ONE-SIDED HOPF ALGEBRAS AND QUANTUM QUASIGROUPS

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Abstract. We study the connections between one-sided Hopf algebras and one-sided quantum quasigroups, tracking the four possible invertibility conditions for the left and right composite morphisms that combine comultiplications and multiplications in these structures. The genuinely one-sided structures exhibit precisely two of the invertibilities, while it emerges that imposing one more condition often entails the validity of all four. A main result shows that under appropriate conditions, just one of the invertibility conditions is sufficient for the existence of a one-sided antipode. In the left Hopf algebra which is a variant of the quantum special linear group of two-dimensional matrices, it is shown explicitly that the right composite is not injective, and the left composite is not surjective.

1. Introduction

Suppose that $\mathbf{V}$ is a strict symmetric monoidal category, for example a category of vector spaces under the tensor product, or a category of sets under the Cartesian product. A *bimagma* (A, ∇, Δ) in $\mathbf{V}$ is an object A of $\mathbf{V}$ equipped with $\mathbf{V}$-morphisms giving a *magma* structure or *multiplication* $\nabla\colon A\otimes A \to A$, and a *comagma* structure or *comultiplication* $\Delta\colon A \to A\otimes A$, such that Δ is a magma homomorphism. (The latter *bimagma condition* is equivalent to its dual: ∇ is a comagma homomorphism.) The bimagma incorporates two dual morphisms in $\mathbf{V}$: the *left composite*

$$\mathsf{G}\colon A \otimes A \xrightarrow{\Delta\otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A\otimes\nabla} A \otimes A \tag{1.1}$$

("G" for "Gauche") and the *right composite*

$$\reflectbox{G}\colon A \otimes A \xrightarrow{1_A\otimes\Delta} A \otimes A \otimes A \xrightarrow{\nabla\otimes 1_A} A \otimes A \tag{1.2}$$

("$\reflectbox{$\mathsf{G}$}$" for "Droite"). The qualifiers "left" and "right" refer to the side of the tensor product on which the comultiplications appear.

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Within a Hopf algebra (whose antipode is not necessarily invertible), the left and right composites are invertible. This observation underlies the formulation of an extension of the concept of a Hopf algebra to accommodate nonassociative multiplications and noncoassociative comultiplications: A bimagma is a *quantum quasigroup* whenever its left and right composites are invertible [21]. The self-dual concept of a quantum quasigroup subsumes various non-self-dual nonassociative generalizations of Hopf algebras that have appeared in the literature, especially in the context of the octonions or function spaces on the 7-sphere [2, 12, 13, 16].

The antipode S of a Hopf algebra $(A, \nabla, \Delta, \eta, \varepsilon, S)$ is specified as a two-sided inverse for the identity mapping 1_A on A within the convolution monoid $\mathbf{V}(A, A)$. *Left* and *right Hopf algebras* are bimonoids $(A, \nabla, \Delta, \eta, \varepsilon)$ equipped with an endomorphism S of A which is only required respectively to be a left or right inverse for 1_A in the convolution monoid [9, 14, 15, 18]. One-sided Hopf algebras are motivated by the physical idea of a boson-fermion correspondence, which finds concrete expression in various versions of the MacMahon master theorem of combinatorics [6, 7, 8, 14]. The rather delicate nature of genuinely one-sided Hopf algebras is readily appreciated by noting that within the category of sets with Cartesian product, left Hopf algebras are necessarily Hopf algebras (in other words, groups with diagonal comultiplication).

A *left* or *right quantum quasigroup* is a bimagma in which the respective left or right composite is invertible [22]. For example, if G is a group, consider the braiding

$$G \otimes G \to G \otimes G; g \otimes h \mapsto h \otimes h^{-1}gh \tag{1.3}$$

in the category of sets under the Cartesian product (here written using tensor product notation, including for ordered pairs at the element level) [11, XIII(1.12)]. The corresponding $\mathcal{R}$-matrix, obtained by applying a final symmetry to the braiding (1.3), may be written as the right composite

$$g \otimes h \xmapsto{1_G \otimes \Delta} g \otimes h \otimes h \xmapsto{\nabla \otimes 1_G} h^{-1}gh \otimes h \tag{1.4}$$

of a bimagma (G, ∇, Δ) with diagonal comultiplication and multiplication $\nabla \colon g \otimes h \mapsto h^{-1}gh$. The right composite (1.4) is invertible, with two-sided inverse $x \otimes y \mapsto yxy^{-1} \otimes y$. The multiplication of (G, ∇, Δ) appears as the conjugation right quasigroup that Joyce has described as "the quandle $\operatorname{Conj} G$" [10]. Indeed, consideration of the right composite (1.4) embeds the use of quandles in knot theory into the more general context of $\mathcal{R}$-matrices.

The aim of the current paper is to study the invertibility properties of left and right composites in one-sided Hopf algebras and quantum quasigroups, clarifying the relationships between these distinct extensions of the Hopf algebra concept. The basic definitions are recorded in Section 2. Although the definitions of one- and two-sided quantum quasigroups do not involve any analogue of the antipode in general, such analogues have been used in earlier nonassociative versions of Hopf algebras. The four so-called *inverse properties* involved, coming in left/right and internal/external flavors, are discussed in Section 3. In this context, Example 3.4 exhibits an elementary nonassociative structure with a right antipode that is not a left antipode.

The key breakdown of the invertibility of the left and right composites is realized by the lemmas of Section 4. For example, the left composite is a section in a left Hopf algebra, and a retract in a right Hopf algebra. Table 1 summarizes the invertibility conditions. Of all four possible one-sided invertibility relationships, precisely two obtain in any genuinely one-sided Hopf algebra or quantum quasigroup.

Section 5 examines those conjunctions of invertibility conditions which are generally sufficient for a one-sided Hopf algebra or quantum loop to become a Hopf algebra. Previous results are recalled in §§5.1–5.2, while new results are presented in §§5.3–5.4. The essence of the new conditions is that often, when three of the four possible invertibility conditions in a row of Table 1 hold, then the fourth follows as a consequence. For example, if the left composite of a left Hopf algebra A is invertible, then A is actually a two-sided Hopf algebra (Proposition 5.5). In turn, Proposition 5.9 shows when a right quantum loop and left Hopf algebra becomes a two-sided Hopf algebra.

Section 6 refines the preceding results in the special situation where the underlying symmetric monoidal category is compact closed, so that objects are paired to dual objects with evaluation and coevaluation morphisms. This happens, for instance, in the case of finite-dimensional vector spaces. A non-linear example is provided by finite semilattices (§6.2). The main result, Theorem 6.9, creates a left antipode for a bimonoid whenever the right composite has a suitable section.

The paper concludes with an examination of the left Hopf algebra $\widetilde{\mathsf{SL}}_q(2)$, the smallest of the left Hopf algebras supporting a quantum version of the MacMahon master theorem [18]. Proposition 7.7 exhibits a specific non-zero element of the kernel of the right composite, while Theorem 7.8 shows that the left composite is not surjective.

Algebraic notation is used throughout the paper, with functions to the right of, or as superfixes to, their arguments. Thus compositions are read from left to right. These conventions serve to minimize the proliferation of brackets.

2. Structures in symmetric monoidal categories

The general setting for the algebras studied in this paper is a (strict) symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$, with a symmetry $\tau\colon A \otimes B \to B \otimes A$ and explicitly written isomorphisms

$$\mathbf{1} \otimes A \xrightarrow{\lambda_A} A \xleftarrow{\rho_A} A \otimes \mathbf{1}$$

for objects A and B. The standard example is provided by the category $\underline{\underline{K}}$ of vector spaces over a field K, or modules over a commutative ring K. This section records some basic definitions applying to objects in $\mathbf{V}$. While these definitions are mostly bread-and-butter for Hopf algebra experts, they may be less familiar to other algebraists.

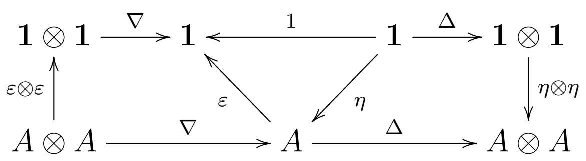
2.1. **Magmas and bimagmas.**

Definition 2.1. Let $\mathbf{V}$ be a symmetric monoidal category.

(a) A magma (A, ∇) is *left unital* if it has a $\mathbf{V}$-morphism $\eta\colon \mathbf{1} \to A$ such that $(\eta \otimes 1_A)\nabla = \lambda_A$. It is *right unital* if it has a $\mathbf{V}$-morphism $\eta\colon \mathbf{1} \to A$ such that $(1_A \otimes \eta)\nabla = \rho_A$. Then it is *unital* if it has a $\mathbf{V}$-morphism $\eta\colon \mathbf{1} \to A$ with respect to which it is both left and right unital.

(b) A comagma (A, Δ) in $\mathbf{V}$ is *counital* if it has a $\mathbf{V}$-morphism $\varepsilon\colon A \to \mathbf{1}$ such that $\Delta(\varepsilon \otimes 1_A) = \lambda_A^{-1}$ and $\Delta(1_A \otimes \varepsilon) = \rho_A^{-1}$.

(c) A bimagma (A, ∇, Δ) is *biunital* if its magma (A, ∇) is unital, its comagma (A, Δ) is counital, and the diagram



commutes.

Remark 2.2. Consider an object A in a concrete monoidal category $\mathbf{V}$.

(a) It is often convenient to write $\lambda_A^{-1}\colon A \to \mathbf{1} \otimes A; a \mapsto 1 \otimes a$, and dually for ρ_A^{-1}.

(b) A magma multiplication on A is often denoted by juxtaposition, namely $(a \otimes b)\nabla = ab$, or with $a \cdot b$ as an infix notation, for elements a, b of A.

(c) If A is a unital magma, write $\eta\colon \mathbf{1} \to A; 1 \mapsto 1$ with the convention of (a). Then with the convention of (b), unitality takes the familiar form $1 \cdot a = a = a \cdot 1$ for a in A.

(d) A comagma comultiplication on A is often denoted by a version of Sweedler notation adapted to the general noncoassociative situation, namely $a\Delta = a^L \otimes a^R$. Note that coassociativity then takes the form

$$a^{LL} \otimes a^{LR} \otimes a^R = a^L \otimes a^{RL} \otimes a^{RR} \tag{2.1}$$

for an element a of A. Given coassociativity, the classical Sweedler notation is recovered by replacing the superscripts, when taken in lexicographic order, by successive subscript numbers. For example, each side of (2.1) is then written as $a_1 \otimes a_2 \otimes a_3$.

2.2. Antipodes.

Definition 2.3. Let $(A, \nabla, \Delta, \eta, \varepsilon)$ be a left- or right-unital and counital bimagma with a morphism $S\colon A \to A$.

(a) The morphism $S\colon A \to A$ is a *left antipode* if $\varepsilon\eta = \Delta(S \otimes 1_A)\nabla$.
(b) The morphism $S\colon A \to A$ is a *right antipode* if $\varepsilon\eta = \Delta(1_A \otimes S)\nabla$.
(c) The morphism $S\colon A \to A$ is an *antipode* if it is both a left and a right antipode.

Remark 2.4. In the Sweedler notation of Remark 2.2, Definition 2.3(a) becomes

$$a^{LS}a^R = a\varepsilon\eta \tag{2.2}$$

for $a \in A$. Similarly, Definition 2.3(b) becomes

$$a\varepsilon\eta = a^L a^{RS} \tag{2.3}$$

for $a \in A$.

2.3. Unilateral and bilateral Hopf algebras.

Definition 2.5. Let $\mathbf{V}$ be a symmetric monoidal category.

(a) A unital magma (A, ∇, η) in $\mathbf{V}$ is said to be a *monoid* if it satisfies the associative law $(\nabla \otimes 1_A)\nabla = (1_A \otimes \nabla)\nabla$.
(b) A counital comagma (A, Δ, ε) in $\mathbf{V}$ is a *comonoid* if it satisfies the coassociative law $\Delta(\Delta \otimes 1_A) = \Delta(1_A \otimes \Delta)$.
(c) A biunital bimagma $(A, \nabla, \Delta, \eta, \varepsilon)$ in $\mathbf{V}$ is a *bimonoid* if (A, ∇, η) is a monoid and (A, Δ, ε) is a comonoid.

Definition 2.6. Let $\mathbf{V}$ be a symmetric monoidal category.

(a) A bimonoid $(A, \nabla, \Delta, \eta, \varepsilon)$ in $\mathbf{V}$ is a *left Hopf algebra* if it has a left antipode $S\colon A \to A$.
(b) A bimonoid $(A, \nabla, \Delta, \eta, \varepsilon)$ in $\mathbf{V}$ is a *right Hopf algebra* if it has a right antipode $S\colon A \to A$.

(c) A bimonoid $(A, \nabla, \Delta, \eta, \varepsilon)$ in $\mathbf{V}$ is said to be a *Hopf algebra* if it has an antipode $S\colon A \to A$. In contrast to the conventions of some authors (compare [5, Def'n. 5.3.10], for example), we do not demand invertibility of the antipode.

2.4. Quantum quasigroups and loops.

Definition 2.7. Let $\mathbf{V}$ be a symmetric monoidal category.

(a) A *left quantum quasigroup* (A, ∇, Δ) is a bimagma in $\mathbf{V}$ for which the left composite morphism (1.1) is invertible.
(b) A *right quantum quasigroup* (A, ∇, Δ) is a bimagma in $\mathbf{V}$ for which the right composite morphism (1.2) is invertible.
(c) A *quantum quasigroup* (A, ∇, Δ) is a bimagma in $\mathbf{V}$ where both the left and right composite morphisms are invertible.

Definition 2.8. Suppose that $(A, \nabla, \Delta, \eta, \varepsilon)$ is a biunital bimagma in a symmetric monoidal category $\mathbf{V}$.

(a) Suppose that (A, ∇, Δ) is a left quantum quasigroup in $\mathbf{V}$. Then $(A, \nabla, \Delta, \eta, \varepsilon)$ is said to be a *left quantum loop*.
(b) Suppose that (A, ∇, Δ) is a right quantum quasigroup in $\mathbf{V}$. Then $(A, \nabla, \Delta, \eta, \varepsilon)$ is said to be a *right quantum loop*.
(c) Let (A, ∇, Δ) be a quantum quasigroup in $\mathbf{V}$. Then $(A, \nabla, \Delta, \eta, \varepsilon)$ is a *quantum loop*.

Remark 2.9. The concepts of these definitions provide genuine extensions of the concept of a Hopf algebra, since Hopf algebras are quantum loops [21, Prop. 4.1]. In particular, consider a Hopf algebra A in the category of vector spaces over a field, under the tensor product. The comultiplication of A provides a coaction that makes A into a right comodule algebra over itself, with a one-dimensional algebra of coinvariants [3, Example 6.2.5(1)]. Then the right composite of the quantum loop A is the *canonical map* of this comodule algebra structure [3, Example 6.4.8(1)].

3. Inverse properties

3.1. Moufang-Hopf algebras and Hopf quasigroups.

The *Moufang-Hopf algebras* of Benkart *et al.* [2, Def'n. 1.2], and the *Hopf quasigroups* of Klim and Majid [12, Def'n. 4.1], are almost bimonoids $(A, \nabla, \Delta, \eta, \varepsilon)$, but not requiring associativity of the multiplication, equipped with an *IP-antipode* morphism $S\colon A\to A$ such that the *external left IP-antipode diagram*

$$\begin{array}{ccccccc} A\otimes A\otimes A & \xrightarrow{S\otimes 1_A\otimes 1_A} & A\otimes A\otimes A & \xrightarrow{1_A\otimes\nabla} & A\otimes A & \xrightarrow{\nabla} & A \\ {\scriptstyle \Delta\otimes 1_A}\uparrow & & & & & \nearrow{\scriptstyle\nabla} & \\ A\otimes A & \xrightarrow[\varepsilon\otimes 1_A]{} & \mathbf{1}\otimes A & \xrightarrow[\eta\otimes 1_A]{} & A\otimes A & & \end{array} \tag{3.1}$$

the *internal right IP-antipode diagram*

$$\begin{array}{ccccccc} A\otimes A & \xrightarrow{1_A\otimes\varepsilon} & A\otimes\mathbf{1} & \xrightarrow{1_A\otimes\eta} & A\otimes A & & \\ {\scriptstyle 1_A\otimes\Delta}\downarrow & & & & & \searrow{\scriptstyle\nabla} & \\ A\otimes A\otimes A & \xrightarrow[1_A\otimes S\otimes 1_A]{} & A\otimes A\otimes A & \xrightarrow[\nabla\otimes 1_A]{} & A\otimes A & \xrightarrow[\nabla]{} & A \end{array} \tag{3.2}$$

the *external right IP-antipode diagram*

$$\begin{array}{ccccccc} A\otimes A\otimes A & \xrightarrow{1_A\otimes 1_A\otimes S} & A\otimes A\otimes A & \xrightarrow{\nabla\otimes 1_A} & A\otimes A & \xrightarrow{\nabla} & A \\ {\scriptstyle 1_A\otimes\Delta}\uparrow & & & & & \nearrow{\scriptstyle\nabla} & \\ A\otimes A & \xrightarrow[1_A\otimes\varepsilon]{} & A\otimes\mathbf{1} & \xrightarrow[1_A\otimes\eta]{} & A\otimes A & & \end{array} \tag{3.3}$$

and *internal left IP-antipode diagram*

$$\begin{array}{ccccccc} A\otimes A & \xrightarrow{\varepsilon\otimes 1_A} & \mathbf{1}\otimes A & \xrightarrow{\eta\otimes 1_A} & A\otimes A & & \\ {\scriptstyle \Delta\otimes 1_A}\downarrow & & & & & \searrow{\scriptstyle\nabla} & \\ A\otimes A\otimes A & \xrightarrow[1_A\otimes S\otimes 1_A]{} & A\otimes A\otimes A & \xrightarrow[1_A\otimes\nabla]{} & A\otimes A & \xrightarrow[\nabla]{} & A \end{array} \tag{3.4}$$

all commute. [For the "IP-antipode" terminology, see Remark 3.2 below.] In Moufang-Hopf algebras, cocommutativity and an additional *left Moufang-Hopf identity* [2, (1.5)] are required. Hopf quasigroups are said to have the *Moufang* property [12, p.3077] if they satisfy the left Moufang-Hopf identity.

3.2. **Inverse-property loops.**

Definition 3.1. Let $(A,\nabla,\Delta,\eta,\varepsilon)$ be a biunital bimagma in a symmetric, monoidal category $\mathbf{V}$, equipped with a $\mathbf{V}$-morphism $S\colon A\to A$.

(a) $(A,\nabla,\Delta,\eta,\varepsilon,S)$ is a *left inverse-property* (or *LIP*) *Hopf loop* in $\mathbf{V}$ if the external and internal left IP-antipode diagrams (3.1) and (3.4) commute. In this case, S is said to be a *left IP-antipode*.

(b) $(A,\nabla,\Delta,\eta,\varepsilon)$ is a *right inverse-property* (or *RIP*) *Hopf loop* in $\mathbf{V}$ if the internal and external right IP-antipode diagrams (3.2) and (3.3) commute. In this case, S is said to be a *right IP-antipode*.

(c) $(A, \nabla, \Delta, \eta, \varepsilon)$ is an *inverse-property* (or *IP*) *Hopf loop* in $\mathbf{V}$ if it is both a left and right inverse-property Hopf loop. In this case, S is said to be an *IP-antipode.*

Remark 3.2. A Hopf quasigroup is a coassociative IP Hopf loop. Thus the "IP-antipode" terminology of §3.1 is consistent with (and motivated by) that of Definition 3.1(c). For the classical inverse properties, see [23, §I.4.1].

A two-sided version of the following appeared as [12, Prop. 4.2(a)], for the case where (A, Δ, ε) is a comonoid.

Proposition 3.3. *Let* $(\mathbf{V}, \otimes, \mathbf{1})$ *be a symmetric, monoidal category.*

(a) *If* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *is a left inverse property Hopf loop in* $\mathbf{V}$*, then* S *is an antipode.*
(b) *If* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *is a right inverse property Hopf loop in* $\mathbf{V}$*, then* S *is an antipode.*

Proof. (a): Biunitality implies

$$\Delta(S \otimes 1_A)\nabla = \rho_A^{-1}(1_A \otimes \eta)(\Delta \otimes 1_A)(S \otimes 1_A \otimes 1_A)(1_A \otimes \nabla)\nabla$$

and $\rho_A^{-1}(1_A \otimes \eta)(\varepsilon \otimes 1_A)(\eta \otimes 1_A)\nabla = \varepsilon\eta$. Thus the commuting of the external left IP-antipode diagram (3.1) implies $\Delta(S \otimes 1_A)\nabla = \varepsilon\eta$. Similarly,

$$\Delta(1_A \otimes S)\nabla = \rho_A^{-1}(1_A \otimes \eta)(\Delta \otimes 1_A)(1_A \otimes S \otimes 1_A)(1_A \otimes \nabla)\nabla$$

and $\rho_A^{-1}(1_A \otimes \eta)(\varepsilon \otimes 1_A)(\eta \otimes 1_A)\nabla = \varepsilon\eta$, so the commuting of the internal left IP-antipode diagram (3.4) implies $\Delta(1_A \otimes S)\nabla = \varepsilon\eta$.

(b) is dual to (a). □

3.3. **A loop example.** Recall that a *quasigroup* is a set Q with a binary multiplication $\nabla \colon Q \times Q \to Q; (x, y) \mapsto xy$ such that the left multiplications

$$L(x) \colon Q \to Q; q \mapsto xq$$

and right multiplications

$$R(x) \colon Q \to Q; q \mapsto qx$$

are bijective for all x in Q. The quasigroup Q is a *loop* if it contains an *identity* element e such that $L(e) = R(e) = \mathrm{id}_Q$ [20, 23].

Example 3.4. Consider the loop Q with identity element 1 and

(3.5)

Q	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	5	4	1	2
4	4	3	2	5	1
5	5	4	1	2	3

for the multiplication table. Take counit $\varepsilon\colon Q \to \{*\}$, comultiplication $\Delta\colon Q \to Q \times Q; x \mapsto (x, x)$, and a unit $\eta\colon \{*\} \to Q; * \mapsto 1$. This builds a quantum loop $(Q, \nabla, \Delta, \eta, \varepsilon)$ in the symmetric, monoidal category $(\mathbf{Set}, \times, \{*\})$ [21], where the 3-cycle $S = (3\ 4\ 5)$ is a right antipode. While $3 \cdot 3^S = 3 \cdot 4 = 1$, note that one has $3^S \cdot 3 = 4 \cdot 3 = 2 \neq 1$, so the right antipode S is not a left antipode. In particular, Proposition 3.3(b) serves to confirm that the loop Q cannot be a right inverse property (Hopf) loop in **Set**. Indeed, $(3 \cdot 3)3^S = 4 \cdot 4 = 5 \neq 3$, so the external right IP-antipode diagram (3.3) does not commute.

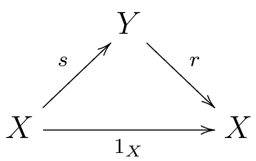
Example 3.4 extends linearly to a quantum loop KQ in the symmetric monoidal category $(\underline{\underline{K}}, \otimes, K)$ for any commutative ring K. The extended right antipode S is not a left antipode (provided K is nontrivial).

4. Invertibility of composites

4.1. **Sections and retracts.** In order to avoid any potential confusion between "right" and "left" inverses, concepts tied to the conventions used for composition of morphisms, we adopt the neutral terminology of sections and retracts. Recall that in the category of sets and functions, sections are injective, while retracts are surjective.

Definition 4.1. Let X and Y be objects of a category $\mathbf{C}$. Consider the triangle

(4.1)



in $\mathbf{C}$.

(a) A morphism $s\colon X \to Y$ is said to be a *section* if there is a morphism $r\colon Y \to X$ such that (4.1) commutes.

(b) A morphism $r\colon Y \to X$ is said to be a *retract* if there is a morphism $s\colon X \to Y$ such that (4.1) commutes.

(c) Suppose that (4.1) commutes for certain morphisms $s\colon X \to Y$ and $r\colon Y \to X$. Then s is a *section of* or *for* r, while r is a *retract of* or *for* s.

4.2. **Composites in one-sided Hopf algebras.**

Lemma 4.2. *Let $(A, \nabla, \Delta, \eta, \varepsilon, S)$ be a left Hopf algebra within a concrete symmetric, monoidal category* $\mathbf{V}$. *Then the left composite* G *is a section, with*

$$\mathsf{G}_S = (\Delta \otimes 1_A)(1_A \otimes S \otimes 1_A)(1_A \otimes \nabla)$$

as a retract.

Proof. Note the maps

$$\begin{array}{ccccc} x \otimes y & \xmapsto{\Delta\otimes 1_A} & x^L \otimes x^R \otimes y & \xmapsto{1_A\otimes\nabla} & x^L \otimes x^R y \\ & & & & \downarrow{\scriptstyle \Delta\otimes 1_A} \\ x^{LL} \otimes x^{LRS} x^R y & \xleftarrow[1_A\otimes\nabla]{} & x^{LL} \otimes x^{LRS} \otimes x^R y & \xleftarrow[1_A\otimes S\otimes 1_A]{} & x^{LL} \otimes x^{LR} \otimes x^R y \end{array}$$

making use of the associativity of the multiplication. By (2.1) in which x replaces a, one has

$$x^L \otimes x^{RL} \otimes x^{RR} = x^{LL} \otimes x^{LR} \otimes x^R \tag{4.2}$$

for y in A. Then for an element y in A, tensoring both sides of (4.2) on the right with y and applying $(1_A \otimes S \otimes 1_A \otimes 1_A)(1_A \otimes 1_A \otimes \nabla)(1_A \otimes \nabla)$ gives

$$\begin{aligned} x^{LL} \otimes x^{LRS} x^R y &= x^L \otimes x^{RLS} x^{RR} y = x^L \otimes x^{R\varepsilon\eta} y \\ &= x^L x^{R\varepsilon\eta} \otimes y = x \otimes y \end{aligned}$$

using (2.2) and counitality for the second and final equalities respectively. Thus $\mathsf{G}\,\mathsf{G}_S = 1_{A\otimes A}$. □

Lemma 4.3. *Let $(A, \nabla, \Delta, \eta, \varepsilon, S)$ be a right Hopf algebra within a concrete symmetric, monoidal category* $\mathbf{V}$. *Then the left composite* G *is a retract, with*

$$\mathsf{G}_S = (\Delta \otimes 1_A)(1_A \otimes S \otimes 1_A)(1_A \otimes \nabla)$$

as a section.

Proof. Note the maps

$$\begin{array}{ccccc} x \otimes y & \xmapsto{\Delta\otimes 1_A} & x^L \otimes x^R \otimes y & \xmapsto{1_A\otimes S\otimes 1_A} & x^L \otimes x^{RS} \otimes y \\ & & & & \downarrow{\scriptstyle 1_A\otimes\nabla} \\ x^{LL} \otimes x^{LR} x^{RS} y & \xleftarrow[1_A\otimes\nabla]{} & x^{LL} \otimes x^{LR} \otimes x^{RS} y & \xleftarrow[\Delta\otimes 1_A]{} & x^L \otimes x^{RS} y \end{array} \tag{4.3}$$

making use of the associativity of the multiplication. For an element y in A, tensoring the equation (4.2) on the right with y and applying the function $(1_A \otimes 1_A \otimes S \otimes 1_A)(1_A \otimes 1_A \otimes \nabla)(1_A \otimes \nabla)$ to each side yields

$$\begin{aligned} x^{LL} \otimes x^{LR} x^{RS} y = x^L \otimes x^{RL} x^{RRS} y &= x^L \otimes x^{R\varepsilon\eta} y \\ &= x^L x^{R\varepsilon\eta} \otimes y = x \otimes y\,, \end{aligned}$$

using (2.3) and counitality for the second and final equalities respectively. Thus $\mathsf{G}_S\,\mathsf{G} = 1_{A\otimes A}$. □

The following two lemmas are the respective chiral duals of Lemmas 4.2 and 4.3.

Lemma 4.4. *Let $(A, \nabla, \Delta, \eta, \varepsilon, S)$ be a right Hopf algebra within a concrete symmetric, monoidal category $\mathbf{V}$. Then the right composite $\reflectbox{$\mathsf{G}$}$ is a section, with*

$$\reflectbox{G}_S = (1_A \otimes \Delta)(1_A \otimes S \otimes 1_A)(\nabla \otimes 1_A)$$

as a retract.

Lemma 4.5. *Let $(A, \nabla, \Delta, \eta, \varepsilon, S)$ be a left Hopf algebra within a concrete symmetric, monoidal category $\mathbf{V}$. Then the right composite $\reflectbox{$\mathsf{G}$}$ is a retract, with*

$$\reflectbox{G}_S = (1_A \otimes \Delta)(1_A \otimes S \otimes 1_A)(\nabla \otimes 1_A)$$

as a section.

Proposition 4.6. *Suppose that $(A, \nabla, \Delta, \eta, \varepsilon)$ is a bimonoid in a concrete symmetric, monoidal category $\mathbf{V}$.*

(a) *The equations*

$$\mathsf{G}\,\mathsf{G}_S = 1_{A\otimes A} \quad \text{and} \quad \reflectbox{G}_S\,\reflectbox{G} = 1_{A\otimes A}$$

hold in $\mathbf{V}$ if A is a left Hopf algebra.

(b) *Dually the equations*

$$\mathsf{G}_S\,\mathsf{G} = 1_{A\otimes A} \quad \text{and} \quad \reflectbox{G}\,\reflectbox{G}_S = 1_{A\otimes A}$$

hold in $\mathbf{V}$ if A is a right Hopf algebra.

(c) [21] *If A is a Hopf algebra, then $(A, \nabla, \Delta, \eta, \varepsilon)$ is a quantum loop.*

Proof. (a): Apply Lemmas 4.2 and 4.5.

(b): Apply Lemmas 4.3 and 4.4.

(c): Combine (a) and (b). □

4.3. **Summary of invertibility conditions.** The following table serves to summarize the left and right invertibility conditions satisfied by the left and right composites in left and right Hopf algebras and quantum quasigroups. In the headings, G' and $\reflectbox{$\mathsf{G}$}'$ respectively denote G_S and $\reflectbox{$\mathsf{G}$}_S$ in left and right Hopf algebras, or the two-sided inverses of G and $\reflectbox{$\mathsf{G}$}$ in left and right quantum quasigroups. A bullet in a table entry signifies that the structure labeling the row of the entry satisfies the invertibility condition labeling its column.

Algebra	$\mathsf{G}\mathsf{G}' = 1_{A\otimes A}$	$\mathsf{G}'\,\mathsf{G} = 1_{A\otimes A}$	$\reflectbox{$\mathsf{G}$}\reflectbox{$\mathsf{G}$}' = 1_{A\otimes A}$	$\reflectbox{$\mathsf{G}$}'\,\reflectbox{$\mathsf{G}$} = 1_{A\otimes A}$
Left Hopf	•			•
Right Hopf		•	•	
Left qgp.	•	•		
Right qgp.			•	•

Table 1. Invertibility conditions.

5. Sufficient conditions for Hopf algebras

This section discusses conditions, old (§§5.1–5.2) and new (§§5.3–5.4), guaranteeing that one-sided Hopf algebras or quantum loops become Hopf algebras. The gist of the new results is that, under many circumstances, the rows of the table of §4.3 will have four bullets as soon as they have three.

5.1. **Previous results.** Let $\underline{\underline{K}}$ be the symmetric, monoidal category of unital modules over a commutative, unital ring K under the tensor product. A comonoid in $\underline{\underline{K}}$ is said to be *simple* if it has exactly two subcomonoids (one trivial, the other improper) [17, Def'n. 2.1.8]. Then the *coradical* of a comonoid is the sum of its simple subcomonoids [17, Def'n. 3.4.1]. A comonoid in $\underline{\underline{K}}$ is *pointed* if each simple subcomonoid is 1-dimensional [17, Def'n. 3.4.4].

Theorem 5.1. *Let $(A, \nabla, \Delta, \eta, \varepsilon, S)$ be a left Hopf algebra in $\underline{\underline{K}}$. Then if any one of these conditions is satisfied:*

(a) *the module A is finitely generated* [9, Prop. 5]*;*
(b) *the monoid (A, ∇, η) is left or right Noetherian as a K-algebra* [9, Prop. 6]*;*
(c) *the monoid (A, ∇, η) is commutative, and K is a field* [9, Th. 3(2)]*;*
(d) *the monoid (A, ∇, η) is commutative, and Noetherian as a K-algebra* [9, Cor. 10]*;*

(e) *the comonoid* (A, Δ, ε) *is pointed* [9, Th. 3(3)]*;*
(f) *the coradical of* (A, Δ, ε) *is cocommutative* [9, Th. 3(4)]*,*

one concludes that $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a Hopf algebra.*

The following result, applied in connection with Theorem 5.1(e), is of independent interest.

Proposition 5.2. [9, Prop. 5] *Let* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *be a left Hopf algebra in* $\underline{\underline{K}}$*. Suppose that the comonoid* (A, Δ, ε) *is pointed. Then the set*

$$A_1 = \{x \in A \mid a\Delta = a \otimes a\,,\ a\varepsilon = 1\}$$

of grouplike elements of A *forms a subgroup of the monoid* (A, ∇, η)*.*

Theorem 5.3. [21, Th. 4.16] *Suppose* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is an associative and coassociative quantum loop in the category* $\underline{\underline{K}}$*, for a field* K*. Then the bimonoid* $(A, \nabla, \Delta, \eta, \varepsilon)$ *uniquely specifies an antipode* S *to yield a Hopf algebra* $(A, \nabla, \Delta, \eta, \varepsilon, S)$*.*

5.2. **The category of sets.** Consider the category $(\mathbf{Set}, \times, \top)$ of sets and functions as a symmetric, monoidal category, with the Cartesian product as tensor product, and a one-element set $\top$ as unit object. Note that if (A, Δ, ϵ) is a comonoid in $(\mathbf{Set}, \times, \top)$, then the comultiplication is $\Delta\colon a \mapsto a \otimes a$ [21, Lemma 3.9]. The simple comonoids are the singleton sets, having just the improper subcomonoid and a trivial subcomonoid (the empty subset). If "dimension" is interpreted as cardinality, then each comonoid is pointed, in the sense that each simple subcomonoid has cardinality 1.

In the notation of Remark 2.2, and with the image of η as $\{1\}$, the antipode conditions reduce to the respective equations $a^S \cdot a = 1$ and $a \cdot a^S = 1$, for each element a of A. The following lemma is a reformulation, in the current Hopf algebra language, of a well-known result: If each element of a monoid has a left inverse, then the monoid is a group. The lemma may be viewed as an extension of Theorem 5.1(e), or of Proposition 5.2.

Lemma 5.4. *Let* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *be a left Hopf algebra in the category* $(\mathbf{Set}, \times, \top)$*. Then* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *is a Hopf algebra.*

Proof. Let S be the left antipode. Then for each element a of A, one has $a^S \cdot a = 1$. In particular, $a^{S^2} \cdot a^S = 1 = a^S \cdot a$, so that a^S is invertible. The equation $a^S \cdot a = 1$ now implies that $a = (a^S)^{-1}$, so that a is invertible, and $a^S = a^{-1}$. Thus $a \cdot a^S = 1$. □

5.3. **One-sided conditions.**

Proposition 5.5. *Let* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *be a left Hopf algebra in a concrete symmetric, monoidal category* $\mathbf{V}$*. If the left composite* G *is invertible, then* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *is a Hopf algebra.*

Proof. Consider an element a of A. Since $(A, \nabla, \Delta, \eta, \varepsilon, S)$ is a left Hopf algebra, Lemma 4.2 shows that $\mathsf{G}\,\mathsf{G}_S = 1_{A\otimes A}$. If G is invertible, then one also has $\mathsf{G}_S\,\mathsf{G} = 1_{A\otimes A}$. Applying these equal morphisms to the image $a \otimes 1^\eta$ of a under $\lambda_A^{-1}(1_A \otimes \eta)$, one obtains

$$a^{LL} \otimes a^{LR}a^{RS}1^\eta = a \otimes 1^\eta \tag{5.1}$$

making use of (4.3) for the left hand side. Additional application of $\varepsilon\eta \otimes 1_A$ to each side of (5.1) yields

$$a^{LL\varepsilon\eta} \otimes a^{LR}a^{RS}1^\eta = a^{\varepsilon\eta} \otimes 1^\eta\,. \tag{5.2}$$

By the counitality, $a^{LL\varepsilon\eta} \otimes a^{LR} = 1^\eta \otimes a^L$, so (5.2) further reduces to

$$1^\eta \otimes a^L a^{RS} = 1^\eta \otimes a^{\varepsilon\eta} \tag{5.3}$$

by unitality. Again, unitality yields $\rho_A^{-1}(\eta \otimes 1_A)\nabla = 1_A$, so $\rho_A^{-1}(\eta \otimes 1_A)$ injects. Since (5.3) takes the form $(a^L a^{RS})\rho_A^{-1}(\eta \otimes 1_A) = a^{\varepsilon\eta}\rho_A^{-1}(\eta \otimes 1_A)$, the requirement (2.3) for S to be a right antipode is satisfied. □

Corollary 5.6. *Consider a left quantum loop in a concrete symmetric, monoidal category* $\mathbf{V}$. *If it forms a left Hopf algebra, then it is actually a Hopf algebra.*

The dual versions of these results are as follows.

Proposition 5.7. *Let* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *be a right Hopf algebra in a concrete symmetric, monoidal category* $\mathbf{V}$. *If the right composite* G *is invertible, then* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *is a Hopf algebra.*

Corollary 5.8. *Consider a right quantum loop in a concrete symmetric, monoidal category* $\mathbf{V}$. *If it forms a right Hopf algebra, then it is actually a Hopf algebra.*

5.4. **Crossed conditions.** The following result presents conditions readily implying that a right quantum loop and left Hopf algebra forms a two-sided Hopf algebra.

Proposition 5.9. *Consider the category* $\underline{\underline{K}}$ *of vector spaces over a field* K. *Suppose that* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a right quantum loop in* $\underline{\underline{K}}$ *that satisfies the following properties:*

(a) *The bimagma* (A, ∇, Δ) *is both associative and coassociative;*
(b) *The comonoid* (A, Δ, ε) *is pointed;*
(c) *The right quantum loop carries the structure* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *of a left Hopf algebra.*

Then $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *forms a Hopf algebra.*

Proof. By Proposition 5.2, the set A_1 of setlike elements forms a subgroup in the monoid (A, ∇, η). Since the comonoid (A, Δ, ε) is pointed, the structure $(A, \nabla, \Delta, \eta, \varepsilon, S)$ then becomes a Hopf algebra [17, Prop. 7.6.3]. □

The dual version of Proposition 5.9 is as follows.

Corollary 5.10. *Consider the category* $\underline{\underline{K}}$ *of vector spaces over a field* K*. Suppose that* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a left quantum loop in* $\underline{\underline{K}}$ *that satisfies the following properties:*

(a) *The bimagma* (A, ∇, Δ) *is both associative and coassociative;*
(b) *The comonoid* (A, Δ, ε) *is pointed;*
(c) *The left quantum loop carries the structure* $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *of a right Hopf algebra.*

Then $(A, \nabla, \Delta, \eta, \varepsilon, S)$ *forms a Hopf algebra.*

In the following chapter, it will be shown (with considerably more work) that the hypotheses (b) and (c) of Proposition 5.9 and Corollary 5.10 are actually superfluous in many situations.

6. Creating one-sided antipodes

This chapter investigates how sections for composites entail the existence of one-sided antipodes. The arguments are developed from the context of multiplier Hopf algebras [24, §4].

6.1. Duality in compact closed categories.

Definition 6.1. A category $\mathbf{V}$ is *compact closed* if it is symmetric monoidal, and if, for each object A, there is a *dual* object A^*, along with an *evaluation* morphism $\mathrm{ev}\colon A \otimes A^* \to \mathbf{1}$ and *coevaluation* morphism $\mathrm{coev}\colon \mathbf{1} \to A^* \otimes A$, such that the diagrams

$$
\begin{array}{ccc}
A \otimes \mathbf{1} & \xrightarrow{1 \otimes \mathrm{coev}} & A \otimes A^* \otimes A \\
{\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle \mathrm{ev} \otimes 1} \\
A & \xleftarrow[\lambda_A]{} & \mathbf{1} \otimes A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathbf{1} \otimes A^* & \xrightarrow{\mathrm{coev} \otimes 1} & A^* \otimes A \otimes A^* \\
{\scriptstyle \lambda_{A^*}}\downarrow & & \downarrow{\scriptstyle 1 \otimes \mathrm{ev}} \\
A^* & \xleftarrow[\rho_{A^*}]{} & A^* \otimes \mathbf{1}
\end{array}
\tag{6.1}
$$

commute.

Remark 6.2. Let A be an object in a concrete compact closed category $\mathbf{V}$.

(a) The evaluation morphism may be written as

$$\mathrm{ev}\colon A \otimes A^* \to \mathbf{1}; x \otimes \chi \mapsto x^\chi .$$

(b) The coevaluation morphism may be written as

$$\mathrm{coev}\colon \mathbf{1} \to A^* \otimes A; 1 \mapsto \epsilon^l \otimes e^r$$

with Sweedler-like conventions for the image. Specific versions are given in (6.2) and (6.5) below.

(c) The elements of A^* are described as *characters* of A.

Example 6.3. Suppose that A is a finitely generated free unital module over a commutative ring K (e.g., a finite-dimensional vector space over a field K). Suppose that $\mathbf{V}$ is the full subcategory of $\underline{\underline{K}}$ on the object class of all finitely generated free modules. Then $A^* = \underline{\underline{K}}(A, K)$. Let $\{e_1, \dots, e_n\}$ be a basis for A, with corresponding dual basis $\{\epsilon_1, \dots, \epsilon_n\}$ given by $e_i^{\epsilon_j} = \delta_{ij}$ for $1 \leq i, j \leq n$. Then

$$\epsilon^l \otimes e^r = \sum_{i=1}^{n} \epsilon_i \otimes e_i \tag{6.2}$$

according to the Sweedler notation of Remark 6.2. The commuting of the diagrams (6.1) corresponds to the equations

$$x\epsilon^l \otimes e^r = \sum_{i=1}^{n} x^{\epsilon_i} e_i = x \quad \text{and} \quad \epsilon^l \otimes e^r \chi = \sum_{i=1}^{n} \epsilon^i e_i^{\chi} = \chi$$

for $x \in A$ and $\chi \in A^*$.

Lemma 6.4. *Let A be an object in a concrete compact closed category $\mathbf{V}$. Consider elements $x_1 \otimes y_1$ and $x_2 \otimes y_2$ of $A \otimes A$. Suppose that*

$$(x_1 \otimes y_1)(1 \otimes \chi) = (x_2 \otimes y_2)(1 \otimes \chi) \tag{6.3}$$

for each character χ of A. Then $x_1 \otimes y_1 = x_2 \otimes y_2$.

Proof. One has

$$\begin{aligned} x_1 \otimes y_1 &= x_1 \otimes y_1 \epsilon^l \otimes e^r = (x_1 \otimes y_1)(1 \otimes \epsilon^l) \otimes e^r \\ &= (x_2 \otimes y_2)(1 \otimes \epsilon^l) \otimes e^r = x_2 \otimes y_2 \epsilon^l \otimes e^r = x_2 \otimes y_2 \end{aligned}$$

by (6.1) and (6.3). □

6.2. **Finite semilattices.** This brief optional section works through an example of duality in a non-linear setting (compare [1, §6]). Let $\mathbf{V}$ be the category of finite join semilattices with lower bounds 0 (commutative, idempotent monoids $(A, +, 0)$, ordered by $x \leq y$ iff $x + y = y$) and semilattice homomorphisms that preserve the lower bounds (i.e., monoid homomorphisms). Then $\mathbf{V}$ is a symmetric, monoidal category, with tensor unit $\mathbf{1} = \{0, 1\}$ [4, 19].

For an object A of $\mathbf{V}$, consider $A^* = \mathbf{V}(A, \mathbf{1})$, with the pointwise bounded semilattice structure inherited from $\mathbf{1}$. Within the theory of ordered sets, elements of A^* are described as *characters* of A, inspiring the terminology of Remark 6.2(c). Each element α of A determines a character $\widehat{\alpha}$ with

$$a\widehat{\alpha} = 1 \quad \Leftrightarrow \quad \alpha \leq a \tag{6.4}$$

for each element a of A. Conversely, each element f of A^* appears as the character $\widehat{\alpha}$ for $\alpha = \min\{a \in A \mid af = 1\}$.

The coevaluation is given by

$$\mathrm{coev}\colon \mathbf{1} \to A^* \otimes A; 1 \mapsto \sum_{\alpha \in A} \widehat{\alpha} \otimes \alpha\,. \tag{6.5}$$

The commuting of the first diagram (6.1) then corresponds to the equation

$$a \sum_{\alpha \in A} \widehat{\alpha} \otimes \alpha = \sum_{\alpha \in A} a\widehat{\alpha} \otimes \alpha = \sum_{\alpha \leq a} \alpha = a$$

for $a \in A$. Now consider

$$(1 \otimes \widehat{\chi})(\mathrm{coev} \otimes 1)(1 \otimes \mathrm{ev}) = \Big(\sum_{\alpha \in A} \widehat{\alpha} \otimes \alpha \otimes \widehat{\chi}\Big)(1 \otimes \mathrm{ev}) = \sum_{\alpha \in A} \widehat{\alpha} \otimes \alpha\widehat{\chi} = \sum_{\chi \leq \alpha} \widehat{\alpha}$$

for $\chi \in A$. For each element a of A, one has

$$a \sum_{\chi \leq \alpha} \widehat{\alpha} = 1 \ \Leftrightarrow \ \exists\ \alpha \geq \chi\,.\ a\widehat{\alpha} = 1 \ \Leftrightarrow \ \exists\ \alpha \geq \chi\,.\ \alpha \leq a \ \Leftrightarrow \ \chi \leq a\,.$$

Comparison with (6.4) then shows that

$$\widehat{\chi} = \sum_{\chi \leq \alpha} \widehat{\alpha}\,,$$

ensuring the commutativity of the second diagram (6.1).

6.3. **Characteristic morphisms.** This section introduces concepts which will be needed in §6.4 below.

Definition 6.5. Let (A, Δ) be a comagma in a concrete compact closed category $\mathbf{V}$.

(a) Set $R_\chi = (1_A \otimes \Delta)(1_A \otimes 1_A \otimes \chi)(1_A \otimes \rho_A)$ for each character χ of A.
(b) Set $L_\chi = (\Delta \otimes 1_A)(\chi \otimes 1_A \otimes 1_A)(\lambda_A \otimes 1_A)$ for each character χ of A.
(c) A $\mathbf{V}$-morphism $M : A \otimes A \to A \otimes A$ is said to be *right characteristic* if it commutes with R_χ for each character χ of A.
(d) Dually, a $\mathbf{V}$-morphism $M : A \otimes A \to A \otimes A$ is described as *left characteristic* if it commutes with L_χ for each character χ of A.

Proposition 6.6. *Let (A, ∇, Δ) be a coassociative bimagma in a concrete compact closed category $\mathbf{V}$.*

(a) The right composite morphism $\reflectbox{$\mathsf{G}$}$ is right characteristic.
(b) Dually, the left composite morphism G is left characteristic.

Proof. For $a\otimes b$ in $A\otimes A$, one has $(a\otimes b)\reflectbox{\textsf{G}}R_\chi = ab^L\otimes b^{RL}b^{RR\chi}$. On the other hand, $(a\otimes b)R_\chi\reflectbox{\textsf{G}} = (a\otimes b^Lb^{R\chi})\reflectbox{\textsf{G}} = ab^{LL}\otimes b^{LR}b^{R\chi}$. Then by coassociativity,

$$\begin{aligned} ab^{LL}\otimes b^{LR}b^{R\chi} &= (a\otimes b^{LL}\otimes b^{LR}\otimes b^R)(\nabla\otimes 1_A\otimes\chi)(1\otimes\rho_A) \\ &= (a\otimes b^L\otimes b^{RL}\otimes b^{RR})(\nabla\otimes 1_A\otimes\chi)(1\otimes\rho_A) = ab^L\otimes b^{RL}b^{RR\chi} \end{aligned}$$

as required for (a). The proof of (b) is dual. □

In the context of Lemmas 4.4 and 4.5, the following result is worthy of note.

Proposition 6.7. (a) *Suppose that* $(A,\nabla,\Delta,\eta,\varepsilon,S)$ *is a left Hopf algebra within a concrete compact closed category* $\mathbf{V}$. *Then the section*

$$\reflectbox{\textsf{G}}_S = (1_A\otimes\Delta)(1_A\otimes S\otimes 1_A)(\nabla\otimes 1_A)$$

of the right composite $\reflectbox{\textsf{G}}$ *is right characteristic.*

(b) *Suppose that* $(A,\nabla,\Delta,\eta,\varepsilon,S)$ *is a right Hopf algebra within a concrete compact closed category* $\mathbf{V}$. *Then the retract*

$$\reflectbox{\textsf{G}}_S = (1_A\otimes\Delta)(1_A\otimes S\otimes 1_A)(\nabla\otimes 1_A)$$

of the right composite $\reflectbox{\textsf{G}}$ *is right characteristic.*

Proof. For $a\otimes b$ in $A\otimes A$, one has $(a\otimes b)\reflectbox{\textsf{G}}_SR_\chi = ab^{LS}\otimes b^{RL}b^{RR\chi}$. On the other hand, $(a\otimes b)R_\chi\reflectbox{\textsf{G}} = (a\otimes b^Lb^{R\chi})\reflectbox{\textsf{G}}_S = (a\otimes b^L\reflectbox{\textsf{G}}_S)\cdot b^{R\chi} = ab^{LLS}\otimes b^{LR}b^{R\chi}$. Then by coassociativity,

$$\begin{aligned} ab^{LLS}\otimes b^{LR}b^{R\chi} &= (a\otimes b^{LL}\otimes b^{LR}\otimes b^R)(1_A\otimes S\otimes 1_A\otimes 1_A)(\nabla\otimes\rho_A) \\ &= (a\otimes b^L\otimes b^{RL}\otimes b^{RR})(1_A\otimes S\otimes 1_A\otimes 1_A)(\nabla\otimes\rho_A) = ab^{LS}\otimes b^{RL}b^{RR\chi} \end{aligned}$$

as required. □

The dual of Proposition 6.7 becomes relevant in the context of Lemmas 4.2 and 4.3.

Corollary 6.8. (a) *Suppose that* $(A,\nabla,\Delta,\eta,\varepsilon,S)$ *is a right Hopf algebra within a concrete compact closed category* $\mathbf{V}$. *Then the section*

$$\mathsf{G}_S = (\Delta\otimes 1_A)(1_A\otimes S\otimes 1_A)(1_A\otimes\nabla)$$

of the left composite G *is left characteristic.*

(b) *Suppose that* $(A,\nabla,\Delta,\eta,\varepsilon,S)$ *is a left Hopf algebra within a concrete compact closed category* $\mathbf{V}$. *Then the retract*

$$\mathsf{G}_S = (\Delta\otimes 1_A)(1_A\otimes S\otimes 1_A)(1_A\otimes\nabla)$$

of the left composite G *is left characteristic.*

6.4. **From right composite sections to left antipodes.** The following result may be viewed as a partial converse to Lemma 4.5. The hypothesis it imposes on the section $\reflectbox{\textsf{G}}_s$ is justified by Proposition 6.7(a).

Theorem 6.9. *Let* $\mathbf{V}$ *be a concrete compact closed category. Suppose that* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a bimonoid in* $\mathbf{V}$ *for which the right composite* $\reflectbox{\textsf{G}}$ *has a right characteristic section* $\reflectbox{\textsf{G}}_s$*, so that* $\reflectbox{\textsf{G}}_s\reflectbox{\textsf{G}} = 1_{A\otimes A}$*. Then*

$$S = \lambda_A^{-1}(\eta \otimes 1_A)\reflectbox{\textsf{G}}_s(1_A \otimes \varepsilon)\rho_A \tag{6.6}$$

is a left antipode for the bimonoid $(A, \nabla, \Delta, \eta, \varepsilon)$.

Proof. For an element a of A, write $(1 \otimes a)\reflectbox{\textsf{G}}_s = a_l \otimes a_r$, using a Sweedler-like notation. Note that

$$aS = (1 \otimes a)\reflectbox{\textsf{G}}_s(1_A \otimes \varepsilon)\rho_A = a_l \cdot a_r^{\varepsilon}\,. \tag{6.7}$$

Also,

$$1 \otimes a = (a_l \otimes a_r)\reflectbox{\textsf{G}} = a_l a_r^L \otimes a_r^R \tag{6.8}$$

implies

$$1 \otimes a^{\varepsilon} = a_l a_r^L \otimes a_r^{R\varepsilon} = a_l a_r^L a_r^{R\varepsilon} \otimes 1 = a_l a_r \otimes 1$$

by counitality. The equation

$$1 \cdot a^{\varepsilon} = a_l a_r \tag{6.9}$$

then results by multiplication.

For a character χ of A, applying $R_\chi \reflectbox{\textsf{G}}_s = \reflectbox{\textsf{G}}_s R_\chi$ to the left hand equation of (6.8) yields

$$(1 \otimes a^L a^{R\chi})\reflectbox{\textsf{G}}_s = a_l \otimes a_r^L a_r^{R\chi}\,. \tag{6.10}$$

Now by counitality and the fact that elements of A^* are $\mathbf{V}$-morphisms:

$$\begin{aligned}(1 \otimes a)\reflectbox{\textsf{G}}_s(1_A \otimes \chi) &= (a_l \otimes a_r)(1_A \otimes \chi) = a_l \otimes (a_r^{L\varepsilon} a_r^R)\chi = a_l \otimes a_r^{L\varepsilon} a_r^{R\chi}\\ &= a_l \otimes (a_r^L a_r^{R\chi})\varepsilon = (a_l \otimes a_r^L a_r^{R\chi})(1_A \otimes \varepsilon) = (1 \otimes a^L a^{R\chi})\reflectbox{\textsf{G}}_s(1_A \otimes \varepsilon)\\ &= (a^L a^{R\chi})S = (a^{LS} a^{R\chi}) = a^{LS} \otimes a^{R\chi} = (a^L \otimes a^R)(S \otimes 1_A)(1_A \otimes \chi)\end{aligned}$$

for each character χ of A. Lemma 6.4 then implies the first equation of

$$(a^L \otimes a^R)(S \otimes 1_A) = (1 \otimes a)\reflectbox{\textsf{G}}_s = a_l \otimes a_r\,.$$

Applying ∇, and recalling (6.9), yields $a^{LS}a^R = 1 \cdot a^{\varepsilon}$, verifying that S is a left antipode. □

The dual of Theorem 6.9 provides a converse to Lemma 4.3, with the hypothesis on G_s being justified by Corollary 6.8(a).

Corollary 6.10. *Let* $\mathbf{V}$ *be a concrete compact closed category. Suppose that* $(A, \nabla, \Delta, \eta, \varepsilon)$ *is a bimonoid in* $\mathbf{V}$ *for which the left composite* G *has a left characteristic section* G_s*, so that* $\mathsf{G}_s\mathsf{G} = 1_{A\otimes A}$*. Then*

$$S = \rho_A^{-1}(1_A \otimes \eta)\mathsf{G}_s(\varepsilon \otimes 1_A)\lambda_A \tag{6.11}$$

is a right antipode for the bimonoid $(A, \nabla, \Delta, \eta, \varepsilon)$*.*

7. A left-sided quantum group

This section will motivate and summarize the construction of the left Hopf algebra that was exhibited in [18], and examine two properties of its composite morphisms. Let K be a field, with a chosen nonzero element q. Consider an alphabetically ordered set $\{a < b < c < d\}$, generating a free monoid $\{a, b, c, d\}^*$, which extends linearly to the free algebra $K\{a, b, c, d\}^*$. The polynomial algebra $K[a, b, c, d]$ is the quotient of $K\{a, b, c, d\}^*$ obtained by imposing mutual relations of commutativity between the elements of $\{a, b, c, d\}$.

The examples of the first two sections are very well-known Hopf algebras. However, it should be noted that their properties are often obtained by repeated Ore extensions [11, §§IV.4–6], whereas we prefer to emphasize the "rewriting rule" approach — see Remark 7.3(b) — which best lends itself to the construction of the left Hopf algebra appearing in §7.3.

7.1. The Hopf algebra SL_2.

Definition 7.1. The *matrix bialgebra* $\mathsf{M}(2)$ is obtained by equipping the polynomial algebra $K[a, b, c, d]$ with the comultiplication summarized by

$$\Delta\colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{7.1}$$

and counit summarized by

$$\varepsilon\colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.2}$$

using matrices as place-holders, and a mnemonic at the end of (7.1).

Now consider the quotient SL_2 of the matrix algebra $(\mathsf{M}(2), \nabla, \eta)$ obtained by imposing the relation

$$ad - bc = 1\,. \tag{7.3}$$

The algebra SL_2 becomes a Hopf algebra with comultiplication (7.1) and counit (7.2), when equipped with the antipode

$$S\colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{7.4}$$

again defined by using matrices as place-holders, and producing a magma homomorphism $S\colon (\mathsf{SL}_2, \nabla) \to (\mathsf{SL}_2, \tau\nabla)$.

7.2. The quantum group $\mathsf{SL}_q(2)$.

Definition 7.2. As an algebra, the *quantum matrix bialgebra* $\mathsf{M}_q(2)$ is the quotient of $K\{a, b, c, d\}^*$ that is obtained by imposing the six commutation relations

$$\begin{aligned} & ba = qab\,,\ dc = qcd\,,\ ca = qac\,,\ db = qbd\,, \\ & cb = bc\,, \quad \text{and} \quad da = ad + (q - q^{-1})bc\,. \end{aligned} \tag{7.5}$$

The coalgebra structure is given by the comultiplication (7.1) and counit (7.2).

Remark 7.3. (a) Note that $\mathsf{M}_q(2)$ reduces to $\mathsf{M}(2)$ when $q = 1$.

(b) The commutation relations (7.5) enable the elements of $\mathsf{M}_q(2)$ to be represented by words in *normal form*: linear combinations of free monoid elements $a^{e_a}b^{e_b}c^{e_c}d^{e_d}$ in which the respective powers of a, b, c, d appear in alphabetical order. To this end, one may express the commutation relations in the form

$$\begin{aligned} & ba \to qab\,,\ dc \to qcd\,,\ ca \to qac\,,\ db \to qbd\,, \\ & cb \to bc\,, \quad \text{and} \quad da \to ad + (q - q^{-1})bc \end{aligned} \tag{7.6}$$

that emphasizes their role as a collection of rewriting rules.

(c) The commutation relations in the top line of (7.5) may be expressed in the symbolic form

$$\begin{bmatrix} b \\ d \end{bmatrix} \circ \begin{bmatrix} a \\ c \end{bmatrix} = q \begin{bmatrix} a \\ c \end{bmatrix} \circ \begin{bmatrix} b \\ d \end{bmatrix} \tag{7.7}$$

and

$$\begin{bmatrix} c & d \end{bmatrix} \circ \begin{bmatrix} a & b \end{bmatrix} = q \begin{bmatrix} a & b \end{bmatrix} \circ \begin{bmatrix} c & d \end{bmatrix} \tag{7.8}$$

using the entrywise or Hadamard product $\circ$ of matrices. The relations (7.7) are known as the *column relations*, while the relations (7.8) are known as the *row relations*.

The quantum group $\mathsf{SL}_q(2)$ is obtained as the quotient of the quantum matrix bialgebra $\mathsf{M}_q(2)$ obtained by imposing the algebra relation

$$ad - q^{-1}bc = 1 \tag{7.9}$$

whose left hand side is described as the *quantum determinant*. The bialgebra $\mathsf{SL}_q(2)$ becomes a Hopf algebra when equipped with the antipode

$$S\colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix} \tag{7.10}$$

defined as usual by using matrices as place-holders, and producing a magma homomorphism $S\colon (\mathsf{SL}_q(2), \nabla) \to (\mathsf{SL}_q(2), \tau\nabla)$. Note that $\mathsf{SL}_q(2)$ reduces to SL_2 when $q = 1$.

7.3. **The left Hopf algebra $\widetilde{\mathsf{SL}}_q(2)$.** Consider the quotient $\widetilde{\mathsf{SL}}_q(2)$ of the free algebra $K\{a, b, c, d\}^*$ that is obtained by imposing the commutation relations

$$ca = qac\,, \quad da - qbc = 1\,, \qquad db = qbd\,, \quad ad - q^{-1}cb = 1\,. \tag{7.11}$$

Note the appearance of the row relations (7.8) and twisted versions of the quantum determinant relation (7.9). The algebra becomes a bialgebra when equipped with the comultiplication (7.1) and counit (7.2) [18, p.157].

The commutation relations (7.11) may be expressed in the form

$$ca \to qac\,, \quad da \to qbc + 1\,, \qquad db \to qbd\,, \quad cb \to qad - q1 \tag{7.12}$$

of rewriting rules, analogous to the rewriting rules (7.6) for $\mathsf{M}_q(2)$.

Definition 7.4. (a) A word in the free monoid $\{a, b, c, d\}^*$ is said to be *irreducible* if it does not contain a subword of the form ca, db, da, or cb.

(b) Let B denote the set of irreducible words in the free monoid $\{a, b, c, d\}^*$.

Lemma 7.5. [18, p.157] *The rewriting rules* (7.12) *are confluent, and each element of* $\widetilde{\mathsf{SL}}_q(2)$ *has a unique irreducible normal form. In particular, the set B forms a basis of the vector space* $\widetilde{\mathsf{SL}}_q(2)$.

Theorem 7.6. [18, p.157] *Define a linear map* $S\colon \widetilde{\mathsf{SL}}_q(2) \to \widetilde{\mathsf{SL}}_q(2)$ *by* (7.10) *and* $S\colon B \to B; x_1x_2 \dots x_{r-1}x_r \mapsto x_r^S x_{r-1}^S \dots x_2^S x_1^S$ *for* $x_i \in \{a, b, c, d\}$. *Then* $\widetilde{\mathsf{SL}}_q(2)$ *is a left Hopf algebra with S as one of the left antipodes, but not a right antipode.*

7.4. **Non-injectivity of the right composite.**

Proposition 7.7. *The nonzero element*

$$(dc - qcd) \otimes a + q^2(cb - bc) \otimes c \tag{7.13}$$

of the tensor square of $\widetilde{\mathsf{SL}}_q(2)$ *lies in the kernel of the right composite.*

Proof. Observe that (7.13) is nonzero, since it reduces as

$$\begin{aligned}
&(dc - qcd) \otimes a + q^2(cb - bc) \otimes c \\
&\quad = (dc - qcd) \otimes a + q^2(qad - q1 - bc) \otimes c \\
&\quad = -q^3(1 \otimes c) + q^3(ad \otimes c) - q^2(bc \otimes c) - q(cd \otimes a) + dc \otimes a
\end{aligned}$$

in terms of the lexicographically ordered basis $B\otimes B$ of the tensor square of $\widetilde{\mathsf{SL}}_q(2)$. Now by (1.2) and (7.1), one has $\big((dc-qcd)\otimes a+q^2(cb-bc)\otimes c\big)\partial =$

$$\begin{aligned}&dca\otimes a - qcda\otimes a + dcb\otimes c - qcdb\otimes c\\ &\qquad + q^2cbc\otimes a - q^2bc^2\otimes a + q^2cbd\otimes c - q^2bcd\otimes c\\ &= (q^2bc^2+qc)\otimes a - (q^3adc - q^3c + qc)\otimes a + q^2bcd\otimes c\\ &\qquad - (q^3ad^2 - q^3d)\otimes c + (q^3adc - q^3c)\otimes a - q^2bc^2\otimes a\\ &\qquad + (q^3ad^2 - q^3d)\otimes c - q^2bcd\otimes c = 0\end{aligned}$$

as required. □

7.5. Non-surjectivity of the left composite.

Theorem 7.8. *The left composite of the left Hopf algebra* $\widetilde{\mathsf{SL}}_q(2)$ *is not surjective.*

Proof. Consider an element x of the set B from Definition 7.4(b). Suppose that $x\otimes 1$ lies in the image of the left composite, say as the image of an element $y_l\otimes y_r$ of the tensor square, written in Sweedler-like notation (and thus not necessarily of tensor rank 1). Since $B\otimes B$ is a basis for the tensor square, the elements y_l may and will be taken from B. Then

$$(y_l\otimes y_r)\mathsf{G} = y_l^L\otimes y_l^R y_r = x\otimes 1\,. \tag{7.14}$$

Applying $1_{\widetilde{\mathsf{SL}}_q(2)}\otimes\varepsilon$ to the final equation of (7.14) yields

$$x\otimes 1 = y_l^L\otimes y_l^{R\varepsilon}y_r^{\varepsilon} = y_l^L y_l^{R\varepsilon}\otimes y_r^{\varepsilon} = y_l\otimes y_r^{\varepsilon} \tag{7.15}$$

with the final equality holding by counitality. Applying $\rho_{\widetilde{\mathsf{SL}}_q(2)}$ to (7.15) yields $x = y_l\cdot y_r^{\varepsilon}$. Thus $y_l\otimes y_r$ is indeed of tensor rank 1, with $x=y_l$ and $y_r^{\varepsilon}=1$.

Now take $x=a$, so that $a\otimes 1 = (a\otimes y_r)\mathsf{G} = a^L\otimes a^R y_r = a\otimes ay_r + b\otimes cy_r$ according to (7.1). Thus $a\otimes(ay_r-1)+b\otimes cy_r = 0$. Respective projections to $a\otimes\widetilde{\mathsf{SL}}_q(2)$ and $b\otimes\widetilde{\mathsf{SL}}_q(2)$ yield $ay_r=1$ and $cy_r=0$. According to (7.11) one then has $q^{-1}c = q^{-1}cay_r = acy_r = 0$, which contradicts Lemma 7.5. □

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