HOUSTON JOURNAL OF MATHEMATICS, Volume 16, No. 3, 1990.

ON THE STRUCTURE OF BARYCENTRIC ALGEBRAS

A. B. ROMANOWSKA AND J. D. H. SMITH

Abstract. Barycentric algebras are decomposed as collections of algebraically open convex sets indexed by their semilattice replicas. They are characterized as subalgebras of Płonka sums of convex sets. Their structure is analyzed completely.

1. Introduction. Let E be a real vector space. For each real number r, one may define a binary operation $\underline{r}$ on E by the affine combination

$$\underline{r} : E \times E \to E; \;\; (x, y) \to xy\underline{r} = x(1-r) + yr. \tag{1.1}$$

This makes $(E, \mathbf{R})$ an idempotent, entropic [**10**, 128(ii)] algebra — a *mode* in the sense of [**10**]. For $(E, \mathbf{R})$, the entropic property just means that

$$xy\underline{r}\ zt\underline{r}\ \underline{s} = xz\underline{s}\ yt\underline{s}\ \underline{r}$$

for any r, s in $\mathbf{R}$ and x, y, z, t in E. Let I^o denote the open unit interval $]0,1[$ (the interior of the closed unit interval $I = [0,1]$). Then subalgebras (C, I^o) of the reduct (E, I^o) of $(E, \mathbf{R})$ are just convex subsets C of E. The Class $\underline{\underline{C}}$ of *convex sets*, i.e., of such subalgebras (C, I^o) of reducts of real vector spaces $(E, \mathbf{R})$ under affine combinations (1.1) with $r \in I^o$, does not form a variety in the sense of universal algebra [**10**, page 17], since it is not closed under the taking of homomorphic images [**10**, Example 122]. The variety generated by $\underline{\underline{C}}$ is the variety $\underline{\underline{B}}$ of *barycentric algebras.* According to [**10**, Theorem 214], an algebra (B, I^o) of type [**10**, page 5] $I^o \to \{2\}$ is a barycentric algebra if and only if it satisfies the identities

$$xx\underline{p} = x \;\; \text{(idempotence)}, \tag{1.2}$$

$$xy\underline{p} = yx\underline{p'} \;\; \text{(skew-commutativity), and} \tag{1.3}$$

$$xy\underline{p}z\underline{q} = xyz\underline{(q/(p'q'))}\,\underline{(p'q')'} \;\; \text{(skew-associativity)}, \tag{1.4}$$

for p, q in I^o, where $p' = 1 - p$. Barycentric algebras have also been studied under the names "semiconvex set" [**2**], [**13**], [**14**] and "convexor" [**3**], [**4**], [**11**], [**12**]. Convex sets (C, I^o) may be recognized amongst all barycentric algebras as being *cancellative*:

$$(1.5) \qquad \forall\, p \in I^o,\ \forall\, x, y, z \in C,\ xy\underline{p} = xz\underline{p} \implies y = z$$

[**10**, Theorem 269].

The significance of barycentric algebras is that they provide a general algebraic framework for the study of convexity. Convexity is closely connected with order, and barycentric algebras give a uniform treatment of convex sets and semilattices. Section 2 below presents a list of examples of barycentric algebras illustrating this uniform treatment. The main purpose of the paper is to present three theorems describing the structure of general barycentric algebras, formulating precisely the way they combine order and convexity. The theorems are stated in the fourth section, after a description in Section 3 of some algebraic construction methods based on semilattices. The theorems are the Decomposition Theorem 4.4, the Characterization Theorem 4.5, and the Structure Theorem 4.6. The Decomposition Theorem is proved in the fifth section, which examines the "algebraically open" barycentric algebras defined in Section 4. The sixth section then gives the proof of the Characterization and Structure Theorems.

2. Examples of barycentric algebras. The examples presented here are of two main types: either theoretically significant, or having such curious structural features that they provide interesting test cases for the theorems of Section 4.

EXAMPLE 2.1 (real affine spaces): Let E be a real vector space. Then the algebra $(E, \mathbf{R})$ of (1.1) describes E as a real affine space [**10**, Corollary 256]. Its reduct (E, I^o) is a barycentric algebra. Note that fixing the base point 0 recovers the real vector space structure from the affine space $(E, \mathbf{R})$.

EXAMPLE 2.2 (convex sets): A convex set (C, I^o) is a subalgebra of the reduct (E, I^o) of an affine space $(E, \mathbf{R})$ as in Example 2.1. Convex sets are also characterized as barycentric algebras that are cancellative in the sense of (1.5). As in [**7**, 4.3] or the proof of [**10**, Theorem 269], each convex set C embeds into a minimal real affine space $(C\mathbf{R}^+V, \mathbf{R})$ called its *affine hull*.

EXAMPLE 2.3 (semilattices): A *semilattice* (H, o) is an idempotent, commutative semigroup. A semilattice becomes a barycentric algebra (H, I^o)

(an *I^o-semilattice* in the terminology of [**10**, page 31]) on defining

$$xy\underline{p} = xoy \tag{2.1}$$

for all p in I^o. The skew-commutativity (1.3) follows from the commutativity of o, and the skew-associativity (1.4) follows from its associativity. Semilattices carry an additional order structure in one of two dual ways. A *join semilattice* $(H, +)$ gives an ordered set $(H, \leq_+)$ with

$$x \leq_+ y \Leftrightarrow x + y = y, \tag{2.2}$$

while a *meet semilattice* $(H, \cdot)$ gives an ordered set $(H, \leq_\cdot)$ with

$$x \leq_\cdot y \Leftrightarrow x \cdot y = x. \tag{2.3}$$

EXAMPLE 2.4 (the extended reals): Part of the usefulness of barycentric algebras in analysis is that they give an algebraic structure to the set $\mathbf{R}^\infty = \mathbf{R} \cup \{\infty\}$ of extended reals. The barycentric algebra $(\mathbf{R}^\infty, I^o)$ has $(\mathbf{R}, I^o)$ of Example 2.1 as a subalgebra, while $x \infty \underline{p} = \infty$ for any x in $\mathbf{R}^\infty$, p in I^o.

EXAMPLE 2.5 (simplicial complexes) [**7**, page 15], [**10**, Exercise 282]: Let K be the (geometric realization of) a simplicial complex. Then there is a barycentric algebra $K \cup \{\infty\}$ with $xy\underline{p} = \infty$ if x and y are not contained in a simplex; if x and y are contained in a simplex, which is a convex set, then $xy\underline{p}$ is the usual convex combination within that simplex.

EXAMPLE 2.6 (the "T"): Let $T = [0,1] \cup]0,1]$. The closed interval $[0,1]$ is called the *crossbar* of the T and $]0,1]$ is called the *vertical.* For x, y both contained within $[0,1]$ or within $]0,1]$, define $xy\underline{p}$ to be the usual convex combination. For x in $[0,1]$ and y in $]0,1]$, define $xy\underline{p} = yp \in]0,1]$. The T is $\mathcal{U}_2$ of [**3**] or the "blocked space" Ξ of [**2**]. It may be coordinatized as a subalgebra of the direct square of the extended reals. The crossbar embeds as $[0,1] \to \mathbf{R}^\infty \times \mathbf{R}^\infty$; $x \mapsto (0, x)$, while the vertical embeds as $]0,1] \to \mathbf{R}^\infty \times \mathbf{R}^\infty$; $y \mapsto (y, \infty)$.

EXAMPLE 2.7 (subalgebra modes): Since barycentric algebras are modes, the general theory of [**10**] applies to them. In particular, given a barycentric algebra (B, I^o), one obtains new barycentric algebras (BS, I^o) and (BP, I^o) using complex products [**10**, (144)] on the sets BS of all subalgebras of (B, I^o) and BP of all finitely-generated subalgebras of (B, I^o) [**10**, Corollary 215].

3. Semilattice sums. Semilattices form the basis for a range of general algebraic construction techniques, known generically as "semilattice sums." The most elegant of these is the Płonka sum [**10**, Definition 236], introduced by Płonka [**9**] under the name "sum of a direct system" as a generalization of A. H. Clifford's "strong semilattice of semigroups" [**1**], [**8**, I.8.7]. The description of Płonka sums in [**10**, 2.3] was based on meet semilattices, but for current purposes it is more natural to consider a join semilattice $(H,+)$, simultaneously an I^o-semilattice (H,I^o) as in Example 2.3. As a partial order $(H,\leq_+)$, the set H is the set of objects of a small category (H) having unique morphism $x \to y$ if and only if $x \leq_+ y$. Consider the category (I^o) whose objects are *I^o-algebras*, algebras (A,I^o) of type [**10**, page 5] $I^o \to \{2\}$ such as barycentric algebras. The morphisms of (I^o) are I^o-homomorphisms. Suppose given a covariant functor $F : (H) \to (I^o)$. Then the *Płonka sum* of the algebras hF (for h in H) is the disjoint union $HF = \dot\bigcup_{h\in H} hF$ of the underlying sets, equipped with the I^o-algebra structure given by

$$(3.1)\quad \underline{p} : hF \times kF \to (h+k)F;\ (x,y) \mapsto x(h \to h+k)F\, y(k \to h+k)F\, \underline{p}$$

for each p in I^o and h,k in H. Note that there is an I^o-algebra homomorphism $\pi_F : (HF,I^o) \to (H,I^o)$, called the *projection*, having restrictions $\pi_F : (hF,I^o) \to (\{h\},I^o)$. The subalgebras (hF,I^o) of the Płonka sum (HF,I^o) are called the *Płonka fibres*. As an example, the barycentric algebra of extended reals (Example 2.4) is a Płonka sum over the 2-element join semilattice $H = \{0 \leq_+ 1\}$. The fibre $0F$ is the barycentric algebra $(\mathbf{R},I^o)$ as in Example 2.1, while $1F$ is the singleton $(\{\infty\},I^o)$. The morphism $(0 \to 1)F$ is the unique mapping $\mathbf{R} \to \{\infty\}$. Płonka sums give a way of constructing many new barycentric algebras.

PROPOSITION 3.1. *A Płonka sum of barycentric algebras is itself a barycentric algebra.*

PROOF: The identities (1.2) — (1.4) specifying barycentric algebras are "regular" in the sense of [**10**, Proposition 238]. A Płonka sum satisfies all the regular identities satisfied by each of its fibres. Thus a Płonka sum of barycentric algebras is a barycentric algebra. ∎

The extended reals, for example, are obtained as the Płonka sum by the functor F from the 2-element semilattice $\{0 \leq_+ 1\}$ with $0F = (\mathbf{R},I^o)$ and $1F = (\{\infty\},I^o)$.

A special case of Proposition 3.1 is that a Płonka sum of convex sets is a barycentric algebra. The Characterization Theorem 4.5 below will characterize all barycentric algebras as subalgebras of Płonka sums of convex sets.

The conditions on a Płonka sum, although natural, are very strong, and do not obtain in general. It is not true that each algebra of type $I^o \to \{2\}$ projecting onto an I^o-semilattice (H, I^o) is a Płonka sum over that semilattice. In [**10**, Definition 623], a more general construction method called the "Lallement sum" was introduced, extending and adapting some semigroup-theoretical work of Lallement [**5**, 2.19]. In fact, as was shown by [**10**, Theorem 624], these Lallement sums have extremely broad applicability. The full generality of Lallement sums is not required here, but the concepts underlying them are needed, in the context of I^o-algebras.

To begin with, define a *sink* [**10**, page 73] S in an I^o-algebra (A, I^o) to be a subset S of A satisfying

$$\forall\, p \in I^o,\ \forall x, y \in A,\ (x \in S \text{ or } y \in S) \Rightarrow xy\underline{p} \in S. \tag{3.2}$$

Sinks are automatically subalgebras. A congruence θ on (A, I^o) is said *to preserve* the sink (S, I^o) if the restriction of the natural projection $A \to A^\theta$; $a \mapsto a^\theta$ to the subalgebra (S, I^o) injects. The algebra (A, I^o) is said to be an *envelope* of a subalgebra (S, I^o) if (S, I^o) is a sink of (A, I^o) such that equality is the only congruence on (A, I^o) preserving (S, I^o). For example, the closed unit interval (I, I^o) is an envelope of the open unit interval (I^o, I^o).

The version of Lallement sums to be used here is as follows (cf. [**10**, 6.2]).

DEFINITION 3.2: Let H be an I^o-semilattice (H, I^o) and a join semilattice $(H, +, \leq_+)$. Suppose given an envelope (E_h, I^o) of an I^o-algebra (C_h, I^o) for each element h of H. For each $h \leq_+ k$ in $(H, \leq_+)$, suppose given an I^o-homomorphism $\phi_{h,k} : (C_h, I^o) \to (E_k, I^o)$ such that:

(a) $\phi_{h,h}$ is the injection of C_h into E_h;

(b) $\forall\, p \in I^o$, $(C_h\phi_{h,h+h'})(C_{h'}\phi_{h',h+h'})\underline{p} \subseteq C_{h+h'}$;

(c) $\forall\, h + h' \leq k$, $\forall\, a \in C_h$, $\forall\, b \in C_{h'}$, $\forall\, p \in I^o$,

$$(a\phi_{h,h+h'})\,(b\phi_{h',h+h'})\,\underline{p}\phi_{h+h',k} = (a\phi_{h,k})\,(b\phi_{h',k})\,\underline{p};$$

(d) $\forall\, k \in H$, $E_k = \{a\phi_{h,k} \mid h \leq_+ k,\ a \in C_h\}$.

Then the disjoint union $B = \cup_{h \in H} C_h$ equipped with the operations

$$(3.3) \qquad \underline{p} : C_h \times C_{h'} \to C_{h+h'};\ (a, b) \mapsto (a\phi_{h,h+h'})(b\phi_{h',h+h'})\underline{p}$$

for each p in I^o is called the *coherent Lallement sum of the algebras* (C_h, I^o) *over the semilattice* H *by the mappings* $\phi_{h,k}$, or more briefly a *coherent Lallement sum.*

Note that the left hand side of the equality in (c) is defined, since $(a\phi_{h,h+h'})(b\phi_{h',h+h'})\underline{p} \in C_{h+h'}$ holds by condition (b). The conditions (a) — (d) are best viewed as generalizations of the functoriality in Płonka sums, where the envelopes coincide with their sinks. A coherent Lallement sum has a *projection* $\pi : (B, I^o) \to (H, I^o)$, an I^o-homomorphism with restrictions $(C_h, I^o) \to (\{h\}, I^o)$. The subalgebras (C_h, I^o) of (B, I^o) are also called the *fibres* of the Lallement sum.

In the proof of the Characterization Theorem 4.5 below to be given in Section 6, a certain extra condition on coherent Lallement sums is considered. This condition is

$$(3.4) \qquad \begin{gathered} \forall\ h, h' \leq_+ k \leq_+ \ell,\ \forall\ a \in C_h,\ \forall\ b \in C_{h'}, \\ a\phi_{h,k} = b\phi_{h',k} \Rightarrow a\phi_{h,\ell} = b\phi_{h',\ell}. \end{gathered}$$

The significance of condition (3.4) resides in the following

PROPOSITION 3.3. *If a coherent Lallement sum satisfies condition* (3.4), *then it is a subalgebra of the Płonka sum of its envelopes.*

PROOF: A functor $F : (H) \to (I^o)$ will be defined. For k in H, the object kF is the envelope (E_k, I^o). For $k \leq_+ \ell$, a morphism $(k \to \ell)F : E_k \to E_\ell$ is needed. Now by (d) of Definition 3.2, each element x of E_k is of the form $a\phi_{h,k}$ for a in C_h with $h \leq_+ k$. Define $x(k \to \ell)F$ to be $a\phi_{h,\ell}$. This is certainly an element of E_ℓ. The definition is good, since if $x = a\phi_{h,k} = b\phi_{h',k}$ for b in $C_{h'}$ with $h' \leq_+ k$, then $a\phi_{h,\ell} = b\phi_{h',\ell}$ by condition (3.4). To see that $(k \to \ell)F$ is an I^o-homomorphism, consider $x = a\phi_{h,k}$ and $y = b\phi_{h',k}$ in E_k. Then for p in I^o,

$$\begin{aligned} x(k \to \ell)F\, y(k \to \ell)F\, \underline{p} &= (a\phi_{h,\ell})\,(b\phi_{h',\ell})\,\underline{p} \\ &= (a\phi_{h,h+h'})\,(b\phi_{h',h+h'})\,\underline{p}\phi_{h+h',\ell} \\ &= (a\phi_{h,h+h'})\,(b\phi_{h',h+h'})\,\underline{p}\phi_{h+h',k}(k \to \ell)F \\ &= (a\phi_{h,k})\,(b\phi_{h',k})\,\underline{p}(k \to \ell)F = xy\underline{p}(k \to \ell)F, \end{aligned}$$

the second and fourth equalities holding by (c) of Definition 3.2 while the third holds by definition of $(k \to \ell)F$. Thus $(k \to \ell)F$ is indeed an I^o-homomorphism. Also $x(k \to k)F = a\phi_{h,k} = x$, so $(k \to k)F$ is the identity on kF. Now suppose $k \leq_+ \ell \leq_+ m$ in H. Then $x(k \to \ell)F(\ell \to m)F = a\phi_{h,\ell}(\ell \to m)F = a\phi_{h,m} = x(k \to m)F$, completing the verification that $F : (H) \to (I^o)$ is a functor. It remains to check that the Lallement sum B is a subalgebra of the Płonka sum by the functor F. Consider elements $a = a\phi_{h,h}$ and $b = b\phi_{h',h'}$ of B. Then for p in I^o, the proeuct $ab\underline{p}$ in the Płonka sum is calculated as $a(h \to h+h')F\, b(h' \to h+h')F\underline{p} = (a\phi_{h,h+h'})(b\phi_{h',h+h'})\underline{p}$, which by (3.3) is just the product $ab\underline{p}$ in the Lallement sum. ∎

4. Structure theorems for barycentric algebras. In this section, the three main theorems describing the structure of barycentric algebras are formulated. The first of these is the Decomposition Theorem 4.4. As its name implies, it describes how barycentric algebras break up into smaller pieces. These pieces are certain convex sets, and they are indexed by a semilattice associated with the barycentric algebra. This semilattice is the *(I^o)-semilattice replica* [**10**, page 17] (BR, I^o) of the barycentric algebra (B, I^o). The semilattice (BR, I^o) is the quotient (B^ρ, I^o) of (B, I^o) by the *semilattice replica congruence* ρ such that any I^o-homomorphism $f : (B, I^o) \to (H, I^o)$ from (B, I^o) to an I^o-semilattice (H, I^o) factors as $f = (\mathrm{nat}\,\rho)f'$ through a unique homomorphism $f' : (B^\rho, I^o) \to (H, I^o)$. Thus the congruence ρ identifies precisely those elements of B which are identified in all I^o-homomorphisms from (B, I^o) to some I^o-semilattice.

The semilattice replica of a barycentric algebra may be given an explicit description in terms of "walls" of the barycentric algebra. Recall that a subset X of a barycentric algebra (B, I^o) is a subalgebra of (B, I^o) iff

$$(4.1) \qquad \forall\, p \in I^o,\ \forall\, x, y \in B,\ (x \in X \text{ and } y \in X) \Rightarrow xy\underline{p} \in X.$$

A subset W of B is said to be a *wall* [**10**, page 61] of (B, I^o) iff

$$(4.2) \qquad \forall\, p \in I^o,\ \forall\, x, y \in B,\ (x \in W \text{ and } y \in W) \Leftrightarrow xy\underline{p} \in W.$$

Thus walls are special subalgebras. Each barycentric algebra has the improper wall (B, I^o). If the barycentric algebra is a convex set, then the walls in the sense of (4.2) are just the walls in the geometric sense [**6**, page 8].

(Some authors also use the term "face," although the word "face" may be used with different meanings. Compare also the concept of "filter" as used in semigroup theory [**8**, I.8.2].) The set of walls of a barycentric algebra (B, I^o) is partially ordered by inclusion. From (4.2), the intersection of a family of walls is again a wall. For a subset X of B, let $[X]$ denote the intersection of all walls of B containing X. Then the set of walls of B forms a join semilattice under the operation $W + W' = [W \cup W']$. For a singleton $X = \{x\}$, write $[\{x\}] = [x]$. Such walls are called *principal walls.*

LEMMA 4.1. *For any x, y in B and p in I^o, one has $[x] + [y] = [xy\underline{p}]$.*

PROOF: On the one hand $xy\underline{p} \in [xy\underline{p}]$ implies $x, y \in [xy\underline{p}]$ by (4.2), so $[x], [y] \subseteq [xy\underline{p}]$ and $[x] + [y] \leq_+ [xy\underline{p}]$. On the other hand, since x and y are contained in the subalgebra $[x] + [y]$, one has $xy\underline{p} \in [x] + [y]$, whence $[xy\underline{p}] \leq_+ [x] + [y]$. ∎

Lemma 4.1 shows that the principal walls of the barycentric algebra (B, I^o) form a subsemilattice $(H, +)$ of the join semilattice of all walls. Moreover, it shows that the mapping

$$B \to H; \ x \mapsto [x] \tag{4.3}$$

is an I^o-homomorphism from the barycentric algebra (B, I^o) onto the I^o-semilattice (H, I^o) of principal walls.

Walls also have a topological significance. If the affine hull of a convex set is a finite-dimensional Euclidean space, then the convex set is open in its affine hull iff it has no proper non-empty walls (see [**10**, Exercise 386] and Proposition 5.1 below, cf. also [**7**, 4.4]). This motivates the following

DEFINITION 4.2: A barycentric algebra is said to be *algebraically open* iff it has no proper non-empty walls.

It also raises the

PROBLEM 4.3: Under what topologies on its affine hull is a convex set open there if and only if it is algebraically open?

It will be shown below (Proposition 5.2) that algebraically open barycentric algebras are ncesssarily convex sets.

THEOREM 4.4 (Decomposition Theorem). *The semilattice replica of a bary-*

centric algebra is its semilattice of principal walls. The fibres over the semilattice are algebraically open convex sets.

The theorem will be proved in Section 5. (It was adumbrated somewhat by [**2**, Example 2.3] and comparable results of semigroup theory [**8**, I.8]. Skornjakov [**12**, page 4] also reports that his pupil "V. V. Ignatov showed that each convexor [barycentric algebra] decomposes as a semilattice of affine convexors [convex sets]," referring to the thesis [**4**].)

The formulation of the Characterization Theorem needs no preliminary definitions, given the concept of Płonka sum from Section 3.

THEOREM 4.5 (Characterization Theorem). *An I^o-algebra is a barycentric algebra if and only if it is a subalgebra of a Płonka sum of convex sets.*

The Characterization Theorem will be proved in Section 6.

The last of the main theorems is the Structure Theorem, which will be given a categorical formulation. Let $\underline{\underline{B}}$ denote the category whose objects are barycentric algebras and whose morphisms are I^o-homomorphisms between them. A new *category* $\underline{\underline{T}}$ *of barycentric structures* will be defined. Its objects are *barycentric structures* $\Phi_H = (\phi_{h,k} : C_h \to E_k \mid h \leq_+ k$ in $H)$. They consist of a join semilattice H, an algebraically open convex set (C_h, I^o) for each h in H, a convex set envelope (E_h, I^o) of (C_h, I^o) for each h in H, and an I^o-homomorphism $\phi_{h,k} : C_h \to E_k$ for each $h \leq_+ k$ in H such that (3.4) and the conditions (a) — (d) of Definition 3.2 are satisfied. The morphisms $f : \Phi_H = (\phi_{h,k} : C_h \to E_k \mid h \leq_+ k$ in $H) \to \Phi'_{H'} = (\phi'_{h',k'} : C'_{h'} \to E'_{k'} \mid h' \leq_+ k'$ in $H')$ of $\underline{\underline{T}}$ are semilattice homomorphisms $f : (H,+) \to (H',+)$ with I^o-homomorphisms $f_h : (E_h, I^o) \to (E'_{hf}, I^o)$ for each h in H, restricting to I^o-homomorphisms $f_h : (C_h, I^o) \to (C'_{hf}, I^o)$, such that the diagrams

$$\text{(4.4)} \qquad \begin{array}{ccc} C_h & \xrightarrow{f_h} & C'_{hf} \\ {\scriptstyle \phi_{h,k}}\downarrow & & \downarrow{\scriptstyle \phi'_{hf,kf}} \\ E_k & \xrightarrow{f_k} & E'_{kf} \end{array}$$

commute for each $h \leq_+ k$ in H.

THEOREM 4.6 (Structure Theorem). *There are functors* $d : \underline{\underline{B}} \to \underline{\underline{T}}$ *(known as* decomposition*) and* $L : \underline{\underline{T}} \to \underline{\underline{B}}$ *(known as* Lallement sum*) by which the category* $\underline{\underline{B}}$ *of barycentric algebras is equivalent to the category* $\underline{\underline{T}}$ *of barycentric structures.*

The Structure Theorem 4.6 gives a complete analysis of the structure of barycentric algebras and their homomorphisms (to within a knowledge of convex sets). It will be proved in Section 6.

5. Walls, sinks, and the Decomposition Theorem. In this section, the Decomposition Theorem 4.4 is proved. The proof is based on the concepts of "sink" as in (3.2) and "wall" as in (4.2). Note that the complement of a wall is a sink. Conversely, although the complement of a sink need not be a subalgebra, the complement is a wall if it is a subalgebra. The concepts are related further, as shown by

PROPOSITION 5.1. *Let* (B, I^o) *be a barycentric algebra. Then the following conditions are equivalent:*

(i) (B, I^o) *is algebraically open;*
(ii) (B, I^o) *has no proper non-empty wall;*
(iii) (B, I^o) *has no proper non-empty sink;*
(iv) *there is no* I^o*-epimorphism* $(B, I^o) \to (\{0,1\}, I^o)$ *onto the two-element join semilattice* $\{0 \leq_+ 1\}$.

PROOF: If $|B| \leq 1$, the equivalence is clear, so assume $|B| > 1$. The equivalence of (i) and (ii) is Definition 4.2. Since the complement of a proper non-empty wall is a proper non-empty sink, (iii) implies (ii). Conversely, assume (ii) holds. Suppose (iii) is false, so that (B, I^o) has a proper non-empty sink S. Consider the set of subalgebras of (B, I^o) contained in the complement of S. This set is partially ordered by inclusion, and is closed under unions of chains. Zorn's Lemma gives a maximal element W. It will be shown that W is a wall of (B, I^o). This contradicts (ii), since W, containing a singleton $\{x\}$ with x in the complement of S, is non-empty. Suppose $xy\underline{p}$ lies in W for x, y in B and p in I^o. For W to be a wall, both x and y must lie in W. Suppose by contradiction that x is not in W. (Skew-commutativity (1.3) shows that no generality is lost by focussing on x rather than y.) By the maximality of W, some element z of the subalgebra of (B, I^o) generated by W and x lies in S. Now $z \neq x$, since $x \in S$ implies $xy\underline{p} \in S \cap W = \varnothing$, a contradiction. Nor does z lie in W, again since $S \cap W$

is empty. Repeated application of idempotence, skew-commutativity and skew-associativity shows that z is of the form $wx\underline{q}$ for some q in I^o and w in W. But for $r = q/(1-p+pq)$ and $s = (1-p)/(1-p+pq)$ one has that W contains $wxy\underline{pr} = ywx\underline{qs} = yz\underline{s} \in S \cap W$, the required contradiction proving that (ii) implies (iii). Now (iv) implies (ii) since a proper non-empty wall W furnishes an I^o-epimorphism $(W \to \{0\}) \cdot ((B - W) \to \{1\})$. Conversely, (ii) implies (iv) since the preimage of 0 under an I^o-epimorphism $(B, I^o) \to (\{0 \leq_+ 1\}, I^o)$ is a proper non-empty wall of (B, I^o). ∎

The next proposition depends on the way sinks arise from a breakdown of cancellativity.

PROPOSITION 5.2. *A barycentric algebra is algebraically open only if it is a convex set.*

PROOF: Let (B, I^o) be an algebraically open barycentric algebra. If (B, I^o) has less than two elements, then it is certainly a convex set. So assume for the rest of the proof that B has at least two elements. Suppose that (B, I^o) is not a convex set. Then by [**10**, Theorem 269], condition (1.5) breaks down, i.e. there are elements $a \neq b$ and x of B such that

$$\exists\, p \in I^o.\ xa\underline{p} = xb\underline{p}. \tag{5.1}$$

By [**7**, Lemma 2], (5.1) is equivalent to

$$\forall\, p \in I^o,\ \ xa\underline{p} = xb\underline{p}. \tag{5.2}$$

For each q in I^o and y in B, define

$$R_q(y) : (B, I^o) \to (B, I^o);\ z \mapsto zy\underline{q}. \tag{5.3}$$

Since barycentric algebras are modes, the mappings (5.3) are I^o-homomorphisms. The truth and equivalence of (5.1) and (5.2) show that the equalizer S of $R_q(a)$ and $R_q(b)$ is a non-empty subalgebra of (B, I^o) independent of the choice of q. It is a proper subalgebra, since $a, b \in S$ implies $a = aR_p(a) = aR_p(b) = ab\underline{p} = ba\underline{p'} = bR_{p'}(a) = bR_{p'}(b) = b$, a contradiction. But S is a sink of (B, I^o), since for s in S, y in B, q in I^o, one has

$$ys\underline{q}a\underline{p} = ysa\underline{(p/(q'p'))}\,\underline{(q'p')'} = ysb\underline{((p/(q'p'))}\,\underline{(q'p')'} = ys\underline{q}b\underline{p},$$

whence $ys\underline{q} \in S$. By Proposition 5.1, the existence of the proper non-empty sink S contradicts the algebraic openness of (B, I^o). ∎

The Decomposition Theorem 4.4 may now be proved, using the notation of Section 4.

PROOF OF THEOREM 4.4: Let (B, I^o) be a barycentric algebra. Let σ be the kernel of the homomorphism (4.3), so that $x\sigma y$ iff $[x] = [y]$. Then σ contains the semilattice replica congruence ρ. The ρ-classes, as barycentric algebras, have no non-trivial semilattice quotients. Thus they are algebraically open by Proposition 5.1(iv). By Proposition 5.2, they are convex sets. It remains to show that σ actually coincides with ρ.

Now for b in B, one has

$$\forall\, a \in b^\rho,\ [a] = [b^\rho]; \tag{5.4}$$

certainly $a \in b^\rho$ implies $[a] \le [b^\rho]$. Conversely, note that $[a] \cap b^\rho$ is a non-empty wall of b^ρ, since $xy\underline{p} \in [a] \cap b^\rho$ with $x, y \in b^\rho$ implies $x, y \in [a]$, so $x, y \in [a] \cap b^\rho$. But b^ρ, being algebraically open, has no proper non-empty walls. Thus $[a] \cap b^\rho = b^\rho$, whence $b^\rho \le [a]$ and $[b^\rho] \le [a]$, completing the verification of (5.4).

Also, considering the semilattice replica (B^ρ, I^o) as a join semilattice $(B^\rho, \le_+)$, one has

$$[b^\rho] = \cup\{c^\rho \mid c^\rho \le_+ b^\rho\} \tag{5.5}$$

for each b in B. Certainly $c^\rho \le_+ b^\rho$ implies $cb\underline{p} \in c^\rho + b^\rho = b^\rho \le [b^\rho]$, whence $c \in [b^\rho]$, so that $[b^\rho] \ge \cup\{c^\rho \mid c^\rho \le_+ b^\rho\}$. On the other hand, the right hand side of (5.5) is the preimage in (B, I^o) under nat ρ of the principal wall $\{c^\rho \mid c^\rho \le_+ b^\rho\}$ of the semilattice (B^ρ, I^o) generated by b^ρ. Now preimages of walls under epimorphisms are walls, so that $[b^\rho]$ is contained in the right hand side of (5.5).

To complete the proof of Theorem 4.4, assume $x\,\sigma\,y$ in B. Then by (5.4), one has $[x^\rho] = [x] = [y] = [y^\rho]$. The expression (5.5) then shows $x^\rho \le_+ y^\rho$ and $y^\rho \le_+ x^\rho$, whence $x^\rho = y^\rho$ or $x\,\rho\,y$. Thus σ is also contained in ρ, so that ρ and σ do indeed coincide. ∎

6. Proof of the Characterization and Structure Theorems. The proof of the Characterization Theorem 4.5 will follow easily in this section during the course of the proof of the Structure Thereom 4.6. The "decomposition" functor $D : \underline{\underline{B}} \to \underline{\underline{T}}$ is based on the decomposition (4.3) of a barycentric algebra (B, I^o) over its I^o-semilattice replica (H, I^o), the join semilattice of principal walls. For k in H, define the convex set C_k to be the pre-image of k under (4.3). Define the *pre-envelope* P_k over k to be the subalgebra (5.5) of (B, I^o) for $b^\rho = C_k$. The envelope E_k will then be obtained as a quotient of the pre-envelope P_k by a certain congruence μ on it. Note that for p in I^o, the mapping $R_p(b) : ([b^\rho], I^o) \to (b^\rho, I^o); x \mapsto xb\underline{p}$ (as in (5.3)) is an I^o-homomorphism. By [7, Lemma 2], i.e. by the equivalence of (5.1) and (5.2), the kernel of $R_p(b)$ is independent of the choice of p in I^o. Define the congruence μ_k or μ on P_k to be the intersection of all these kernels as b ranges through C_k.

LEMMA 6.1. *The congruence μ_k on P_k is the maximal congruence on P_k preserving C_k.*

PROOF: Since the convex set C_k is cancellative, the kernel of each $R_p(b)$ preserves C_k, whence their intersection μ_k does also. Now suppose that a congruence θ on P_k preserves C_k. If $x \,\theta\, y$, then for each b in C_k and p in I^o one has $b \,\theta\, b \Rightarrow xb\underline{p} \,\theta\, yb\underline{p}$. But $xb\underline{p}, yb\underline{p} \in C_k$. Since θ preserves C_k, the equality $xb\underline{p} = yb\underline{p}$ or $xR_p(b) = yR_p(b)$ holds. Hence $x \,\mu_k\, y$ and $\theta \leq \mu_k$. ∎

COROLLARY 6.2. *The kernel of $R_p(b) : (P_k, I^o) \to (C_k, I^o)$ is independent of the choice of b in C_k.*

PROOF: The kernel preserves C_k, and thus is contained in μ_k by Lemma 6.1. But the kernel also contains μ_k, by the definition of the latter. ∎

Consider the quotient $E_k = P_k^\mu$ of P_k. Lemma 6.1 shows that C_k (identified with C_k^μ) is a subalgebra of E_k preserved only by equality. Since C_k is a sink of P_k, it is also a sink of E_k. Thus E_k is an envelope of the algebraically open convex set C_k.

LEMMA 6.3. *The envelope E_k is a convex set.*

PROOF: It will be shown that the barycentric algebra $E_k = P_k^\mu$ is cancellative. Suppose $x^\mu z^\mu \underline{p} = y^\mu z^\mu \underline{p}$ for x, y, z in P_k, so that $xz\underline{p} \,\mu\, yz\underline{p}$. Then for all q in I^o, b in C_k, one has $xR_q(b)zR_q(b)\underline{p} = xz\underline{p}R_q(b) = yz\underline{p}R_q(b) = yR_q(b)zR_q(b)\underline{p}$. But $P_kR_q(b)$ is contained in the cancellative algebra C_k, so that $xR_q(b) = yR_q(b)$, whence $x^\mu = y^\mu$ as required. ∎

For each $h \leq_+ k$ in H, there is an I^o-homomorphism

$$\mu_{h,k} : C_h \rightarrow E_k;\ x \mapsto x^{\mu}, \tag{6.1}$$

the restriction to C_h of the natural projection nat $\mu_k : P_k \rightarrow E_k$. This completes the definition of the barycentric structure $\Phi_H = (\phi_{h,k} : C_h \rightarrow E_k \mid h \leq_+ k \text{ in } H)$ associated by the decomposition functor D with the barycentric algebra (B, I^o) having semilattice H of principal walls.

LEMMA 6.4. *The I^o-homomorphisms* (6.1) *satisfy* (3.4) *and the conditions* (a) — (d) *of Definition* 3.2.

PROOF: Condition (a) is immediate, since μ_h preserves C_h. For (b), consider elements a of C_h and b of $C_{h'}$. Then $ab\underline{p}$ lies in $C_{h+h'}$. But

$$\begin{aligned}(a\phi_{h,h+h'})(b\phi_{h',h+h'})\underline{p} &= (a \operatorname{nat} \mu_{h+h'})(b \operatorname{nat} \mu_{h+h'})\underline{p} \\ &= ab\underline{p} \operatorname{nat} \mu_{h+h'} \\ &= ab\underline{p},\end{aligned}$$

verifying (b). The left hand side of (c) becomes

$$ab\underline{p} \operatorname{nat} \mu_k = (a \operatorname{nat} \mu_k)(b \operatorname{nat} \mu_k)\underline{p},$$

and thus coincides with the right hand side of (c). Condition (d) is immediate from the definition of E_k. To verify (3.4), consider c in C_k and d in D_ℓ, along with p, q in I^o. Then $a\phi_{h,k} = b\phi_{h',k'}$ implies $ac\underline{p} = bc\underline{p}$. Thus

$$aR_q(d)R_p(cd\underline{q}) = ad\underline{q}\, cd\underline{q}\, \underline{p} = ac\underline{p}\, dd\underline{p}\, \underline{q} = bc\underline{p}\, dd\underline{p}\, \underline{q} = bR_q(d)R_p(cd\underline{q}),$$

the second equality holding by the entropic law [**10**, 128(ii)]. Now $aR_q(d)$, $bR_q(d)$, and $cd\underline{q}$ lie in the cancellative algebra (C_k, I^o). Thus $aR_q(d) = bR_q(d)$, whence $a\mu_\ell b$ and $a\phi_{h,\ell} = b\phi_{h',\ell}$, as required. ∎

For a barycentric algebra morphism $\theta : (B, I^o) \rightarrow (B', I^o)$, the $\underline{\underline{T}}$-morphism $f = \theta D$ is defined as follows. The semilattice homomorphism $f : (H, I^o) \rightarrow (H', I^o)$ is given by the commutative diagram

$$\begin{array}{ccc} (B, I^o) & \xrightarrow{\ \theta\ } & (B', I^o) \\ {\scriptstyle \operatorname{nat}\rho}\downarrow & & \downarrow{\scriptstyle \operatorname{nat}\rho'} \\ (H, I^o) & \xrightarrow[\ f\ ]{} & (H', I^o) \end{array} \tag{6.2}$$

in $\underline{\underline{B}}$, where the vertical arrows are the projections onto the semilattice replicas. The I^o-homomorphisms $f_h : (E_h, I^o) \to (E'_{hf}, I^o)$ are given by the commutative diagrams

$$
(6.3) \qquad
\begin{array}{ccc}
(P_h, I^o) & \xrightarrow{\ \theta\ } & (P'_{hf}, I^o) \\
{\scriptstyle \mathrm{nat}\,\mu_h}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{nat}\,\mu'_{hf}} \\
(E_h, I^o) & \xrightarrow[f_h]{} & (E'_{hf}, I^o)
\end{array}
$$

in $\underline{\underline{B}}$, where $\mathrm{nat}\,\mu'_{hf} : (P'_{hf}, I^o) \to (E'_{hf}, I^o)$ denotes the natural projection of the pre-envelope P'_{hf} onto the corresponding envelope E'_{hf} in $B'D$. To see that f_h is well-defined by (6.3), suppose $a \,\mu_h\, b$. Choose an element $c\theta$ of C'_{hf} with c in C_h (this is always possible, by the definition of f). Then for p in I^o, the equality $ac\underline{p} = bc\underline{p}$ implies $a\theta c\theta \underline{p} = b\theta c\theta \underline{p}$, so that $a\theta \,\mu'_h\, b\theta$ as required. To check that $\overline{f} = \theta\overline{D}$ is a $\underline{\underline{T}}$-morphism, the commutativity of the diagrams (4.4) remains to be checked. But (4.4) becomes

$$
(6.4) \qquad
\begin{array}{ccc}
(C_h, I^o) & \xrightarrow{\ \theta\ } & (C'_{hf}, I^o) \\
{\scriptstyle \mathrm{nat}\,\mu_k}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{nat}\,\mu'_{kf}} \\
(E_k, I^o) & \xrightarrow[f_k]{} & (E'_{kf}, I^o)
\end{array}
\quad ,
$$

whose commutativity follows from that of (6.3). It is straightforward to verify that $D : \underline{\underline{B}} \to \underline{\underline{T}}$ is actually a functor.

The next step in the proof of the Structure and Characterization Theorems begins with the definition of the "Lallement sum" functor $L : \underline{\underline{T}} \to \underline{\underline{B}}$. Given a barycentric structure $\Phi_H = (\phi_{h,k} : C_h \to E_k \mid h \leq_+ k \text{ in } \overline{H})$, form the Lallement sum $\phi_H L = (B, I^o)$ with $B = \dot{\bigcup}_{h \in H} C_h$ and the I^o-operations defined as in (3.3). By Proposition 3.3, this Lallement sum $\Phi_H L$ is a subalgebra of a Płonka sum of its envelopes, which as convex sets are barycentric algebras. By Proposition 3.1, it follows that $\Phi_H L = (B, I^o)$ is a barycentric algebra. Given a $\underline{\underline{T}}$-morphism $f : \Phi_H \to \Phi_{H'}$, a mapping $fL : \Phi_H L \to \Phi_{H'} L$ is defined as the disjoint union $\dot{\bigcup}_{h \in H} (f_h : C_h \to C'_{hf})$.

Suppose given elements a of C_h, b of $C_{h'}$, and p of I^o. Then

$$\begin{aligned} ab\underline{p}(fL) &= (a\phi_{h,h+h'})(b\phi_{h',h+h'})\underline{p}f_{h+h'} \\ &= (a\phi_{h,h+h'}f_{h+h'})(b\phi_{h',h+h'}f_{h+h'})\underline{p} \\ &= (af_h\phi'_{hf,hf+h'f})(bf_{h'}\phi'_{h'f,hf+h'f})\underline{p} \\ &= a(fL)b(fL)\underline{p}. \end{aligned}$$

Here the first and fourth equalities hold by (3.3) and the definition of fL, the second equality holds since $f_{h+h'} : E_{h+h'} \to E'_{hf+h'f}$ is an I^o-homomorphism, and the third holds by (4.4). Thus fL is a $\underline{\underline{B}}$-morphism. It is then straightforward to see that $L : \underline{\underline{T}} \to \underline{\underline{B}}$ is functorial.

The proof of the Characterization Theorem 4.5 is a corollary of the following

LEMMA 6.5. *A barycentric algebra B is naturally isomorphic to BDL.*

PROOF: Suppose given a barycentric algebra B. Then for a in C_h, b in $C_{h'}$, p in I^o, the product $ab\underline{p}$ in BDL is given as $(a\phi_{h,h+h'})(b\phi_{h',h+h'})\underline{p} = a^\mu b^\mu \underline{p}$, where μ is the maximal congruence on the pre-envelope $P_{h+h'}$ preserving $C_{h+h'}$. But $a^\mu b^\mu \underline{p} = ab\underline{p}^\mu = ab\underline{p}$, the original product in (B, I^o). Thus (B, I^o) and (BDL, I^o) are naturally isomorphic. ∎

PROOF OF THEOREM 4.5: As observed above, the Lallement sum $\Phi_H L$ of a barycentric structure Φ_H is a subalgebra of a Płonka sum of convex sets. But a barycentric algebra B is naturally isomorphic to the Lallement sum $\Phi_H L$ of the barycentric structure $\Phi_H = BD$.

PROOF OF THEOREM 4.6: It follows from Lemma 6.5 that the composite functor DL is naturally equivalent to the identity on $\underline{\underline{B}}$. It thus remains to be shown that the composite LD is naturally equivalent to the identity on $\underline{\underline{T}}$. Let $\Phi_H = (\phi_{h,k} : C_h \to E_k \mid h \leq_+ k \text{ in } H)$ be a barycentric structure. Form the Lallement sum $\Phi_H L = (B, I^o)$. It projects onto the I^o-semilattice (H, I^o), with fibres (C_h, I^o) for h in H. Since these fibres are algebraically open, Proposition 5.1(iv) shows that H is the semilattice replica of $\Phi_H L$. For k in H, form the pre-envelope $P_k = \cup\{C_h \mid h \leq_+ k \text{ in } H\}$, a subalgebra of (B, I^o). By (3.3) and condition (c) of Definition 3.2, the mapping $\biguplus_{h \leq_+ k} (\phi_{h,k} : C_h \to E_k) : P_k \to E_k$ is an I^o-homomorphism. By Definition 3.2(d), it surjects. Let λ be its kernel. Thus P_k^λ is isomorphic to E_k. By condition (a) of Definition 3.2, the congruence λ preserves C_k. By

Lemma 6.1, λ is contained in the congruence μ on P_k. Conversely, suppose $x\mu y$ for some x in C_h and y in $C_{h'}$. Then there is some z in C_k and p in I^o such that the equation $xz\underline{p} = yz\underline{p}$ holds in $\phi_H L$. By (3.3), this means that the relation $(x\phi_{h,k})(z\phi_{k,k})\underline{p} = (y\phi_{h',k})(z\phi_{k,k})\underline{p}$ holds in the convex set (E_k, I^o). But (E_k, I^o) is cancellative, so $x\phi_{h,k} = y\phi_{h',k}$, i.e. $x\lambda y$. Thus the congruences λ and μ on P_k agree. This shows that the envelope P_k^μ built in $\Phi_H LD$ agrees with the algebra $E_k = P_k^\lambda$ in the barycentric structure ϕ_H. Furthermore, the homomorphisms $\operatorname{nat}\mu : C_h \to P_k^\mu$ built in $\Phi_H LD$ agree with the homomorphisms $\operatorname{nat}\lambda = \phi_{h,k} : C_h \to E_k$ of the barycentric structure Φ_H. ∎

ACKNOWLEDGEMENT: This paper was written while the first author was visiting the Department of Mathematics at Iowa State University during 1986/87.

REFERENCES

1. A. H. Clifford, *Semigroups admitting relative inverses*, Ann. Math. **42** (1941), 1037–1049.
2. J. Flood, *Semiconvex geometry*, J. Austral. Math. Soc. **30** (1981), 496–510.
3. V. V. Ignatov, *Quasivarieties of convexors*, (in Russian, translated in Sov. Math. (Iz. VUZ) 29(1985), 13–17), Izv. Vyssh. Uchebn. Zaved. Mat. **29** (1985), 12–14.
4. ______, *The structure of convexors*, (in Russian), VINITI No. 8018-84 (1984).
5. G. Lallement, *Demi-groupes réguliers*, Ann. Mat. Pura Appl. **77** (1967), 47–129.
6. B. Mielnik, *Theory of filters*, Commun. Math. Phys. **15** (1969), 1–46.
7. W. D. Neumann, *On the quasivariety of convex subsets of affine spaces*, Arch. Math. **21** (1970), 11–16.
8. M. Petrich, "Lectures in Semigroups," Akademie-Verlag, Berlin, 1977.
9. J. Płonka, *On a method of construction of abstract algebras*, Fund. Math. **61** (1967), 183–189.
10. A. B. Romanowska and J. D. H. Smith, "Modal Theory," Heldermann Verlag, Berlin, 1985.
11. L. A. Skornjakov, *Convexors*, Studia Sci. Math. Hungar. **16** (1981), 25–34.
12. ______, *Stochastic algebra*, (in Russian, translated in Sov. Math. (Iz. VUZ) 29 (1985), 1–12, Izv. Vyssh. Uchebn. Zaved. Mat. **29** (1985), 3–11.
13. T. Świrszcz, *Monadic functors and convexity*, Bull. Acad. Pol. Sci. Ser. Sci. Mat. Astron. et Phys. **22** (1974), 39–42.
14. ______, *Monadic functors and categories of convex sets*, Preprint 70, Institute of Mathematics, Polish Academy of Sciences (1975).

Warsaw Technical University
00661 Warsaw, Poland

Iowa State University
Ames, Iowa 50011

Received July 10, 1987