-m-tic*) [9, 10, see also Ref. 11]. In other words, the original version took perceived space to be represented by a planar, rectilinear three-web, in particular by its simplest possible form – as three projective pencils of lines having one of the latter in common. This represents the most degenerate form of a line-cubic [see, e.g. Ref. 12]. So, it is only natural to take a step further and define space more broadly, namely to be *any* rectilinear three web in the projective plane. However, having abandoned the constraint that a spatial dimension is "isomorphic" to a pencil of lines, we must also modify our definition of the time coordinate. Here our intuition suggests, and the subsequent reasoning fully justifies, replacing a particular pencil of conics (the old definition [1, 3]) by a specific pencil of point-cubics.

Thus it is cubic curves that will play a key role in the present extension of the notion of a P-G space-time configuration. It is, therefore, reasonable to begin our "space-time endeavour" with a compact, but sufficiently deep exposition of their most pronounced algebro-geometric properties; the reader who wishes to gain further knowledge of the subject may find it useful to consult e.g. [13–16].

2.0 Basics of the Theory of Planar Cubic Curves

2.1 A Point-Cubic and its First Polar

Consider a planar *point-cubic*, i.e. the curve given by the third-order equation [1]

$$C_{\breve{x}} \equiv \sum_{i,j,k=1}^{3} c_{ijk}\breve{x}_i\breve{x}_j\breve{x}_k = \quad + c_{111}\breve{x}_1^3 + c_{222}\breve{x}_2^3 + c_{333}\breve{x}_3^3 + 3c_{112}\breve{x}_1^2\breve{x}_2 + 3c_{113}\breve{x}_1^2\breve{x}_3 +$$
$$+ 3c_{122}\breve{x}_1\breve{x}_2^2 + 3c_{133}\breve{x}_1\breve{x}_3^2 + 3c_{223}\breve{x}_2^2\breve{x}_3 + 3c_{233}\breve{x}_2\breve{x}_3^2 +$$
$$+ 6c_{123}\breve{x}_1\breve{x}_2\breve{x}_3 = 0, \tag{1}$$

where $\breve{x}_i$ are the homogeneous coordinates of points of the projective plane, P_2, and c_{ijk} represent quantities completely symmetric in their indices, that is

$$\begin{aligned} c_{112} = c_{121} = c_{211}, \quad c_{113} = c_{131} = c_{311},\\ c_{122} = c_{212} = c_{221}, \quad c_{133} = c_{313} = c_{331},\\ c_{223} = c_{232} = c_{322}, \quad c_{233} = c_{323} = c_{332}, \end{aligned} \tag{2}$$

and

$$c_{123} = c_{132} = c_{213} = c_{231} = c_{312} = c_{321}. \tag{3}$$

The cubic C is irreducible (proper) if the cubic ternary form of eqn. (1) is incapable of being resolved into factors of lower order(s); otherwise C is reducible (composite).

Our first concern is with the polar properties of the cubic C. Thus [13–16], the p-th polar of a point $\breve{z}_l$ with respect to a curve Γ of n-th order is the curve whose equation results from p applications of the so-called polar operator

$$\sum_{l=1}^{3} \breve{z}_l \frac{\partial}{\partial \breve{x}_l} = \breve{z}_1 \frac{\partial}{\partial \breve{x}_1} + \breve{z}_2 \frac{\partial}{\partial \breve{x}_2} + \breve{z}_3 \frac{\partial}{\partial \breve{x}_3} \tag{4}$$

[1] The symbols and notation used in this paper are identical with those of [3]. The ground field assumed throughout the paper is that of real numbers, except for Sec. 4 where it is of characteristic two.

on the equation of Γ. Hence, applying this operator once on eqn. (1), we find that the first polar of $\breve{z}_l$ with respect to C

$$0 = \sum_{l=1}^{3} \breve{z}_l \frac{\partial}{\partial \breve{x}_l} \left(\sum_{i,j,k=1}^{3} c_{ijk} \breve{x}_i \breve{x}_j \breve{x}_k \right) = 3 \sum_{i,j,k=1}^{3} \breve{z}_k c_{kij} \breve{x}_i \breve{x}_j \tag{5}$$

represents a point-*conic* $\hat{Q}$ [see, e.g. Ref. 3],

$$\hat{Q}_{\breve{x}} \equiv \sum_{i,j=1}^{3} q_{ij} \breve{x}_i \breve{x}_j = 0, \tag{6}$$

where

$$\varrho q_{ij} = \sum_{k=1}^{3} \breve{z}_k c_{kij}, \quad \varrho \neq 0. \tag{7}$$

There is a two-fold infinity of points in P_2, so that there exists a doubly infinite system of conics that can occur as polars of points with respect to C. And this aggregate of conics represents a *net*, as can easily be seen if we formally put $\breve{z}_i = \vartheta_i$ in eqn. (5) and rewrite this equation as

$$\begin{aligned} 0 &= \sum_{i,j=1}^{3} \left(\vartheta_1 c_{1ij} + \vartheta_2 c_{2ij} + \vartheta_3 c_{3ij} \right) \breve{x}_i \breve{x}_j = \\ &= \vartheta_1 \sum_{i,j=1}^{3} c_{1ij} \breve{x}_i \breve{x}_j + \vartheta_2 \sum_{i,j=1}^{3} c_{2ij} \breve{x}_i \breve{x}_j + \vartheta_3 \sum_{i,j=1}^{3} c_{3ij} \breve{x}_i \breve{x}_j = \\ &\equiv \vartheta_1 \hat{Q}_{\breve{x}}^{(1)} + \vartheta_2 \hat{Q}_{\breve{x}}^{(2)} + \vartheta_3 \hat{Q}_{\breve{x}}^{(3)}. \end{aligned} \tag{8}$$

2.2 The Hessian and Cayleyan of a Point-Cubic

Clearly, not all the conics of this net are proper. Some are degenerate, i.e. consisting of a pair of (distinct or coincident) lines, or of a single point. A question thus naturally arises: what is the locus of points whose polar conics with respect to C are degenerate? A given conic is degenerate iff its determinant vanishes; this, in our case, reads

$$\det q_{ij} = \det \left(\sum_{k=1}^{3} \breve{z}_k c_{kij} \right) = 0. \tag{9}$$

whence it follows that the locus in question, called the *Hessian*, H, of C [13–16], is a curve of third degree in $\breve{z}_k$, i.e. a *point-cubic*. Because the Hessian is a singly infinite system of points, so also is the assembly of the degenerate polar conics. And the latter have a remarkable property, namely they are the tangent lines to a curve of the third class, i.e. to a *line-cubic*; this curve is known [13–16] as the *Cayleyan*, K, of C. The equation of K can be found as follows. If the polar conic corresponding to $\breve{z}_k$ is degenerate, then

$$\sum_{i,j,k=1}^{3} \breve{z}_k c_{kij} \breve{x}_i \breve{x}_j = \sum_{i=1}^{3} \zeta_i \breve{x}_i \sum_{j=1}^{3} \eta_j \breve{x}_j. \tag{10}$$

This involves six equations

$$\sum_{k=1}^{3} \breve{z}_k c_{kij} = \left(\zeta_i \eta_j + \zeta_j \eta_i \right)/2, \tag{11}$$

which, using Kronecker's symbol δ_{ij} (being 1 for $i = j$ and zero otherwise), can be rearranged as follows:

$$0 = \sum_{k=1}^{3} \breve{z}_k c_{kij} - \sum_{l=1}^{3} \zeta_l \left(\delta_{li} \eta_j + \delta_{lj} \eta_i \right)/2 \equiv \sum_{\sigma=1}^{6} V_\sigma M_{\kappa\sigma}, \tag{12}$$

where

$$V_\sigma \equiv (\breve{z}_1, \breve{z}_2, \breve{z}_3, \zeta_1, \zeta_2, \zeta_3) \tag{13}$$

and the six by six matrix $M_{\sigma\kappa}$, with $\kappa = 1 \Leftrightarrow ij = 11$, $\kappa = 2 \Leftrightarrow ij = 22$, $\kappa = 3 \Leftrightarrow ij = 33$, $\kappa = 4 \Leftrightarrow ij = 23$, $\kappa = 5 \Leftrightarrow ij = 13$, and $\kappa = 6 \Leftrightarrow ij = 12$, is of the form

$$M_{\kappa\sigma} = \begin{pmatrix} c_{111} & c_{211} & c_{311} & -\eta_1 & 0 & 0 \\ c_{122} & c_{222} & c_{322} & 0 & -\eta_2 & 0 \\ c_{133} & c_{233} & c_{333} & 0 & 0 & -\eta_3 \\ c_{123} & c_{223} & c_{323} & 0 & -\eta_3/2 & -\eta_2/2 \\ c_{113} & c_{213} & c_{313} & -\eta_3/2 & 0 & -\eta_1/2 \\ c_{112} & c_{212} & c_{312} & -\eta_2/2 & -\eta_1/2 & 0 \end{pmatrix}. \tag{14}$$

Hence, eqn. (12) has a non-trivial solution(s) only if $\det M_{\sigma\kappa} = 0$, that is when

$$K \equiv \det \begin{pmatrix} c_{111} & c_{211} & c_{311} & \eta_1 & 0 & 0 \\ c_{122} & c_{222} & c_{322} & 0 & \eta_2 & 0 \\ c_{133} & c_{233} & c_{333} & 0 & 0 & \eta_3 \\ 2c_{123} & 2c_{223} & 2c_{323} & 0 & \eta_3 & \eta_2 \\ 2c_{113} & 2c_{213} & 2c_{313} & \eta_3 & 0 & \eta_1 \\ 2c_{112} & 2c_{212} & 2c_{312} & \eta_2 & \eta_1 & 0 \end{pmatrix} = 0, \tag{15}$$

the last equation being, indeed, of degree three in the line-coordinates η_i. [2] The Cayleyan is thus a curve of *class* three and represents, as mentioned in the Introduction, a general hexagonal three-web [9, 10].

2.3 Singular and Inflexion Points on a Point-Cubic

Before making a definite move towards the new definition of time, we have to expand further our formalism by discussing the properties of *singular* and other exceptional points that a cubic C may contain. We shall consider [13–16] the intersection properties of C with a line L, given by two distinct points $\breve{x}_i$ and $\breve{z}_i$,

$$\varrho \breve{y}_i = \lambda \breve{x}_i + \breve{z}_i. \tag{16}$$

Here ϱ is, as before, a non-zero proportionality factor, and λ is a variable parameter. Substituting the last equation into eqn. (1) and performing a Taylor expansion we get

$$\varrho^3 C_{\breve{y}} = C_{\breve{z}} + \lambda \sum_{i=1}^{3} \breve{x}_i \frac{\partial C_{\breve{z}}}{\partial \breve{z}_i} + \frac{\lambda^2}{2!} \sum_{i,j=1}^{3} \breve{x}_i \breve{x}_j \frac{\partial^2 C_{\breve{z}}}{\partial \breve{z}_i \partial \breve{z}_j} + \frac{\lambda^3}{3!} \sum_{i,j,k=1}^{3} \breve{x}_i \breve{x}_j \breve{x}_k \frac{\partial^3 C_{\breve{z}}}{\partial \breve{z}_i \partial \breve{z}_j \partial \breve{z}_k}. \tag{17}$$

Now, suppose that both $\breve{y}_i$ and $\breve{z}_i \in C$. Then eqn. (17) reduces to

$$0 = \lambda \sum_{i=1}^{3} \breve{x}_i \frac{\partial C_{\breve{z}}}{\partial \breve{z}_i} + \frac{\lambda^2}{2!} \sum_{i,j=1}^{3} \breve{x}_i \breve{x}_j \frac{\partial^2 C_{\breve{z}}}{\partial \breve{z}_i \partial \breve{z}_j} + \frac{\lambda^3}{3!} \sum_{i,j,k=1}^{3} \breve{x}_i \breve{x}_j \breve{x}_k \frac{\partial^3 C_{\breve{z}}}{\partial \breve{z}_i \partial \breve{z}_j \partial \breve{z}_k}, \tag{18}$$

which has a single root $\lambda = 0$. If one keeps the point $\breve{z}_i$ fixed and varies the position of $\breve{x}_i$, one can, however, find such a position of the line L that

$$0 = \sum_{i=1}^{3} \breve{x}_i \frac{\partial C_{\breve{z}}}{\partial \breve{z}_i}, \tag{19}$$

in which case $\lambda = 0$ is a *double* root of eqn. (18); this means that the line L has a *two*-point contact with C at $\breve{z}_i$, i.e. it is the *tangent* line to C at $\breve{z}_i$. It may also happen that eqn. (19) is satisfied regardless of the choice of the point $\breve{x}_i$, which is possible only if

$$0 = \frac{\partial C_{\breve{z}}}{\partial \breve{z}_k} = 3 \sum_{i,j=1}^{3} c_{kij} \breve{z}_i \breve{z}_j \tag{20}$$

[2] The same must, of course, hold for ζ_i, because both the quantities, ζ_i and η_i, enter eqn. (11) in a symmetric way.

for each $k = 1, 2, 3$; in other words, the line L has at $\breve{z}_i$ a two-point contact with C whatever its position – the point $\breve{z}_i$ being a *singular* (or double) point of C [13–16]. Incidentally, comparing eqns. (20) and (5) we see that a singular point of C lies on every first polar with respect to the latter.

Next, assume that the fixed point $\breve{z}_i$ is singular, and consider again the varying line L passing through it. This line may well assume such a position that also

$$\sum_{i,j=1}^{3} \breve{x}_i \breve{x}_j \frac{\partial^2 C_{\breve{z}}}{\partial \breve{z}_i \partial \breve{z}_j} = 0 \tag{21}$$

so that $\lambda = 0$ is a *triple* root of eqn. (18), L and C thus having a three-point contact at $\breve{z}_i$. The last equation is the equation of a point-conic Q^*,

$$Q^*_{\breve{x}} = \sum_{i,j=1}^{3} q^*_{ij}(\breve{z}) \breve{x}_i \breve{x}_j = 0, \tag{22}$$

where

$$q^*_{ij}(\breve{z}) \equiv \frac{\partial^2 C_{\breve{z}}}{\partial \breve{z}_i \partial \breve{z}_j}. \tag{23}$$

We shall show that this conic is degenerate. To this end we recall that any homogeneous form Γ of the n-th order satisfies Euler's Theorem [see, e.g., Ref. 14, p.6]

$$n\Gamma_{\breve{x}} = \sum_{i=1}^{3} \breve{x}_i \frac{\partial \Gamma_{\breve{x}}}{\partial \breve{x}_i}. \tag{24}$$

For $\partial C_{\breve{z}}/\partial \breve{z}_j$, which is of the 2-nd order, this reads

$$2\frac{\partial C_{\breve{z}}}{\partial \breve{z}_j} = \sum_{i=1}^{3} \breve{z}_i \frac{\partial^2 C_{\breve{z}}}{\partial \breve{z}_i \partial \breve{z}_j}. \tag{25}$$

Because $\breve{z}_i$ is a singular point, the left-hand side of the last equation is zero for each j, and this is only possible if

$$\det \frac{\partial^2 C_{\breve{z}}}{\partial \breve{z}_i \partial \breve{z}_j} = 0. \tag{26}$$

Hence, the conic Q^* is indeed degenerate. Since it comprises a pair of distinct *real*, distinct *imaginary*, or real *coincident* lines, there correspondingly exist three different species of singular points on point-cubics, usually described as *crunodes*, *acnodes*, and *cusps*, [3] respectively – see Fig. 1.

The remaining class of exceptional points exhibited by irreducible point-cubics C are the so-called *inflexional* points, or *flexes* for short. These points $\breve{z}_i$ are non-singular, and the tangents at them have a triple intersection with C, i.e. they obey eqns. (19) and (21), but not eqn. (20). As the variable points $\breve{x}_i$ lie simultaneously on the tangent line L (eqn. (19)) and the conic Q^* (eqn. (21)), L must be a factor of Q^*. This implies that Q^* is also degenerate in this case, i.e. it satisfies eqn. (26).

To conclude this section, we note that both singular and inflexional points of C satisfy eqn. (26), which when combined with eqn. (1) is equivalent to

$$\det \left(\sum_{k=1}^{3} c_{ijk} \breve{z}_k \right) = 0. \tag{27}$$

The last equation is identical with eqn. (9). This means that both singular points and flexes of C also lie on its Hessian H_C.

[3] The latter being also known as spinodes, or stationary points.

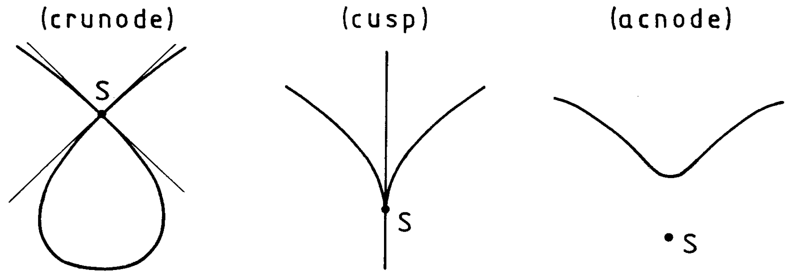

Figure 1: The three distinct types of singular (double) points S featured by irreducible planar curves of the third order.

3.0 A "Cubic Web" Model of Space-Time

3.1 Singular Proper Point-Cubics and the Arrow of Time

At this point a sufficiently large amount of information about the theory of cubic curves has been gathered for us to be able to introduce a fully projective version of the pencil concept of the arrow of time [1–3]. Our starting point is the existence of *three* species of singular points on point-cubics. Taking into account that a proper point-cubic cannot contain more than one singular point [see, e.g. Ref. 15, p. 26], this implies that there are just *three* distinct groups of singular proper point-cubics, namely *crunodal* (i.e. those possessing crunodes), *acnodal* (featuring acnodes) and *cuspidal* (endowed with cusps). This immediately reminds us of a similar structure found within the group of all affine proper conics in P_2, where we have hyperbolae (i.e. conics having a couple of real points of intersection with the ideal line), ellipses (characterized by two imaginary points of intersection) and parabolas (in which case the two points are coincident (and, so, real)), respectively [1–3]. Such a resemblance is, however, not accidental, but has a deep origin in the concept of so-called inversion of a conic [see, for example, Ref. 13, Ch. XIII]. Without going into unnecessary details we here just note that the point-cubics in question are the inverses of affine conics with respect to a point situated on the latter; in particular, crunodal/acnodal cubics are the inverses of hyperbolae/ellipses, whilst a cuspidal cubic is obtained by inverting a parabola. Recalling futher that it is a specific pencil of affine proper conics which was taken in our original theory to define the arrow of time, with the domain of hyperbolae/ellipses representing events of the past/future, and with a single parabola standing for the moment of the present [1–3], the possible directions for extending this model are not difficult to spot. We are simply to find a pencil of point-cubics featuring a domain of crunodal cubics (past), a region of acnodal cubics (future), and a single cuspidal cubic (present), representing the borderline between the two.

One of the pencils endowed with such a property is the following:

$$\begin{aligned} C_{\breve{x}}(\vartheta) &\equiv \breve{x}_3\breve{x}_2^2 - \breve{x}_1\left(\breve{x}_1 - \kappa\breve{x}_3\right)^2 + \vartheta\breve{x}_3\left(\breve{x}_1 - \kappa\breve{x}_3\right)^2 = \\ &= -\breve{x}_1^3 + \vartheta\kappa^2\breve{x}_3^3 - \kappa(\kappa+2\vartheta)\breve{x}_1\breve{x}_3^2 + (2\kappa+\vartheta)\breve{x}_1^2\breve{x}_3 + \breve{x}_3\breve{x}_2^2 = 0, \end{aligned} \tag{28}$$

where ϑ is a variable parameter and κ – an arbitrary, fixed constant. The pencil has two real base points

$$B: \quad \breve{x}_2 = 0 \ \wedge \ \breve{x}_1 = \kappa\breve{x}_3 \tag{29}$$

and

$$B': \quad \breve{x}_1 = 0 \ \wedge \ \breve{x}_3 = 0, \tag{30}$$

the former being, at the same time, the *singular* point of all the proper cubics contained in (28) – as one can easily verify substituting eqns. (28) and (29) into eqn. (20). Let us check that the pencil is really endowed with the requisite structure.

To this end we shall examine the character of the degenerate conics defined via eqns. (22) and (23) that correspond to individual point-cubics of pencil (28) at the singular point B. Thus, inserting eqns. (28) and (29) into eqn. (23) we obtain

$$\begin{aligned} &q^*_{11} = -2(\kappa - \vartheta)\breve{z}_3, \quad q^*_{12} = 0, \quad q^*_{13} = 2\kappa(\kappa - \vartheta)\breve{z}_3, \\ &q^*_{21} = q^*_{12} = 0, \quad q^*_{22} = 2\breve{z}_3, \quad q^*_{23} = 0, \\ &q^*_{31} = q^*_{13} = 2\kappa(\kappa - \vartheta)\breve{z}_3, \quad q^*_{32} = q^*_{23} = 0, \quad q^*_{33} = -2\kappa^2(\kappa - \vartheta)\breve{z}_3, \end{aligned} \tag{31}$$

so that the equation of $Q^*(\vartheta)$ reads

$$Q^*_{\breve{x}}(\vartheta) = (\vartheta - \kappa)\left(\breve{x}_1 - \kappa\breve{x}_3\right)^2 + \breve{x}_2^2 = 0, \tag{32}$$

and represents a pencil of degenerate conics sharing B and consisting of imaginary line-pairs for $\vartheta > \kappa$, a repeated (real) line when $\vartheta = \kappa$, and real line-pairs if $\vartheta < \kappa$. Consequently, the family of proper point-cubics of (28) breaks up into the domains of acnodal ($\vartheta > \kappa$) and crunodal ($\vartheta < \kappa$) cubics, the two being separated from each other by a single cuspidal cubic [4] ($\vartheta = \kappa$) – as depicted in Fig. 2. We have thus succeeded in finding a fully projective model of the psychological arrow of time, replacing (a pencil of proper) conics by (that of proper) point-cubics, and the affine properties of the former by the singular characteristics of the latter; in a succinct form, the difference between the two models is displayed in Table 1.

Table 1: Relation between the old (conic) and new (cubic) pencil-definitions of the structure of the psychological arrow of time.

TEMPORAL	THE ARROW OF TIME GENERATED BY A PENCIL OF	
DOMAIN	AFFINE PROPER CONICS	SINGULAR PROPER CUBICS
past	hyperbolae	crunodal cubics
present	parabola	cuspidal cubic
future	ellipses	acnodal cubics

In order to gain deeper insight into the structure of this "cubic" time arrow, let us take a look at the properties of the Cayleyans that correspond to the cubics of eqn. (28). Thus, comparison of eqns. (28) and (1) shows that

$$c_{111} = -1, \quad c_{333} = \vartheta\kappa^2, \quad c_{133} = -\frac{\kappa(\kappa + 2\vartheta)}{3}, \quad c_{113} = \frac{2\kappa + \vartheta}{3}, \quad c_{223} = \frac{1}{3}, \tag{33}$$

other c_{ijk} being zero, which after substitution into eqn. (15) yields

$$\begin{aligned} K(\vartheta) \quad &\equiv \quad \det\begin{pmatrix} -1 & 0 & (2\kappa+\vartheta)/3 & \eta_1 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & \eta_2 & 0 \\ -\kappa(\kappa+2\vartheta)/3 & 0 & \vartheta\kappa^2 & 0 & 0 & \eta_3 \\ 0 & 2/3 & 0 & 0 & \eta_3 & \eta_2 \\ 2(2\kappa+\vartheta)/3 & 0 & -2\kappa(\kappa+2\vartheta)/3 & \eta_3 & 0 & \eta_1 \\ 0 & 0 & 0 & \eta_2 & \eta_1 & 0 \end{pmatrix} = \\ &= \quad -\frac{1}{3}\left(\eta_3 + \kappa\eta_1\right)\left[\frac{2}{3}(\kappa - \vartheta)^2\eta_2^2 + \eta_1\eta_3 + \frac{2\vartheta + \kappa}{3}\eta_1^2\right] = 0. \end{aligned} \tag{34}$$

[4] As well as by the cubic $\vartheta = \infty$, which is, however, *reducible* (being composed of a single line, $\breve{x}_3 = 0$, and a double line, $\breve{x}_1 - \kappa\breve{x}_3 = 0$ – see eqn. (28)).

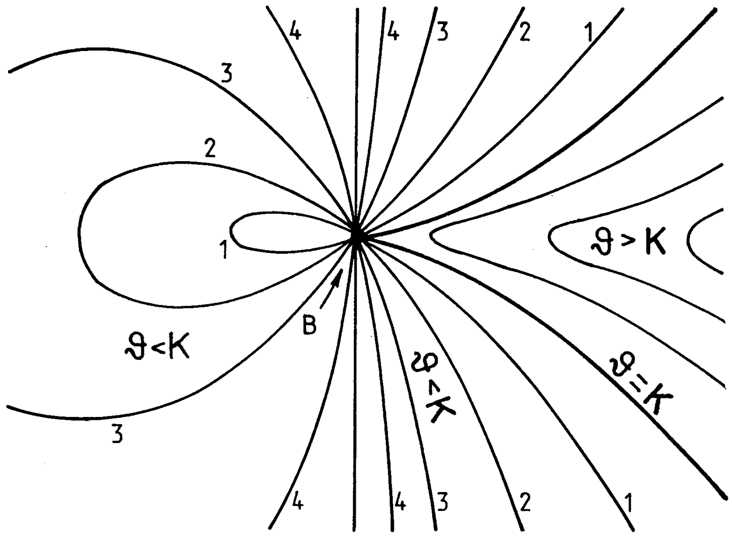

Figure 2: An image of the pencil of singular point-cubics defined by eqn. (28) in the affine plane with the Cartesian coordinates $x \equiv \check{x}_1/\check{x}_3$ and $y \equiv \check{x}_2/\check{x}_3$. Crunodal cubics ($\vartheta < \kappa$) consist of a loop (situated leftward from B) and an infinite branch (rightward from B); for better visualization, the numbers are given showing which infinite branch belongs to a given loop. In the case of the cuspidal cubic ($\vartheta = \kappa$, thick) the loop has shrunk to a point (B). Finally, acnodal cubics ($\vartheta > \kappa$) feature an infinite branch and the point B, which is detached (isolated) from the former.

From the last equation it follows that the Cayleyans of (28) represent *reducible* line-cubics, comprising a *point*,

$$L_\eta = -\frac{1}{3}\left(\eta_3 + \kappa\eta_1\right) = 0, \tag{35}$$

and a *conic*,

$$Q_\eta(\vartheta) = \frac{2}{3}(\kappa - \vartheta)^2\eta_2^2 + \eta_1\eta_3 + \frac{2\vartheta + \kappa}{3}\eta_1^2. \tag{36}$$

This conic is proper (i.e. non-degenerate) for every finite value of the parameter ϑ *except for* the $\vartheta = \kappa$ case, when it breaks up into a pair of distinct real points, viz.

$$Q_\eta(\vartheta) = \eta_1\eta_3 + \kappa\eta_1^2 = \eta_1\left(\eta_3 + \kappa\eta_1\right) = 0, \tag{37}$$

one of which is identical with the point given by eqn. (35) (easily recognized to be nothing but the singular point B). The Cayleyan $K(\vartheta = \kappa)$ thus consists only of *two* distinct pencils of lines, and thus does not represent a genuine three-web. Mathematically, this "peculiarity" is a direct consequence of the fact that whereas "the crunode and the acnode are the varieties of the node, and varieties of the same generality, the difference being that of real and imaginary, the cusp is ... really a distinct singularity" [15, p. 24]. [5] Physically, this means that the moment of the present, the "now," stands on a different footing than an event of the past/future – the fact well-established and firmly supported by our everyday experience. This is a property which is absent in our original, conic model of space-time [1, 3]. A seemingly trivial conic to cubic extension of our theory thus leads to an important conceptual advantage, as it will become evident in subsequent section(s).

[5] This is most explicitly seen when singular points are analyzed in terms of the so-called (super)linear branches. While the nodes are found to be (represented by) a union of two linear branches, a cusp is a *uni*branch singularity [see, for example, Ref. 13, Chapt. VI]. A more detailed account of this topic would take us, however, too far afield from the central idea of the paper.

3.2 Space as a Specific Hexagonal Three-Web

As already mentioned on several occasions, the tangent lines to a line-m-tic form a system referred to as a linear (hexagonal) m-web. Generally speaking [see e.g. Refs. 9 and 10 for a more detailed account], a d-web of dimension k in a neighbourhood $\mathcal{U} \in P_n$ (a projective space of dimension n, $n \geq 2$) is a family of d foliations of dimension k in $\mathcal{U}$ which are in general position. There exists a special class of so-called *linear* webs in which the leaves (elements) of all the foliations represent (pieces of) linear subspaces of P_n. Recalling the definition of the class m of a planar curve as the number of tangents (real or imaginary) that can be drawn to the curve from a general point P we clearly get, with varying P, an m-web of dimension $k = 1$; in the particular case of a line-cubic we have $m = 3$.

Our purpose here is to examine the structural properties of generic m-webs generated by the three distinct species of singular proper point-cubics comprising pencil (28). Rephrased, we are to find the class of of singular point-cubics contained in (28). To this end, we shall consider a generic point-cubic, whose equation we rewrite – with a view towards future discussion of the characteristic two case – in a form that slightly differs from (1)

$$\tilde{C}_{\breve{x}} \equiv \sum_{i \leq j \leq k}^{3} \tilde{c}_{ijk}\breve{x}_i\breve{x}_j\breve{x}_k = \quad + \tilde{c}_{111}\breve{x}_1^3 + \tilde{c}_{222}\breve{x}_2^3 + \tilde{c}_{333}\breve{x}_3^3 + \tilde{c}_{112}\breve{x}_1^2\breve{x}_2 + \tilde{c}_{113}\breve{x}_1^2\breve{x}_3 +$$
$$+ \tilde{c}_{122}\breve{x}_1\breve{x}_2^2 + \tilde{c}_{133}\breve{x}_1\breve{x}_3^2 + \tilde{c}_{223}\breve{x}_2^2\breve{x}_3 + \tilde{c}_{233}\breve{x}_2\breve{x}_3^2 +$$
$$+ \tilde{c}_{123}\breve{x}_1\breve{x}_2\breve{x}_3 = 0. \tag{38}$$

The class of this cubic, i.e. the maximum number of tangents (real or imaginary) that can be drawn to $\tilde{C}_{\breve{x}}$ from an arbitrary point $\breve{z}_i$ not on $\tilde{C}_{\breve{x}}$, is found as follows. We take the coordinate system in such a way that $\breve{z}_i$ coincides with one of the vertices of the fundamental triangle, say V_1

$$\varrho\breve{z}_i = (1, 0, 0), \qquad \varrho \neq 0. \tag{39}$$

Then, the lines through $\breve{z}_i$ are given by

$$\breve{x}_2 = \lambda\breve{x}_3, \tag{40}$$

where λ is a variable parameter. Inserting the last equation into eqn. (38) and putting $x \equiv \breve{x}_1/\breve{x}_3$, we get

$$0 = \quad \tilde{c}_{111}x^3 + (\tilde{c}_{112}\lambda + \tilde{c}_{113})\,x^2 + \left(\tilde{c}_{122}\lambda^2 + \tilde{c}_{123}\lambda + \tilde{c}_{133}\right)x +$$
$$\tilde{c}_{222}\lambda^3 + \tilde{c}_{223}\lambda^2 + \tilde{c}_{233}\lambda + \tilde{c}_{333} \equiv f_0(\lambda)x^3 + f_1(\lambda)x^2 + f_2(\lambda)x + f_3(\lambda), \tag{41}$$

where $f_s(\lambda)$ stands for the polynomial of the s-th degree in λ. Now, a given line $\lambda = \lambda_0$ of pencil (40) is a tangent to $\tilde{C}_{\breve{x}}$ iff it has at least a two-point contact with $\tilde{C}_{\breve{x}}$. This is equivalent to saying that eqn. (41) has a repeated root for $\lambda = \lambda_0$, which is the case when [see, e.g., Ref. 17]

$$T \equiv \frac{a^3}{27} + \frac{b^2}{4} = 0, \tag{42}$$

where

$$a \equiv \frac{3f_0f_2 - f_1^2}{3f_0^2} \tag{43}$$

and

$$b \equiv \frac{2f_1^3}{27f_0^3} - \frac{f_1f_2}{3f_0^2} + \frac{f_3}{f_0}, \tag{44}$$

that is, when

$$\Delta \equiv f_1^2f_2^2 - 4f_0f_2^3 - 4f_1^3f_3 - 27f_0^2f_3^2 - 18f_0f_1f_2f_3 = 0. \tag{45}$$

This polynomial is of the sixth order, so the class of a point-cubic is, in general, six [13–16]. This limit is, however, attained by non-singular cubics; the class of singular cubics is less than six.

To see that, we shall again change our coordinate system so that the singular point $\breve{z}_i^{\odot}$ coincides with one of the remaining two vertices, say V_2

$$\varrho \breve{z}_i^{\odot} = (0,1,0), \qquad \varrho \neq 0. \tag{46}$$

As

$$\frac{\partial \tilde{C}_{\breve{x}}}{\partial \breve{x}_1} = 3\tilde{c}_{111}\breve{x}_1^2 + 2\left(\tilde{c}_{112}\breve{x}_1\breve{x}_2 + \tilde{c}_{113}\breve{x}_1\breve{x}_3\right) + \tilde{c}_{122}\breve{x}_2^2 + \tilde{c}_{123}\breve{x}_2\breve{x}_3 + \tilde{c}_{133}\breve{x}_3^2, \tag{47}$$

$$\frac{\partial \tilde{C}_{\breve{x}}}{\partial \breve{x}_2} = 3\tilde{c}_{222}\breve{x}_2^2 + 2\left(\tilde{c}_{122}\breve{x}_1\breve{x}_2 + \tilde{c}_{223}\breve{x}_2\breve{x}_3\right) + \tilde{c}_{112}\breve{x}_1^2 + \tilde{c}_{123}\breve{x}_1\breve{x}_3 + \tilde{c}_{233}\breve{x}_3^2, \tag{48}$$

$$\frac{\partial \tilde{C}_{\breve{x}}}{\partial \breve{x}_3} = 3\tilde{c}_{333}\breve{x}_3^2 + 2\left(\tilde{c}_{133}\breve{x}_1\breve{x}_3 + \tilde{c}_{233}\breve{x}_2\breve{x}_3\right) + \tilde{c}_{113}\breve{x}_1^2 + \tilde{c}_{123}\breve{x}_1\breve{x}_2 + \tilde{c}_{223}\breve{x}_2^2, \tag{49}$$

must all vanish at $\breve{x}_i = \breve{z}_i^{\odot}$, we have

$$\tilde{c}_{122} = \tilde{c}_{222} = \tilde{c}_{223} = 0. \tag{50}$$

Hence, both

$$f_2 \equiv \tilde{c}_{122}\lambda^2 + \tilde{c}_{123}\lambda + \tilde{c}_{133} = \tilde{c}_{123}\lambda + \tilde{c}_{133} \equiv \hat{f}_1 \tag{51}$$

and

$$f_3 \equiv \tilde{c}_{222}\lambda^3 + \tilde{c}_{223}\lambda^2 + \tilde{c}_{233}\lambda + \tilde{c}_{333} = \tilde{c}_{233}\lambda + \tilde{c}_{333} \equiv \bar{f}_1 \tag{52}$$

become *first* degree polynomials, which means that the polynomial

$$\Delta \equiv \Delta^{\odot} = f_1^2\hat{f}_1^2 - 4f_0\hat{f}_1^3 - 4f_1^3\bar{f}_1 - 27f_0^2\bar{f}_1^2 - 18f_0f_1\hat{f}_1\bar{f}_1 = 0 \tag{53}$$

is of the *fourth* order only. We thus see that a singular proper point-cubic is of the fourth class. Strictly speaking, the last statement is only true for crunodal and acnodal cubics; cuspidal cubics are of the *third* class. This is not difficult to see. Thus, the polar conic $\hat{Q}_{\breve{x}}(\breve{z}_i^{\odot})$ of the singular point $\breve{z}_i^{\odot}$ with respect to $\tilde{C}_{\breve{x}}^{\odot}$ meeting the constraints given by eqn. (50) is

$$\hat{Q}_{\breve{x}}(\breve{z}_i^{\odot}) \equiv \sum_{i=1}^{3} \breve{z}_i^{\odot} \frac{\partial \tilde{C}_{\breve{x}}^{\odot}}{\partial \breve{x}_i} = \frac{\partial \tilde{C}_{\breve{x}}^{\odot}}{\partial \breve{x}_2} = \tilde{c}_{112}\breve{x}_1^2 + \tilde{c}_{123}\breve{x}_1\breve{x}_3 + \tilde{c}_{233}\breve{x}_3^2 = 0. \tag{54}$$

For $\breve{z}_i^{\odot}$ to be a cusp, this polar conic must be a repeated line, which is the case when

$$\hat{\delta} \equiv \tilde{c}_{123}^2 - 4\tilde{c}_{112}\tilde{c}_{233} = 0. \tag{55}$$

Now, as the only terms in $\Delta^{\odot}$ that contain λ in the fourth power are

$$f_1^2\hat{f}_1^2 - 4f_1^3\bar{f}_1 = \tilde{c}_{112}^2\left(\tilde{c}_{123}^2 - 4\tilde{c}_{112}\tilde{c}_{233}\right)\lambda^4 + g_3(\lambda), \tag{56}$$

where $g_3(\lambda)$ is a *third* order polynomial, we see that the class of a cuspidal point-cubic $\tilde{C}_{\breve{x}}^{\odot}$ is indeed three.

If we combine these observations with those of Subsection 3.1, we arrive at the remarkable fact that while the webs associated with past/future events are *four*-webs ($m_{\rm cr} = m_{\rm ac} = 4$), that corresponding to the moment of the present is a *three*-web ($m_{\rm cu} = 3$). This is nothing but a direct manifestation of the above-mentioned distinguishing role that the present moment plays in the structure of our cubic-borne temporal dimension. And as it is only the present that we can directly perceive, it is natural to assume that it will be only the present-related web which is accessible to our senses. So, identifying each foliation of this three-web with a single dimension of the perceived space gives us a very natural and elegant qualitative explanation of the observed dimensionality of space! Moreover, in our approach this fact is closely linked with the very existence of the arrow of time, for proper cubic curves are the lowest order curves featuring singular points (let us recall that a proper curve of the second order, a conic, has no singular point(s)). Thus, our theory goes much farther in this respect than any other of the current physical theories [18-19, and references therein]. There is another crucial point which illustrates and justifies the potential significance of the web view of space. If one takes the group of all differentiable transformations of a manifold on which an m-web is given, and studies local web invariants relative to transformations of this group, one finds [9, 10] that $m = 3$ is the *first* case where there are local invariants, i.e. where the structure is locally *non*-trivial. Thus, confronted with the old enigma of why observed space is endowed with just three dimensions, the web picture offers a solid conceptual framework within which an answer to this question can be given: space is three-dimensional simply because it represents nothing but the lowest possible non-trivial web! In other words, Nature loves not only non-triviality ($m \geq 3$), but simplicity ($m = 3$) as well.

4.0 Cubic Web Space-Times over Galois Fields of Characteristic Two

4.1 Basic Properties of Galois Fields of Characteristic Two

In order to appreciate the intricacy of the link between the internal structure of the time dimension and the multiplicity of spatial coordinates, we shall discuss in some detail the case when the projective plane is defined over a Galois field of characteristic two. We begin by reviewing briefly some of the necessary background concerning such fields; the reader who is interested in further details should consult, for example, [20; see also Ref. 5].

A set of elements F that is closed under two operations called addition and multiplication is a *field* if:

1. it is an Abelian (i.e. consisting of commuting elements) group under addition with identity 0;
2. all the elements excluding zero form an Abelian group under multiplication with identity 1; and
3. $u(v+w) = uv + uw$, $(u+v)w = uw + vw$ for all u, v, and $w \in \mathsf{F}$.

Since the elements of a field F form a group under addition, every element $w \in \mathsf{F}$ has an additive order, namely the least positive integer r for which $r.w = 0$. This additive order, being the same for all non-zero elements of F, is called the *characteristic* of F; it is said to be zero if all non-zero elements have infinite order. If F has a finite characteristic, this must be a prime p. The number of elements of a field with a finite characteristic p is a power p^n, n being a positive integer. For every prime p and every positive integer n, there exists a field having exactly p^n elements; this field is called the *Galois field* of order p^n and denoted by $\mathsf{GF}(p^n)$. Every element w of $\mathsf{GF}(p^n)$ satisfies the equation

$$w^{p^n} - w = 0, \tag{57}$$

i.e. the set of elements of the field is formed by the zeros of the polynomial $w^{p^n} - w$. For any field $\mathsf{GF}(p^n)$ there exists an element s, called the primitive root, such that

$$\mathsf{GF}(p^n) = \left\{0, 1, s, s^2, \ldots, s^{p^n-2} | s^{p^n-1} = 1\right\}. \tag{58}$$

Thus, for $p = 2$ eqn. (57) reads

$$w^{2^n} - w = 0 \tag{59}$$

and it follows from there that *every* element of $\mathsf{GF}(2^n)$ is a square,

$$w = \left(w^{2^{n-1}}\right)^2. \tag{60}$$

Moreover,

$$0 = 2w = w + w \qquad \Leftrightarrow \qquad +w = -w, \tag{61}$$

which implies that an element of $\mathsf{GF}(2^n)$ has only *one* square root.

The above situation is diametrically different from the case $p > 2$, for among the $p^n - 1$ non-zero elements of a field of odd order there are $(p^n-1)/2$ non-squares and the same number of squares – the latter being the even powers of the primitive element s. To see this, first note that the primitive element itself satisfies eqn. (57), so that

$$s^{p^n-1} = 1, \tag{62}$$

and, as a consequence,

$$s^{(p^n-1)/2} = -1. \tag{63}$$

Now, for the element s^ℓ (ℓ being a positive integer) to be a square, i.e. $s^\ell = w^2$, we find from eqn. (57) the constraint

$$\begin{aligned} +1 = w^{p^n-1} &= \left(w^2\right)^{(p^n-1)/2} = \\ &= \left(s^\ell\right)^{(p^n-1)/2} = \left(s^{(p^n-1)/2}\right)^\ell = (-1)^\ell , \end{aligned} \tag{64}$$

which is indeed met only for ℓ even.

Next, let us cast a brief glance at the properties of the binomial theorem over finite fields. As is well-known, the binomial theorem describes how to expand the sum of two elements u and v raised to a positive integer power $\ell \geq 1$

$$(u+v)^\ell = u^\ell + \sum_{k=1}^{\ell-1} \frac{\ell!}{k!(\ell-k)!} u^{\ell-k} v^k + v^\ell. \tag{65}$$

Here it is crucial to observe that if ℓ is a prime, then it divides $\ell!/k!(\ell-k)!$ for all k, $1 \leq k \leq \ell - 1$, so that $(u+v)^\ell = u^\ell + v^\ell \pmod{\ell}$; hence, in a field of characteristic p the binomial theorem simply reads

$$(u+v)^p = u^p + v^p. \tag{66}$$

As an interesting corollary we have,

$$(u+v+\ldots+w)^{p^q} = u^{p^q} + v^{p^q} + \ldots + w^{p^q}, \tag{67}$$

where q is a positive integer; in characteristic two

$$(u+v+\ldots+w)^{2^q} = u^{2^q} + v^{2^q} + \ldots + w^{2^q}. \tag{68}$$

Having compiled some basic facts about finite fields we are now able to examine the solutions of a quadratic equation over $\mathsf{GF}(2^n)$.

4.2 Quadratic Equations over $\mathsf{GF}(2^n)$

In this section our primary attention will be focused on finding solutions to the quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0, \tag{69}$$

In the case of a field F of odd and/or zero characteristic this task is quite easy as the equation can be completed to the square

$$\left(x + \frac{b}{2a}\right)^2 - \frac{\delta}{(2a)^2} = 0, \qquad \delta \equiv b^2 - 4ac \tag{70}$$

from which it follows that it has:

– one (double) root if $\delta = 0$;
– two distinct roots if δ is a non-zero square in F; and
– no solutions if δ is a non-square in F.

It is, however, immediately evident that this strategy cannot be employed for fields of characteristic two, where eqn. (70) becomes singular. The strategy pursued in this case is as follows ([20], pp. 3–4; see also [5]). If $b = 0$, eqn. (69) can be rewritten, in light of eqns. (60) and (68), as a perfect square,

$$\left(\sqrt{a}x + \sqrt{c}\right)^2 = 0, \tag{71}$$

from which it is obvious that it has just one (double) root

$$x = \sqrt{\frac{c}{a}} \tag{72}$$

due to eqn. (61). For $b \neq 0$ we introduce the new variables

$$z \equiv \frac{a}{b}x, \qquad \Theta \equiv \frac{ac}{b^2}, \tag{73}$$

in which eqn. (69) acquires the form

$$z^2 + z + \Theta = 0. \tag{74}$$

Next, we define

$$D(w) \equiv w + w^2 + w^4 + \ldots + w^{2^{n-1}}, \tag{75}$$

and exploiting eqns. (59), (61) and (68) verify that

$$D^2(w) + D(w) = 0, \tag{76}$$

that is

$$D(w) = 0, \quad \text{or} \quad D(w) = 1. \tag{77}$$

On the other hand, we note that

$$\begin{aligned} D(z^2+z+\Theta) &= (z^2+z+\Theta) + (z^2+z+\Theta)^2 + \ldots + (z^2+z+\Theta)^{2^{n-1}} \\ &= (z^2+z+\Theta) + (z^4+z^2+\Theta^2) + \ldots + \left(z^{2^n} + z^{2^{n-1}} + \Theta^{2^{n-1}}\right) \\ &= \left(z^2 + \ldots + \left(z^2\right)^{2^{n-1}}\right) + \left(z + \ldots + z^{2^{n-1}}\right) + \left(\Theta + \ldots + \Theta^{2^{n-1}}\right) \\ &= D(z^2) + D(z) + D(\Theta) \end{aligned} \tag{78}$$

and

$$\begin{aligned} D^2(z) \equiv \left(z + z^2 + \ldots + z^{2^{n-1}}\right)^2 &= z^2 + \left(z^2\right)^2 + \ldots + \left(z^2\right)^{2^{n-1}} \\ &= D\left(z^2\right) \end{aligned} \tag{79}$$

so that

$$D(z^2+z+\Theta) = D^2(z) + D(z) + D(\Theta) = D(\Theta). \tag{80}$$

Since $D(0) = 0$, the last expression implies that eqn. (74) can only be solved provided that

$$D(\Theta) = 0. \tag{81}$$

Furthermore, it is a simple matter to check that if $z = z_0$ is a solution to eqn. (74), so also is $z_0 + 1$:

$$(z_0+1)^2 + (z_0+1) + \Theta = z_0^2 + 1 + z_0 + 1 + \Theta = z_0^2 + z_0 + \Theta = 0, \tag{82}$$

where we have again used eqn. (61). So, eqn. (74) has in $\mathsf{GF}(2^n)$:

– two distinct roots if $D(\Theta) = 0$, and

– no solutions if $D(\Theta) = 1$.

All in all, we see that in a finite field of characteristic two, the quadratic equation (69) is characterized by *two* distinct, mutually complementary "modes" depending on whether b is zero or not; in the former case it has just one (double) solution (eqn. (72)), whereas in the latter case it yields no or two distinct roots according as $D(ac/b^2)$ (see eqn. (75)) is unity or zero, respectively. At this point our characteristic two background is strong enough to handle the question of the fine structure of corresponding space-times.

4.3 "Peculiar" Properties of Time and Space over $\mathsf{GF}(2^n)$

We return to eqn. (32), which now acquires the form

$$Q^*_{\breve{x}}(\vartheta) = \left(\sqrt{(\vartheta - \kappa)}\,(\breve{x}_1 - \kappa\breve{x}_3) + \breve{x}_2\right)^2 = 0 \tag{83}$$

telling us that, in $\mathsf{GF}(2^n)$, the conic $Q^*_{\breve{x}}(\vartheta)$ is a repeated line *ir*respective of the value of the parameter ϑ. This means that all the proper cubics of the pencil (28) are cuspidal i.e. that the time dimension loses its arrow and consists solely of the moments of the present! This is, indeed, a very peculiar metamorphosis of the temporal, and in order to get a deeper understanding of what is going on here, we shall give a general account of the properties of a proper singular point-cubic $\tilde{C}^{\odot}_{\breve{x}}$ in $\mathsf{GF}(2^n)$. If the coordinate system is selected as before, i.e. the singular point coincides with V_2 (eqn. (46)), then the cubic is given by eqns. (38) and (50), and the polar conic of the singular point with respect to it is given by eqn. (54), which can be rewritten as a quadratic equation

$$\tilde{c}_{112}x^2 + \tilde{c}_{123}x + \tilde{c}_{233} = 0, \tag{84}$$

where $x \equiv \breve{x}_1/\breve{x}_3$; obviously, if this equation has two distinct roots, one double root or no root, then the polar conic is a pair of lines, a repeated line or just the singular point $\breve{z}^{\odot}_i$, and, accordingly, the singular point is a crunode, a cusp, or an acnode, respectively. From the comparison of eqns. (84) and (69) we see that $\tilde{c}_{123}$ plays in the former equation the same role as b in the latter, and from what we found in the previous section it then follows that in $\mathsf{GF}(2^n)$ there exist two different kinds of pencils of point-cubics depending on whether the equations of the cubics of the pencil do or do not lack the completely "skew" term, i.e. the term proportional to $\breve{x}_1\breve{x}_2\breve{x}_3$,: while in the former case the pencil features cuspidal cubics only, in the latter case it in general consists of both crunodal ($D\left(\tilde{c}_{112}\tilde{c}_{233}/(\tilde{c}_{123})^2\right) = 0$) and acnodal ($D\left(\tilde{c}_{112}\tilde{c}_{233}/(\tilde{c}_{123})^2\right) = 1$) cubics, but is devoid of any cuspidal one. Correspondingly (see Sect. 3.1), we have two distinct structures that the time dimension over a finite field of characteristic two may exhibit: the homogeneous, "pure present" pattern ($\tilde{c}_{123} = 0$) and the "presentless" arrow ($\tilde{c}_{123} \neq 0$). In particular, as eqn. (28) has no completely skew term, the corresponding time dimension is thus of the first kind. At this point, it is worth recalling [see Ref. 5, Sects. 3–5] that also time dimensions borne by a specific, 2%1 type of pencil of conics exhibit the same structural pattern in $\mathsf{GF}(2^n)$, according as the common nucleus of the conics lies on or off the ideal line.

It is, however, not only time whose structure undergoes a dramatic change as we switch from the field of reals to $\mathsf{GF}(2^n)$; space is even subject to a more pronounced transformation, for it is found to lose a couple of its dimensions! In order to verify this statement, as well as to see what kind of dimensional reduction is taking place, it is sufficient to return to the polynomial (53), whose order is equal to the class of a singular point-cubic, $\tilde{C}^{\odot}_{\breve{x}}$. As we are now in characteristic two, i.e. in a "world" where any multiple of two is equivalent to zero and where there is no distinction between "+" and "−" (eqn. (61)), this polynomial becomes the square of a *quadratic* polynomial,

$$\Delta^{\odot}_{p=2} = \left(f_1\hat{f}_1 + f_0\bar{f}_1\right)^2 = \left(\tilde{c}_{123}\tilde{c}_{112}\lambda^2 + g_1(\lambda)\right)^2 = 0. \tag{85}$$

Table 2: Basic properties of the cubic web space-times in the field of real numbers, R, as well as in Galois fields of characteristic two, $\mathsf{GF}(2^n)$.

Temporal event	Dimensionality of Space		
	R	$\mathsf{GF}(2^n)$	
		$\tilde{c}_{123} = 0$	$\tilde{c}_{123} \neq 0$
past	4	–	2
present	3	1	–
future	4	–	2

Hence, a singular proper point-cubic is of class *two* in $\mathsf{GF}(2^n)$. Again, this is valid only for crunodal and acnodal cubics. The class of a cuspidal cubic is *one*; for $\tilde{C}^{\odot}_{\breve{x}}$ has a cusp if and only if $\tilde{c}_{123} = 0$, in which case eqn. (85) represents the square of a *linear* polynomial. And thus, in a finite field of characteristic two, the tangent lines to a crunodal or acnodal cubic form a two-web ($m_{cr} = m_{ac} = 2$), whilst those enveloping a cuspidal cubic represent just a one-web ($m_{cu} = 1$). All in all, we find that two distinct "pathological" space-time structures live in a $\mathsf{GF}(2^n)$ environment: the $\tilde{c}_{123} = 0$ space-time, whose time dimension contains only present moments and where an event-related space has a single dimension, and the $\tilde{c}_{123} \neq 0$ configuration, where time is endowed with the domains of past and future, but lacks the present, and each event-related space is two-dimensional – as depicted in a succinct form in Table 2. We thus find out that the difference between (the moment of) the present and (an event of) the past/future is here more marked than in the field of real numbers: for not only have the corresponding event-related spaces a different dimensionality, but the two kinds of event cannot even share one and the same temporal dimension!

As the attentive reader may already have noticed, one of the most important messages the above discussion tries to impart is, undoubtedly, the existence of a profound, albeit subtle and intricate, connection between the *multiplicity* of spatial coordinates and the *intrinsic* structure of the time dimension. (This finding is not completely new for it is already inherent, although in a slightly different context, in our original conic model of space-time [2, 3, 5]; in the present work, however, it is given a new twist for it is formulated, in contradistinction to the conic model, in a *projectivelly* invariant way.) In other words, a clue to a plausible explanation of the observed dimensionality of space lies with the internal structure of time! Hence, it is quite obvious why even the most sophisticated physical theories, in particular superstrings and supermembranes, "have yet no answer to the question of why our universe appears to be four-dimensional, let alone why it appears to have signature (1, 3)" [21, see also Ref. 18, 19]; it is simply because all these theories treat the time dimension as being structureless and homogeneous, like a spatial coordinate.

5.0 "Cuspidal" Space-Times

5.1 Self-Duality of a Cuspidal Cubic

A projective plane, as a geometric object in its own right, is endowed with many interesting properties, one of which, the so-called *principle of duality*, opens up for us a new, conceptually different – yet no less attractive and intriguing – perspective from which to address the question of the dimensionality of space-time. The principle, loosely speaking, says that a point, $\breve{x}_i$, and a (straight)line, ζ_i, are mutually dual concepts. That means that instead of viewing the points of the projective plane as the fundamental entities, and the lines as loci of points, we may equally well take the lines as primary geometric entities and define points in terms of lines, characterizing a point by the complete set (pencil) of lines passing through it. Going back to our discussion of the class of a point-cubic (Sect. 3.2.), it thus follows that a cuspidal cubic in the field of real numbers, being of class three, is a *self-dual* curve, i.e. the tangent lines to it form an aggregate described in terms of line-coordinates by an equation of the third degree – the degree of the original equation (eqn. (1) or (38)). To make this feature visible to the eye, as well as to facilitate our subsequent reasoning, we shall take the equation of a cuspidal cubic, $C_{\breve{x}}^{\prec}$, in its canonical form

$$C_{\breve{x}}^{\prec} = \breve{x}_1 \breve{x}_2^2 + \kappa \breve{x}_3^3 = 0, \quad \kappa \neq 0. \tag{86}$$

This cubic has the cusp, $\breve{z}_i^{\prec}$, located at V_1,

$$\varrho \breve{z}_i^{\prec} = (1, 0, 0), \quad \varrho \neq 0, \tag{87}$$

the inflexional point, $\breve{z}_i^{\sim}$, identical with V_2,

$$\varrho \breve{z}_i^{\sim} = (0, 1, 0), \quad \varrho \neq 0, \tag{88}$$

and its cuspidal and inflexional tangents coincide with the lines

$$\breve{x}_2 = 0 \tag{89}$$

and

$$\breve{x}_1 = 0, \tag{90}$$

respectively – see Fig. 3. Our task is to find the tangential equation of this cubic, i.e. to rewrite eqn. (86) in line-coordinates ζ_i.

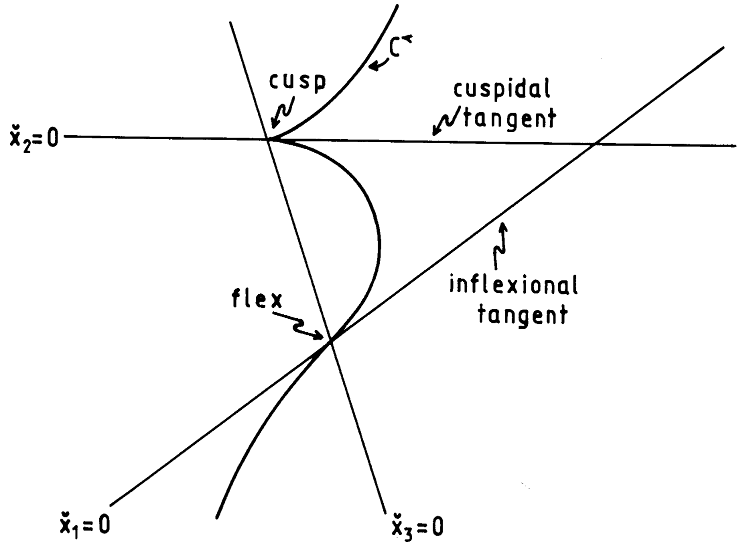

Figure 3: The shape of the cuspidal point-cubic defined by eqn. (86).

This can be achieved in the following way (see, for example, Ref. [15]). Let us consider a generic point-cubic $C_{\breve{x}}$, eqn. (1). We look for the coordinates ζ_i of the polar line to a generic point $\breve{z}_i$ with respect to $C_{\breve{x}}$, for this line becomes a tangent to $C_{\breve{x}}$ if $\breve{z}_i$ lies on the latter. From Sect. 2.1. it follows that the equation of this line can be found by twice applying the polar operator, eqn. (4), to eqn. (1),

$$\begin{aligned} 0 &= \sum_{k,l=1}^{3} \breve{z}_k \breve{z}_l \frac{\partial^2 C_{\breve{x}}}{\partial \breve{x}_k \partial \breve{x}_l} = \sum_{l=1}^{3} \breve{z}_l \frac{\partial}{\partial \breve{x}_l} \left(\sum_{k=1}^{3} \breve{z}_k \frac{\partial C_{\breve{x}}}{\partial \breve{x}_k} \right) = \\ &= \sum_{l=1}^{3} \breve{z}_l \frac{\partial}{\partial \breve{x}_l} \left(3 \sum_{i,j,k=1}^{3} \breve{z}_k c_{kij} \breve{x}_i \breve{x}_j \right) = 6 \sum_{j=1}^{3} \left(\sum_{k,l=1}^{3} \breve{z}_k \breve{z}_l c_{klj} \right) \breve{x}_j, \end{aligned} \tag{91}$$

which after comparison with the equation of a line,

$$0 = \sum_{j=1}^{3} \zeta_j \breve{x}_j, \tag{92}$$

yields

$$\varrho \zeta_j = \sum_{k,l=1}^{3} c_{jkl} \breve{z}_k \breve{z}_l, \quad \varrho \neq 0, \tag{93}$$

or, for the points on $C_{\breve{x}}$,

$$\varrho \zeta_j = \sum_{k,l=1}^{3} c_{jkl} \breve{x}_k \breve{x}_l, \quad \varrho \neq 0. \tag{94}$$

Now, as eqn. (86) is a particular form of eqn. (1) with the only non-zero coefficients being

$$c_{122} = \frac{1}{3} \quad \text{and} \quad c_{333} = \kappa, \tag{95}$$

eqn. (94) implies

$$\varrho \zeta_1 = \frac{1}{3} \breve{x}_2^2, \quad \varrho \zeta_2 = \frac{2}{3} \breve{x}_1 \breve{x}_2, \quad \varrho \zeta_3 = \kappa \breve{x}_3^2. \tag{96}$$

The last step is to rewrite eqn. (86) as

$$(\breve{x}_1\breve{x}_2)^2\,\breve{x}_2^2 = \kappa^2\breve{x}_3^6 \tag{97}$$

and combine the latter with eqns. (96) to get

$$C_\zeta^{\prec} = \frac{27}{4}\zeta_1\zeta_2^2 - \frac{1}{\kappa}\zeta_3^3 = 0, \tag{98}$$

a line-*cubic*.

5.2 A Pencil of Cuspidals and its Dual

Although cuspidal point-cubics are self-dual objects when considered individually, this is not the case with their aggregates. Thus, for example, their pencils, i.e. *linear* one-parametric sets, emerge, in general, as *non*-linear aggregates when described in terms of line-coordinates.

As a simple illustrative example, we shall consider a pencil

$$C_{\breve{x}}^{\prec}(\vartheta) = \breve{x}_1\breve{x}_2^2 + \vartheta\left(\breve{x}_2^3 + \breve{x}_3^3\right) = 0, \tag{99}$$

which features two reducible cubics, viz.

$$\vartheta = 0: \quad \breve{x}_1 = 0 \ \vee \ \breve{x}_2^2 = 0 \tag{100}$$

and

$$\vartheta = \pm\infty: \quad \breve{x}_2^3 + \breve{x}_3^3 = (\breve{x}_2 + \breve{x}_3)\left(\breve{x}_2^2 - \breve{x}_2\breve{x}_3 + \breve{x}_3^2\right) = 0, \tag{101}$$

and a couple of base points,

$$B_1: \ \varrho\breve{x}_i = (1,0,0), \quad \varrho \neq 0, \tag{102}$$

and

$$B_2: \ \varrho\breve{x}_i = (0,1,-1), \quad \varrho \neq 0. \tag{103}$$

In order to see that the (proper) cubics of this pencil are all cuspidal, we put eqn. (99) into the form

$$C_{\breve{x}}^{\prec}(\vartheta) = (\breve{x}_1 + \vartheta\breve{x}_2)\breve{x}_2^2 + \vartheta\breve{x}_3^3 = 0, \tag{104}$$

which is formally identical to eqn. (86) for

$$\varrho\breve{x}'_1 = \breve{x}_1 + \vartheta\breve{x}_2, \quad \varrho\breve{x}'_2 = \breve{x}_2, \quad \varrho\breve{x}'_3 = \breve{x}_3, \tag{105}$$

and $\vartheta = \kappa$. From this comparison it further follows that these cuspidals share their cusps (B_1) as well as their cuspidal tangents ($\breve{x}_2 = 0$), while their flexes are situated on the line $\breve{x}_3 = 0$ and have coordinates $\varrho\breve{z}_i^{\sim}(\vartheta) = (\vartheta,-1,0)$, and their inflexional tangents coincide with the "variable" line $\breve{x}_1 + \vartheta\breve{x}_2 = 0$ – see Fig. 4a. From eqns. (1) and (104) we find

$$c_{122} = \frac{1}{3}, \quad c_{222} = c_{333} = \vartheta, \quad \text{other } c_{ijk} = 0, \tag{106}$$

which after insertion into eqn. (94) yield

$$\varrho\zeta_1 = \frac{1}{3}\breve{x}_2^2, \quad \varrho\zeta_2 = \frac{2}{3}\breve{x}_1\breve{x}_2 + \vartheta\breve{x}_2^2, \quad \varrho\zeta_3 = \kappa\breve{x}_3^2. \tag{107}$$

Combining these equations with eqn. (104) squared as

$$(\breve{x}_1 + \vartheta\breve{x}_2)^2\breve{x}_2^4 = \vartheta^2\breve{x}_3^6, \tag{108}$$

we arrive, after a little algebra, at

$$C_\zeta^{\prec}(\vartheta) = \vartheta\zeta_1(\zeta_2 - \vartheta\zeta_1)^2 - \frac{4}{27}\zeta_3^3 = 0, \tag{109}$$

which is clearly *not* a pencil since it depends on ϑ in a non-linear way. Reversing the procedure, we find that the pencil

$$\bar{C}_{\zeta}^{\prec}(\vartheta) = (\zeta_1 + \vartheta\zeta_2)\zeta_2^2 + \vartheta\zeta_3^3 = 0, \tag{110}$$

the dual counterpart to pencil (104), acquires in the point-representation the form

$$\bar{C}_{\breve{x}}^{\prec}(\vartheta) = \vartheta\breve{x}_1(\breve{x}_2 - \vartheta\breve{x}_1)^2 - \frac{4}{27}\breve{x}_3^3 = 0. \tag{111}$$

This last equation is a cubic equation in the variable ϑ and this means that through a generic point of the plane there pass, in general, *three* distinct cubics of the aggregate, i.e. that this aggregate represents a *three*-web – see Fig. 4b.

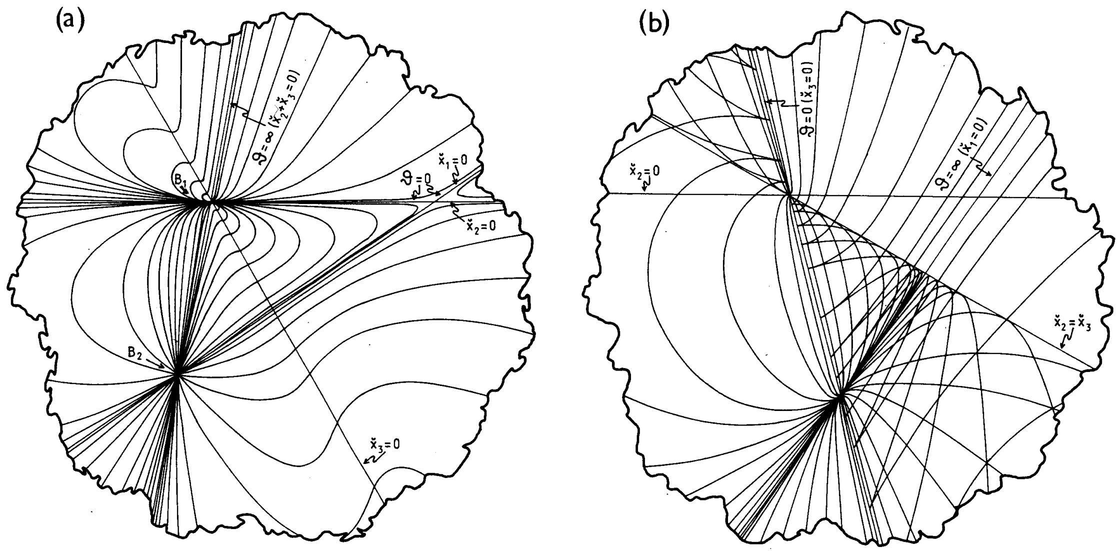

Figure 4: The images (a) of the pencil of cuspidal cubics defined by eqn. (99) (⇔ eqn. (104)) and (b) of its dual pencil in a point-representation, eqn. (111). The symbols and notation are explained in the text.

5.3 Cuspidal Space-Times

From what has just been observed, a conceptually new idea for a possible coupling between time and space emerges. At the very beginning, we take time and space to be *both* represented by pencils of cuspidal cubics. Time is represented by a point-pencil, while space is represented by the dual line-pencil. So, we view both time and space originally as *one*-dimensional configurations, dual to each other. It is only in the second step, when this appealing symmetry (duality) is broken by treating both the sets in *one and the same* representation, that of points, where the difference in dimensionality between the two concepts comes into play; while the point-pencil remains intact, the line-pencil becomes, in general, a non-linear assembly of point-cubics. So, whereas time retains its single dimension, space acquires some extra dimensions – their total number amounting to the degree of the non-linearity in question, which obviously cannot exceed three. Thus, the line-pencil of eqn. (110), being equivalent to a three-web, eqn. (111), generates three spatial coordinates. However, there exist also pencils that give rise to only two- or even just one-dimensional spaces.

To justify the last statement, we shall consider a point-pencil of the form

$$C_{\breve{x}}^{\prec}(\vartheta) = (\breve{x}_1 + \vartheta\breve{x}_2)\breve{x}_2^2 + \breve{x}_3^3 = 0. \tag{112}$$

Although this equation varies only slightly from eqn. (104), the structure of the two pencils is quite different, for the present pencil has only one degenerate (the triple line $\breve{x}_2 = 0$) and a single base point (V_1, the cusp).

Following the same procedure as before, we find that its dual,

$$\bar{C}_{\zeta}^{\prec}(\vartheta) = (\zeta_1 + \vartheta\zeta_2)\zeta_2^2 + \zeta_3^3 = 0, \tag{113}$$

acquires in a point-representation the form

$$\bar{C}_{\breve{x}}^{\prec}(\vartheta) = \breve{x}_1 \left(\breve{x}_2 - \vartheta\breve{x}_1\right)^2 - \frac{4}{27}\breve{x}_3^3 = 0, \tag{114}$$

showing quadratic dependence on ϑ; hence, it represents a two-web and, consequently, induces a two-dimensional space. Another illustrative example is provided by the point-pencil

$$C_{\breve{x}}^{\prec}(\vartheta) = (\breve{x}_1 + \vartheta\breve{x}_3)\breve{x}_2^2 + \breve{x}_3^3 = 0 \tag{115}$$

whose cuspidals share both the cusps (V_1) and flexes (V_2), and whose only degenerate cubic consists of the lines $\breve{x}_2 = 0$ (counted twice) and $\breve{x}_3 = 0$. Its dual,

$$\bar{C}_{\zeta}^{\prec}(\vartheta) = (\zeta_1 + \vartheta\zeta_3)\zeta_2^2 + \zeta_3^3 = 0, \tag{116}$$

when considered as a locus of points, appears as

$$\bar{C}_{\breve{x}}^{\prec}(\vartheta) = \frac{27}{4}\breve{x}_1\breve{x}_2^2 - \left(\breve{x}_3 - \vartheta\breve{x}_1\right)^3 = 0. \tag{117}$$

Although this equation is again a cubic equation in ϑ, it does not represent a genuine three-web as we would expect, but only a one-web. For, if we put

$$\sqrt[3]{\frac{27}{4}\breve{x}_1\breve{x}_2^2} \equiv a \tag{118}$$

and

$$\breve{x}_3 - \vartheta\breve{x}_1 \equiv x, \tag{119}$$

we can rewrite it as

$$0 = a^3 - x^3 = (a - x)\left(a^2 + ax + x^2\right) \tag{120}$$

which is seen to have only one real root ($x = a$) for each a, the other two solutions being imaginary ($x_{\pm} = a(-1 \pm i\sqrt{3})/2$). From eqn. (119) it follows that the same holds for ϑ. So, out of the three foliations of the web represented by eqn. (117) there is only one that is real, and the "effective" dimensionality of the corresponding space is thus one.

It is striking to observe that the number of foliations of a given web, i.e. the multiplicity of spatial dimensions, is very sensitive to the structure of the original point-pencil to which the web is dually related, i.e. sensitive to the properties of the corresponding time coordinate. Hence, we again confirm our earlier claim (Sect. 4.3.) about an intimate link between the intrinsic properties of time and the dimensionality of space. This connection is here, however, of a different origin than that discussed in Sects. 3 and 4: whereas in the first approach we stick to one and the same point-pencil (eqn. (28)) and change the ground field ($\mathsf{R} \rightarrow \mathsf{GF}(2^n)$) to reveal this feature, here we keep the field fixed (R) and vary instead the shape of the point-pencil (eqns. (104), (112) and (115)).

There is, however, another important message stemming from the comparison of the two approaches. While in the cubic web model time stands on a different footing from space – the former being composed of cubics (Sect. 3.1.), the latter of lines (Sect. 3.2.), in the present model both concepts are put on an equal footing, as they both have cubics as their representations (Sect. 5.3.). But the price to be paid for this is that the time dimension loses its arrow; indeed, as all the point-pencils (eqns. (104), (122) and (115)) consist only of cuspidal cubics, looking back at Sect. 3.1. (Table 1) we see that the time dimensions they generate exhibit only one kind of event, the present. In this respect, the cuspidal approach is more akin to a physicist's view of time and space, while the cubic web approach is found to be more appropriate for considering how we human beings perceive or experience time and space [22–28].

6.0 Conclusion

We have constructed an extension of the theory of pencil-generated space-times, based on specific pencils of singular proper point-cubics. This generalized theory, while leaving all the key features of the original concept of the arrow of time intact, features the moment of the present as standing on a different footing than an event of the past/future – the fact well-established and firmly supported by our everyday experience. However, its major contribution to the study of space-time is the profound link that it provides between the intrinsic structure of the time dimension and the dimensionality of space: it certainly has something to offer to current attempts towards the ultimate unification of fundamental interactions, and it may even turn out to lend us a helping hand in our quest for a possible reconciliation between quantum mechanics and general relativity. Moreover, the theory as a whole provides a broader setting for the investigation of psychological space-times, also offering the possibility of further extension to spaces generated by *non*-hexagonal m-webs. Finally, it opens up an entirely new area of application of web geometry.

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