Algebra Colloquium **11**:2 (2004) 287–296

Algebra Colloquium

Orthogonal Ternary Algebras and Thomas Sums*

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Received 30 November 2001
Revised 18 September 2002

Communicated by K.P. Shum

Abstract. A comtrans algebra is said to decompose as a Thomas sum of two subalgebras if it is a direct sum at the module level and its algebra structure is obtained from the subalgebras and their mutual interactions. Internal and external versions of the Thomas sum are defined, and shown to be equivalent. The construction yields a duality that generalizes Weyl's unitary trick. Orthogonal algebras are shown to decompose as iterated Thomas sums of Euclidean spaces.

2000 Mathematics Subject Classification: 17D99, 53C35

Keywords: comtrans algebra, Lie algebra, symmetric space, Thomas decomposition

1 Introduction

The ternary algebras of the title are *comtrans algebras* described formally in Section 2 below. They are modules equipped with two basic trilinear operations, a left alternative *commutator* and a *translator* satisfying the Jacobi identity. The two operations are connected by the *comtrans identity*. Comtrans algebras were originally introduced in [12] in answer to a problem from differential geometry, asking for the algebraic structure in the tangent bundle corresponding to the coordinate n-ary loop of an $(n+1)$-web (cf. [3]). In this context, the role played by comtrans algebras is analogous to that played by the Lie algebra of a Lie group. As described in Section 6 below,

*The first author acknowledges support from the Korea Research Foundation Grant KRF-2000-015-DP0007.

or more fully in [10], taking repeated commutators in a Lie algebra $\mathfrak{L}$ yields a comtrans algebra $CT(\mathfrak{L})$. Furthermore, the Lie algebra $\mathfrak{L}$ is simple if and only if the comtrans algebra $CT(\mathfrak{L})$ is simple (see [10]). Other simple comtrans algebras are furnished by spaces equipped with bilinear forms (see [9]), spaces of Hermitian operators (see [11]), and certain more general spaces of matrices (see [10]).

The motivation for the current paper comes from the use of *symmetric* or *involutive* Lie algebras to describe symmetric spaces (see [2, 4, 5]). If a (real) Lie algebra $\mathfrak{g}$ is equipped with an involutive automorphism σ, then the concept of a symmetric Lie algebra describes the decomposition

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{k} \tag{1.1}$$

of $\mathfrak{g}$ into the eigenspaces of σ belonging respectively to the eigenvalues -1 and 1. The eigenspace $\mathfrak{k}$ in (1.1), as the fixed point set of σ, is a Lie subalgebra of $\mathfrak{g}$. On the other hand, the eigenspace $\mathfrak{p}$ has no autonomous existence as a Lie algebra since it is not closed under the binary Lie bracket. The lack of closure of $\mathfrak{p}$ in (1.1) is a parity phenomenon, traceable to the evenness of the arity of the Lie bracket. Within Lie theory, one has to resort to Lie triple systems in order to achieve closure (cf. Sec. IV.7 of [4] or Sec. XI.4 of [5]). But using comtrans algebras, in which the basic operations are ternary, the closure is automatic. One is led to the main technical idea in this paper, the concept of a *Thomas sum*

$$G = E \textcircled{T} F \tag{1.2}$$

of comtrans algebras. (The name comes from the special case $n = 3$ of (7.4), namely the comtrans algebra of the Lorentz group. Compare [2, p. 502].) In (1.2), the comtrans algebra G is a module direct sum of subalgebras E and F, each of which acts on the other. The algebra structure of G is recovered from the subalgebras and their mutual interactions. Thus (1.2) does not just give a decomposition of a known algebra G, but it may also be used to construct the algebra G from the smaller algebras E and F.

Just like their specializations, direct sums, Thomas sums come in two versions: internal and external. Internal Thomas sums are defined in Section 3. The main results of that section (Propositions 3.2 and 3.4) relate internal Thomas sums to the existence of involutive automorphisms. External Thomas sums are introduced in Section 4. There is an equivalence between internal and external Thomas sums, enabling one to speak simply of Thomas sums. Section 5 describes duality for Thomas sums. The dual of a Thomas sum (1.2) is obtained by negating the internal algebra structure of E and the action of E on F (Definition 5.2). It is defined over all rings, regardless of the characteristic. In the real case, it specializes to yield the duality of symmetric Lie algebras (Proposition 5.3), better known to physicists as "Weyl's unitary trick" (see [2, 5]).

The comtrans algebras obtained from Lie algebras and from symmetric bilinear forms lie in the variety of *monic* comtrans algebras, in which the comtrans identity reduces to simple equality between the commutator and the translator. Thomas sums of monic algebras are treated in Section 6. Section 7 then uses Thomas sums to analyze real orthogonal algebras. Theorem 7.1 shows how the comtrans algebra O^n of the real orthogonal group $\mathrm{O}(n,\mathbb{R})$ is obtained as an iterated Thomas sum of Euclidean spaces E^r of decreasing dimensions r. One normally thinks of groups as being "active", while the spaces on which they act remain "passive". Now the spaces have been endowed with algebra structure, however, it turns out that they are able to encode the (infinitesimal) group structure. The cascading sequence (7.1) of Euclidean spaces is reminiscent of the structure of support functions of non-compact convex sets (see [1, 6]); it might prove fruitful to examine this relationship further. The corollaries to Theorem 7.1 demonstrate the irreducibility of the actions of the orthogonal algebras and Euclidean spaces in the Thomas sum decomposition of O^n. The final Corollary 7.4 uses duality to convert the Thomas decomposition (7.2) of Theorem 7.1 into a Thomas decomposition of the comtrans algebra of the real orthogonal group $\mathrm{O}(n,1)$.

For concepts and conventions of algebra that are not otherwise explained here, refer to [13].

2 Comtrans Algebras

A *comtrans algebra* E over a unital commutative ring R is an R-module E equipped with two trilinear ternary operations, a *commutator* $[x,y,z]$ and a *translator* $\langle x,y,z\rangle$, such that the commutator satisfies the *left alternative identity*

$$[x,x,y]=0, \tag{2.1}$$

the translator satisfies the *Jacobi identity*

$$\langle x,y,z\rangle+\langle y,z,x\rangle+\langle z,x,y\rangle=0, \tag{2.2}$$

and together the commutator and translator satisfy the *comtrans identity*

$$[x,y,x]=\langle x,y,x\rangle. \tag{2.3}$$

Note that the class $\mathfrak{CT}_R$ of all comtrans algebras over a fixed ring R forms a variety in the sense of universal algebra. This variety becomes (the class of objects of) a bicomplete category whose morphisms are the homomorphisms between comtrans algebras (cf. Theorems 2.1.3 and 2.2.3 of [13, Chpt. IV]). For a member E of $\mathfrak{CT}_R$, let $E[X]$ denote the coproduct of E in $\mathfrak{CT}_R$ with the free $\mathfrak{CT}_R$-algebra on a singleton $\{X\}$. For x,y in E, there are R-module homomorphisms

$$K(x,y):E[X]\to E[X];\quad z\mapsto[z,x,y], \tag{2.4}$$

$$R(x,y):E[X]\to E[X];\quad z\mapsto\langle z,x,y\rangle, \tag{2.5}$$

and

$$L(x,y) : E[X] \to E[X]; \quad z \mapsto \langle y, x, z \rangle. \tag{2.6}$$

The *universal enveloping algebra* $U(E)$ of E (see [8]) is the R-subalgebra of the endomorphism ring of the R-module $E[X]$ generated by

$$\{K(x,y), R(x,y), L(x,y) \mid x, y \in E\}.$$

Note that the maps $(x,y) \mapsto K(x,y)$, $(x,y) \mapsto R(x,y)$, and $(x,y) \mapsto L(x,y)$ from $E \times E$ to $U(E)$ are bilinear.

Proposition 2.1. *In the enveloping algebra $U(E)$ of a comtrans algebra E, one has*

$$K(x,x) - R(x,x) - L(x,x) = 0. \tag{2.7}$$

Proof. Apply the left-hand side of (2.7) to an element z of $E[X]$ and simplify by consecutive use of (2.1)–(2.3). □

A comtrans algebra E is said to *act* on another comtrans algebra F if the R-module F is a module over the enveloping algebra $U(E)$ of E. The action is *irreducible* if F is an irreducible $U(E)$-module. The action is *trivial* if $fK(e,e') = fR(e,e') = fL(e,e') = 0$ for all f in F and e, e' in E. The algebras E and F are said to *interact mutually* if each acts on the other.

3 Internal Thomas Sums

Definition 3.1. A comtrans algebra G is said to be the *internal Thomas sum* of subalgebras E and F if

(i) as a module, G is the internal direct sum of its submodules E and F; and

(ii) the following containments are satisfied:

$$[E,F,F] \subseteq E, \quad [F,E,F] \subseteq E, \quad [F,F,E] \subseteq E, \tag{3.1}$$

$$\langle E,F,F \rangle \subseteq E, \quad \langle F,E,F \rangle \subseteq E, \quad \langle F,F,E \rangle \subseteq E, \tag{3.2}$$

$$[F,E,E] \subseteq F, \quad [E,F,E] \subseteq F, \quad [E,E,F] \subseteq F, \tag{3.3}$$

$$\langle F,E,E \rangle \subseteq F, \quad \langle E,F,E \rangle \subseteq F, \quad \langle E,E,F \rangle \subseteq F. \tag{3.4}$$

Proposition 3.2. *Let σ be an involutive automorphism of a comtrans algebra G over a ring R in which 2 is invertible. Let E be the set of elements of G negated by σ. Let F be the set of elements of G that are fixed by σ. Then*

(i) *E and F are subalgebras of G.*

(ii) *G is the internal Thomas sum of E and F.*

Proof. As eigenspaces of σ, the subsets E and F of G are submodules. The module G decomposes as the direct sum of E and F, since each element

x of G is the sum of elements $x(1-\sigma)/2$ of E and $x(1+\sigma)/2$ of F. The verifications of (3.1)–(3.4) are straightforward. □

Corollary 3.3. *Let E and F be comtrans algebras. Then the direct product $E \times F$ is the internal Thomas sum of its subalgebras $E \times \{0\}$ and $\{0\} \times F$.*

Proof. If 2 is invertible in R, then one may apply Proposition 3.2 to the automorphism σ of $E \times F$ that negates $E \times \{0\}$ and fixes $\{0\} \times F$. In any case, identifying E with $E \times \{0\}$ and F with $\{0\} \times F$, condition (i) in Definition 3.1 is standard, while the commutators and translators on the left-hand side of the containments (3.1)–(3.4) are all $\{0\}$. □

The idea used for the quick proof of Corollary 3.3 in the case where 2 is invertible may also be used to obtain a converse to Proposition 3.2.

Proposition 3.4. *Over a ring R in which 2 is invertible, let G be a comtrans algebra that decomposes as the internal Thomas sum of subalgebras E and F. Then there is an involutive automorphism σ of G such that E and F are the sets of elements of G respectively negated and fixed by σ.*

Proof. Define σ to be the direct sum of the module automorphisms $-1_E : E \to E$ and $1_F : F \to F$. Then σ, clearly an involutive module automorphism, is readily verified to be a comtrans algebra automorphism of G. □

Corollary 3.5. *Let R be a commutative unital ring in which 2 is invertible. Then the class of comtrans algebras G over R that decompose as internal Thomas sums forms a variety $\mathfrak{CTh}_R$ of universal algebras.*

Proof. Take $\mathfrak{CTh}_R$ to be the variety of comtrans algebras G over R whose type has been enriched by an involutive comtrans algebra automorphism σ (cf. [13, p. 287]). □

4 External Thomas Sums

Definition 4.1. Let E and F be two mutually interacting comtrans algebras over a ring R. Their *external Thomas sum* is defined to be the external direct sum $G = E \oplus F$ of the R-modules E and F, equipped with a commutator

$$\begin{aligned}[][e_1+f_1, e_2+f_2, e_3+f_3] &= [e_1,e_2,e_3] + [f_1,f_2,f_3] \\ &+ e_1K(f_2,f_3) - e_2K(f_1,f_3) + e_3\{L(f_2,f_1)+R(f_2,f_1)-K(f_2,f_1)\} \\ &+ f_1K(e_2,e_3) - f_2K(e_1,e_3) + f_3\{L(e_2,e_1)+R(e_2,e_1)-K(e_2,e_1)\}\end{aligned}$$

and a translator

$$\begin{aligned}\langle e_1+f_1, e_2+f_2, e_3+f_3\rangle &= \langle e_1,e_2,e_3\rangle + \langle f_1,f_2,f_3\rangle \\ &+ e_1R(f_2,f_3) - e_2\{R(f_3,f_1)+L(f_1,f_3)\} + e_3L(f_2,f_1) \\ &+ f_1R(e_2,e_3) - f_2\{R(e_3,e_1)+L(e_1,e_3)\} + f_3L(e_2,e_1)\end{aligned}$$

defined using elements e_i of E and f_i of F.

Proposition 4.2. *The external Thomas sum G of two mutually interacting comtrans algebras E and F over R is a comtrans algebra. This comtrans algebra G is the internal Thomas sum of its subalgebras $E \oplus \{0\}$ and $\{0\} \oplus F$. Conversely, suppose a comtrans algebra G is the internal Thomas sum of subalgebras E and F. Then G is isomorphic to the external Thomas sum of E and F.*

Proof. The first two statements of the theorem are straightforward. For the final part, use (2.4)–(2.6) to define an action of F on E. This is possible by (3.1) and (3.2). Similarly, on the strength of (3.3) and (3.4), use (2.4)–(2.6) to define an action of E on F. The external direct sum $E \oplus F$ is then specified by Definition 4.1 with this mutual interaction. The map $\theta : E \oplus F \to G$ given by $e \oplus f \mapsto e + f$ is readily seen to be a comtrans algebra isomorphism. □

Once Proposition 4.2 is established, one may abuse language and suppress the distinction between internal and external Thomas sums, speaking simply of the *Thomas sum* of two mutually interacting comtrans algebras. This usage is similar to the usage for direct sums. Indeed, in the context of Corollary 3.3, Thomas sums specialize to direct sums in the case where the mutual interaction is trivial. It is convenient to write $G = E \circledT F$ to denote that a comtrans algebra G is a Thomas sum of E and F, although this notation does not record the specific mutual interaction of the subalgebras.

5 Duality

Definition 5.1. The *negation* $\bar{E}$ of a comtrans algebra $(E, [\ ,\ ,\ ], \langle\ ,\ ,\ \rangle)$ is the comtrans algebra $(E, -[\ ,\ ,\ ], -\langle\ ,\ ,\ \rangle)$.

Note that the negation of $\bar{E}$ is E itself. Although the negation of a comtrans algebra E is not generally isomorphic to E (unless R contains a square root of -1), the universal enveloping algebras of E and $\bar{E}$ coincide.

Definition 5.2. Let $G = E \circledT F$ be a Thomas sum of E and F given by actions $\eta : U(E) \to \mathrm{End}_R(F)$ and $\varphi : U(F) \to \mathrm{End}_R(E)$. Then the *dual* $G^* = \bar{E} \circledT F$ is the Thomas sum of $\bar{E}$ and F given by the action $-\eta$ of $U(E) = U(\bar{E})$ on F and the action φ of $U(F)$ on the equal R-modules $\bar{E} = E$.

Note that $G^{**} = G$, justifying the terminology of Definition 5.2. If R is of characteristic 2, then $G^* = G$.

Consider the variety $\mathfrak{CT}_{\mathbb{C}}$ of complex comtrans algebras. There is a so-called *real form functor* $R : \mathfrak{CT}_{\mathbb{C}} \to \mathfrak{CT}_{\mathbb{R}}$ to the variety of real comtrans algebras, obtained by forgetting the imaginary scalar multiplications. Since this functor preserves underlying sets, it possesses a left adjoint $C : \mathfrak{CT}_{\mathbb{R}} \to$

$\mathfrak{CT}_{\mathbb{C}}$ known as *complexification* (cf. Corollary 3.4.8 of [13, Chpt. IV]).

Let $G = E \circledT F$ be a Thomas sum of real comtrans algebras, with involutive automorphism $\sigma = -1_E \oplus 1_F$ given by Proposition 3.4. Then the complexification G^C of G decomposes as the Thomas sum $G^C = E^C \circledT F^C$ obtained via Proposition 3.2 from the involutive automorphism σ^C of G^C. Consider the real subspace $iE \oplus F$ of G^{CR}.

Proposition 5.3. *The space $iE \oplus F$ forms a subalgebra of G^{CR}, decomposing as the Thomas sum $G^* = iE \circledT F$ dual to $G = E \circledT F$.*

Proof. For e_j in E and f_j in F, one has

$$\begin{aligned}
&[ie_1 + f_1, ie_2 + f_2, ie_3 + f_3] \\
&= i\{-[e_1, e_2, e_3] + [e_1, f_2, f_3] + [f_1, e_2, f_3] + [f_1, f_2, e_3]\} \\
&\quad + \{[f_1, f_2, f_3] - [f_1, e_2, e_3] - [e_1, f_2, e_3] - [e_1, e_2, f_3]\} \\
&= -i[e_1, e_2, e_3] + [f_1, f_2, f_3] \\
&\quad + ie_1 K(f_2, f_3) - ie_2 K(f_1, f_3) + ie_3\{L(f_2, f_1) + R(f_2, f_1) - K(f_2, f_1)\} \\
&\quad - f_1 K(e_2, e_3) + f_2 K(e_1, e_3) - f_3\{L(e_2, e_1) + R(e_2, e_1) - K(e_2, e_1)\}.
\end{aligned}$$

In similar vein, one may verify that the translator on $iE \oplus F$ is dual to the translator on G. □

Example 5.4. Let A be a real comtrans algebra. Define $\delta : A \to A^2 : a \mapsto (a, a)$, $\varepsilon : A \to A^2 : a \mapsto (-a, a)$, and $\tau : A^2 \to A^2 : (a, b) \mapsto (b, a)$. Since τ negates A^ε and fixes A^δ, Proposition 3.2 shows that

$$A^2 = A^\varepsilon \circledT A^\delta. \tag{5.1}$$

Similarly, the complexification A^C of A is obtained as a Thomas sum

$$A^C = iA \circledT A \tag{5.2}$$

by virtue of the "complex conjugation"

$$A^C \to A^C; \quad a + ia' \mapsto a - ia'.$$

Then one may readily verify that the direct square (5.1) is dual to the complexification (5.2).

6 Monic Algebras

Definition 6.1. A comtrans algebra is said to be *monic* if its commutator and translator agree (so that it has just one rather than two basic trilinear operations). In other words, monic algebras are characterized by the *monic identity* $[x, y, z] = \langle x, y, z\rangle$. Let $\mathfrak{M}_R$ denote the variety of monic comtrans algebras over the commutative unital ring R.

If β is a symmetric bilinear form on an R-module E, then

$$[x, y, z] = \langle x, y, z\rangle = y\beta(x, z) - x\beta(y, z) \tag{6.1}$$

defines a monic comtrans algebra $(E, [\ ,\ ,\ ], \langle\ ,\ ,\ \rangle)$ (cf. Example 2.3 of [10]). Let E^n denote the monic real comtrans algebra obtained via (6.1) from the n-dimensional Euclidean space. The algebra E^n is called the *n-dimensional Euclidean (comtrans) algebra.* If $(L, [\ ,\ ])$ is a Lie algebra, then

$$[x, y, z] = \langle x, y, z\rangle = [[x, y], z] \tag{6.2}$$

defines a monic comtrans algebra $(L, [\ ,\ ,\ ], \langle\ ,\ ,\ \rangle)$ (cf. Example 2.2 of [10]). Note that if R is a field and L is finite-dimensional over R, then the Killing form of the Lie algebra $(L, [\ ,\ ])$ is just the trace of the restriction of (2.4) to L. Let O^n denote the monic real comtrans algebra obtained via (6.2) from the Lie algebra $\mathfrak{o}(n, \mathbb{R})$ of the real orthogonal group $\mathrm{O}(n)$, the group of automorphisms of E^n (cf. Corollary 3.7 of [9]). The algebra O^n is called the *nth real orthogonal (comtrans) algebra.* Note that the Euclidean algebra E^3 and orthogonal algebra O^3 coincide, each denoting the comtrans algebra of the real vector triple product. By convention, $\mathrm{O}^1 = \{0\}$, while O^2 and E^1 both coincide with the 1-dimensional real abelian algebra $\mathbb{R}$. The (complexification of the) comtrans algebra E^2 appears as the type $E(I, I)$ in [7]. One may readily verify the following.

Theorem 6.2. *A Thomas sum $G = E \circledT F$ of monic E and F is monic itself if and only if the identities*

$$R(x, y) = K(x, y) \quad \text{and} \quad L(x, y) = K(x, y) - K(y, x)$$

hold in each of the mutual actions of E on F and vice versa.

7 Orthogonal Algebras

Theorem 7.1. *For each positive integer n, the orthogonal real comtrans algebra O^{n+1} decomposes as an iterated Thomas sum*

$$\mathrm{O}^{n+1} = \mathrm{E}^n \circledT (\mathrm{E}^{n-1} \circledT (\cdots \circledT (\mathrm{E}^2 \circledT \mathrm{E}^1)\cdots)) \tag{7.1}$$

of Euclidean algebras. In particular, for each positive integer n, the orthogonal real comtrans algebra O^{n+1} decomposes as a Thomas sum

$$\mathrm{O}^{n+1} = \mathrm{E}^n \circledT \mathrm{O}^n. \tag{7.2}$$

Proof. The proof of (7.1) goes by induction on n. The induction basis is the observation that $\mathrm{O}^2 = \mathbb{R} = \mathrm{E}^1 \oplus \mathrm{O}^1 = \mathrm{E}^1$. The induction step is (7.2). Let $\{e^i_1, \ldots, e^i_i\}$ be the standard basis for E^i. In each algebra O^n of real skew-symmetric matrices for $n > i$, the basis element e^i_j of E^i appears as the difference $E^{(i+1),j} - E^{j,(i+1)}$ of elementary matrices. According to

Theorem 6.2, the action of E^n on O^n in the monic Thomas sum (7.2) is described entirely by the specification of the action of the maps $K(e_i^n, e_j^n)$ on e_k^t for $0 < t < n$ and $0 < k < t$. By comparing with the various repeated commutators in the Lie algebra $\mathfrak{o}(n+1, \mathbb{R})$ of skew-symmetric matrices, these actions are seen to be zero except for $e_i^s K(e_i^n, e_j^n) = e_{s+1}^{j-1}$ and $e_{s+1}^{i-1} K(e_i^n, e_j^n) = -e_{s+1}^{j-1}$ in the case $s < j-1$, together with $e_i^s K(e_i^n, e_j^n) = -e_j^s$ and $e_{s+1}^{i-1} K(e_i^n, e_j^n) = e_j^s$ in the case $s > j-1$. In similar fashion, the action of O^n on E^n is described entirely by the specification of the action of the maps $K(e_j^l, e_k^m)$ on e_i^n for $0 < j \leq l < n$, $0 < k \leq m < n$, and $0 < i \leq n$. Again comparing with the various repeated commutators in the Lie algebra $\mathfrak{o}(n+1, \mathbb{R})$ of skew-symmetric matrices, these actions are seen to be zero except for $e_i^n K(e_i^j, e_{j+1}^{s-1}) = e_s^n$, $e_i^n K(e_i^j, e_s^j) = -e_s^n$, $e_i^n K(e_j^{i-1}, e_j^{s-1}) = -e_s^n$, and $e_i^n K(e_j^{i-1}, e_s^{j-1}) = e_s^n$. This completes the description of O^{n+1} as the Thomas sum (7.2). □

Corollary 7.2. *In the Thomas sum* (7.2), *the action of* E^n *on* O^n *is irreducible.*

Proof. Let x be a non-zero element of a non-trivial submodule J of O^n. Suppose the expression of x as a linear combination of standard basis elements of E^k for $k < n$ includes a particular standard basis element e_i^t with a non-zero coefficient. Then $-xK(e_i^n, e_i^n)$ is the orthogonal projection of x onto the subspace of $\mathrm{E}^{n-1} \oplus \cdots \oplus \mathrm{E}^1$ spanned by

$$\{e_i^{t'} \mid i \leq t' \leq n-1\} \cup \{e_j^{i-1} \mid 1 \leq j \leq i-1\}, \tag{7.3}$$

and then $xK(e_i^n, e_i^n)K(e_{t+1}^n, e_{t+1}^n)$ is the orthogonal projection of x onto the subspace of $\mathrm{E}^{n-1} \oplus \cdots \oplus \mathrm{E}^1$ spanned by e_i^t. Thus e_i^t lies in J.

For $i > j \geq 1$, one has $e_i^t K(e_{t+1}^n, e_j^n) = e_j^{i-1}$. Similarly, for $i < j \leq n$, one has $e_i^t K(e_{t+1}^n, e_j^n) = -e_i^{j-1}$. Thus J contains (7.3).

Finally, let e_j^s be a standard basis element of E^s for $1 \leq s < n$. For $i \leq s$, one has e_i^s in (7.3), and then the equation $e_i^s K(e_i^n, e_j^n) = -e_j^s$ shows that $e_j^s \in J$. For $i > s$, implying $j \leq i-1$, one has e_j^{i-1} in (7.3), and then the equation $e_j^{i-1} K(e_i^n, e_{s+1}^n) = -e_j^s$ shows that $e_j^s \in J$. Thus J is improper. □

Corollary 7.3. *In the Thomas sum* (7.2), *the action of* O^n *on* E^n *is irreducible.*

Proof. Let x be a non-zero element of a non-trivial submodule J of E^n. Suppose

$$x = r_i e_i^n + \cdots + r_n e_n^n,$$

where i is the smallest index such that $x \cdot e_i^n \neq 0$. It will first be shown that $e_i^n \in J$. If $i = n$, the result is immediate. If $i = n-1$, it follows from $xK(e_1^{n-2}, e_1^{n-2}) = -r_i e_{n-1}^n$. If $i \leq n-2$, it follows from $xK(e_i^i, e_i^i) = -r_i e_i^n$. Next, since $e_{n-1}^n K(e_1^{n-2}, e_1^{n-1}) = -e_n^n$ and $e_i^n K(e_i^i, e_{i+1}^{n-1}) = e_n^n$ for

$1 \leq i \leq n-2$, one finds that $e_n^n \in J$ in any case. Finally, the equations $e_n^n K(e_2^{n-1}, e_1^1) = e_1^n$ and $e_n^n K(e_1^{n-1}, e_1^{s-1}) = -e_s^n$ for $1 < s < n$ show that each standard basis element e_j^n, and thus each element of E^n, lies in the submodule J. □

Let $\mathrm{O}(n,1)$ denote the group of automorphisms of the $(n+1)$-dimensional real space equipped with the quadratic form of signature $(+,\ldots,+,-)$. Let $\mathfrak{o}(n,1)$ denote the Lie algebra of $\mathrm{O}(n,1)$, and $\mathrm{O}^{n,1}$ the comtrans algebra obtained from $\mathfrak{o}(n,1)$ via (6.2).

Corollary 7.4. *The comtrans algebra* $\mathrm{O}^{n,1}$ *decomposes as the Thomas sum*

$$\mathrm{O}^{n,1} = \bar{\mathrm{E}}^n \, \text{Ⓣ} \, \mathrm{O}^n \tag{7.4}$$

dual to (7.2).

Proof. Using Proposition 5.3, the result follows from the symmetric space duality between $\mathfrak{o}(n,1)$ and $\mathfrak{o}(n+1,\mathbb{R})$ (cf. Sec. XI.10 of [5]). □

References

[1] D.-H. Choi, J.D.H. Smith, Support functions of general convex sets, *Algebra Universalis* **49** (2003) 305–319.

[2] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Wiley, New York, 1974.

[3] V.V. Goldberg, *Theory of Multicodimensional* $(n+1)$*-Webs*, Kluwer, Dordrecht, 1988.

[4] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

[5] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Wiley, New York, 1969.

[6] A.B. Romanowska, J.D.H. Smith, Support functions and ordinal products, *Geom. Dedicata* **30** (1989) 281–296.

[7] X.R. Shen, Classification of the comtrans algebras of low dimensionalities, *Comm. Algebra* **23** (1995) 1751–1780.

[8] X.R. Shen, J.D.H. Smith, Representation theory of comtrans algebras, *J. Pure Appl. Algebra* **80** (1992) 177–195.

[9] X.R. Shen, J.D.H. Smith, Comtrans algebras and bilinear forms, *Arch. Math.* **59** (1992) 327–333.

[10] X.R. Shen, J.D.H. Smith, Simple multilinear algebras, rectangular matrices and Lie algebras, *J. Algebra* **160** (1993) 424–433.

[11] X.R. Shen, J.D.H. Smith, Simple algebras of Hermitian operators, *Arch. Math.* **65** (1995) 534–539.

[12] J.D.H. Smith, Multilinear algebras and Lie's theorem for formal n-loops, *Arch. Math.* **51** (1988) 169–177.

[13] J.D.H. Smith, A.B. Romanowska, *Post-Modern Algebra*, Wiley, New York, 1999.