ON HOPF ALGEBRAS IN ENTROPIC JÓNSSON-TARSKI VARIETIES

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Abstract. Comonoid, bi-algebra, and Hopf algebra structures are studied within the universal-algebraic context of entropic varieties. Attention focuses on the behavior of setlike and primitive elements. It is shown that entropic Jónsson-Tarski varieties provide a natural universal-algebraic setting for primitive elements and group quantum couples (generalizations of the group quantum double). Here, the set of primitive elements of a Hopf algebra forms a Lie algebra, and the tensor algebra on any algebra is a bi-algebra. If the tensor algebra is a Hopf algebra, then the underlying Jónsson-Tarski monoid of the generating algebra is cancellative. The problem of determining when the Jónsson-Tarski monoid forms a group is open.

1. Introduction

The aim of this paper is to consider comonoid, bi-algebra, and Hopf algebra structure within the context of universal algebra. The general categorical setting for such structure is provided by symmetric monoidal categories (or "symmetric tensor categories", compare [11, p. 69]), with an associative tensor product and a unit object. Now an algebra (A, Ω) of type $\tau \colon \Omega \to \mathbb{N}$ is said to be *entropic* if the operation

$$\omega \colon A^{\omega\tau} \to A; (a_1, \ldots, a_{\omega\tau}) \mapsto a_1, \ldots, a_{\omega\tau}\omega$$

is a homomorphism for each operator ω in Ω. With homomorphisms as morphisms and the free algebra on one generator as unit, a variety **V** of universal algebras forms a symmetric monoidal category if its algebras are entropic, as discussed by Davey and Davis some thirty years ago [1]. In many senses,

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the current paper is a natural sequel to their work. Its focus is on those aspects of the bi-algebra and Hopf algebra structures which are peculiar to the universal-algebraic setting of entropic varieties, most notably concentrating on the concepts of setlike and primitive elements. Nevertheless, some standard Hopf algebra arguments are occasionally repeated here, both because they may not be familiar to universal algebraists, and because the set-theoretical details may differ in the current setting where a ring structure is being replaced by the free algebra on one generator.

The main novelty of the paper lies in the universal-algebraic treatment of primitive elements. Here, the natural context is provided by entropic Jónsson-Tarski varieties, which are characterized by their inclusion of a commutative monoid structure defined by derived operations [2]. The absence of general invertibility in these monoids is the major issue that arises, for example in the construction of a Lie bracket of primitive elements in Corollary 4.8. Again, while Corollary 5.4 shows that the commutative monoid of a Jónsson-Tarski algebra A is cancellative if the tensor algebra of A (free monoid over A in the sense of [1]) is a Hopf algebra, the question of determining when that cancellative commutative monoid is an abelian group emerges as the open Problem 5.5.

The plan of the paper is as follows. Section 2 gives a quick review of those categorical and elementary properties of tensor products in entropic varieties (beyond what is discussed in [1]) that are needed for the purposes of the paper, including a definition of the rank of a tensor (§2.1). Basic definitions and properties of monoids, comonoids, bi-algebras, and Hopf algebras in an entropic variety are presented in Section 3. These include a universal-algebraic version of Sweedler notation (also suitable for noncoassociative comultiplications), and a discussion of setlike elements of comonoids in an entropic variety (§3.4). The keystone of the paper consists of Section 4, focused on the concept of primitive elements of a comonoid in an entropic Jónsson-Tarski variety. Theorem 4.9 shows that the primitive elements of any Hopf algebra in an entropic Jónsson-Tarski variety form a Lie algebra. Theorem 5.2 exhibits the tensor algebra AT over an entropic Jónsson-Tarski algebra A as a bi-algebra where each element of A is primitive. The tensor algebra becomes a Hopf algebra when the Jónsson-Tarski monoid of A is an abelian group (Corollary 5.3). In Section 6 it is shown that entropic Jónsson-Tarski varieties form a natural setting for *group quantum couple* Hopf algebras, which are constructed in Theorem 6.1. The significance of these examples is that they are neither commutative nor cocommutative in general. Furthermore, they embrace a number of known constructions: group algebras, dual group algebras, and group quantum doubles.

In general, the paper uses the algebraic notations and conventions of [10]. For classical Hopf algebras (or "quantum groups"), one may refer to [3, 5, 11].

2. Tensor products

Let $\mathbf{V}$ be a variety of entropic algebras, often considered as a category with homomorphisms as morphisms. As examples, one may bear in mind the category $\mathbf{Set}$ of sets (no nontrivial operations), the category $\mathbf{B}$ of barycentric algebras [6, 7], the category $\underline{K}$ of unital modules over a commutative, unital ring K, and the categories of commutative monoids or bounded semilattices. For algebras A, B in $\mathbf{V}$, the set $\mathbf{V}(A, B)$ of homomorphisms from A to B is a subalgebra of the power $\mathbf{Set}(A, B)$ with pointwise operations inherited from B.

Consider a fixed algebra Y in $\mathbf{V}$. There is a functor $\mathbf{V}(Y, \underline{\quad})$ or $R_Y \colon \mathbf{V} \to \mathbf{V}$ with morphism part taking $f \colon X \to X'$ to

$$\mathbf{V}(Y, X) \to \mathbf{V}(Y, X'); g \mapsto gf \,.$$

This functor has a left adjoint S_Y [1, 4], [10, §IV.2.4].

Definition 2.1. For algebras Z, Y in $\mathbf{V}$, the *tensor product* $Z \otimes Y$ of Z and Y is the image ZS_Y of Z under the object part of the left adjoint S_Y to the functor $R_Y \colon \mathbf{V} \to \mathbf{V}; X \mapsto \mathbf{V}(Y, X)$.

The adjoint relationship between S_Y and R_Y may be summarized by the natural isomorphism

$$(2.1) \qquad \mathbf{V}(Z \otimes Y, X) \cong \mathbf{V}\big(Z, \mathbf{V}(Y, X)\big)$$

for algebras X, Y, Z in $\mathbf{V}$. The commutativity of the tensor product, in the form of an isomorphism

$$(2.2) \qquad \tau \colon Z \otimes Y \to Y \otimes Z$$

for algebras Y, Z in $\mathbf{V}$, is obtained as a consequence [10, III(3.6.6)]. The general associativity of the tensor product in an entropic variety is discussed in [1, §3], and will be used implicitly throughout this paper.

Example 2.2. In the category $\mathbf{Set}$ of sets, the general tensor product $Z \otimes Y$ of Definition 2.1 is the cartesian product $Z \times Y$, and the adjoint relationship (2.1) amounts to Currying [10, p. 12].

Example 2.3. In $\underline{K}$, the general tensor product $Z \otimes Y$ of Definition 2.1 reduces to the usual tensor product of K-modules [10, §III.3.6].

Example 2.4. In the category $\mathbf{B}$ of barycentric algebras, the tensor product $Z \otimes Y$ is a special case of the tensor product for modes [9, Proposition 3.5].

Proposition 2.5. *Let* $\mathbf{1}$ *be the free algebra in* $\mathbf{V}$ *on one generator. Then there are natural isomorphisms*

$$(2.3) \qquad \mathbf{1} \otimes A \xrightarrow{\lambda_A} A \xleftarrow{\rho_A} A \otimes \mathbf{1}$$

for each algebra A in $\mathbf{V}$.

Proof. There is a natural isomorphism $\mathbf{V}(\mathbf{1}, U) \cong U$ for each algebra U of $\mathbf{V}$. The adjoint relationship (2.1) then gives

$$\mathbf{V}(\mathbf{1} \otimes A, X) \cong \mathbf{V}\big(\mathbf{1}, \mathbf{V}(A, X)\big) \cong \mathbf{V}(A, X)$$

for each algebra X of $\mathbf{V}$. The uniqueness of adjoints yields $\mathbf{1} \otimes A \cong A$. The remainder of (2.3) follows from the commutativity (2.2) of the tensor product. $\square$

2.1. Tensor rank

Let $\mathbf{V}$ be a variety of entropic algebras.

Definition 2.6. Consider Z, Y, X in $\mathbf{V}$.

(a) A *bihomomorphism* is a function

$$f \colon Z \times Y \to X; (z, y) \mapsto (z, y)f$$

such that $Z \to X; z \mapsto (z, y)f$ is a homomorphism for each y in Y, and $Y \to X; y \mapsto (z, y)f$ is a homomorphism for each z in Z.

(b) The set of all bihomomorphisms from $Z \times Y$ to X is written as $\mathbf{V}(Z, Y; X)$.

Currying (compare Example 2.2) from the right-hand side of (2.1) yields a bijection

$$\mathbf{V}\big(Z, \mathbf{V}(Y, X)\big) \cong \mathbf{V}(Z, Y; X)$$

with the set of bihomomorphisms, so (2.1) may be rewritten as

$$(2.4) \qquad\qquad \mathbf{V}(Z \otimes Y, X) \cong \mathbf{V}(Z, Y; X) \,.$$

Setting $X = Z \otimes Y$ in (2.4) yields a bihomomorphism

$$(2.5) \qquad\qquad \otimes \colon Z \times Y \to Z \otimes Y; (z, y) \mapsto z \otimes y \,,$$

corresponding to $1_{Z \otimes Y}$ on the left of (2.4). Elements of the image of (2.5) inside $Z \otimes Y$ are described as having *tensor rank* 1. More generally, a *tensor* or element t of $Z \otimes Y$ is said to have *tensor rank* r if

$$(2.6) \qquad\qquad (z_1 \otimes y_1) \cdots (z_r \otimes y_r)w = t$$

for some $\mathbf{V}$-word w of minimal arity r such that an expression of t in the form (2.6) is available. (The existence of such an expression for each tensor t is noted in [1, p. 70]. Simply put, $Z \otimes Y$ is generated by the set of elements of tensor rank 1.)

Example 2.7. The variety $\mathbf{Set}_0$ of pointed sets, with a unique nullary operation selecting a constant 0, and no other basic operations, is entropic. In a pointed set A with pointed element 0, denote $A \smallsetminus \{0\}$ by $A^\sharp$. Call $A^\sharp$ the set of *non-zero* elements of A. For pointed sets Z and Y, the tensor product $Z \otimes Y$ is the disjoint union $\{0\} \cup (Z^\sharp \times Y^\sharp)$. The element 0 has tensor rank 0, while non-zero elements of $Z \otimes Y$ have tensor rank 1. In homotopy theory, this tensor product is known as the *smash product* or *reduced join* [12, §III.2].

Proposition 2.5 may now be reformulated in "elementary" terms (compare [1, §2]).

Proposition 2.8. *Let* $\mathbf{1}$ *be the free* $\mathbf{V}$*-algebra on a single generator* x.

(a) *For* $a \in A \in \mathbf{V}$, *the isomorphisms* (2.3) *are implemented as*

$$x \otimes a \xmapsto{\ \lambda_A\ } a \xmapsfrom{\ \rho_A\ } a \otimes x \ .$$

(b) *In particular,* $\mathbf{1}$ *carries a commutative monoid multiplication*

$$\nabla \colon \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}; x \otimes x \mapsto x$$

and a comultiplication

$$\Delta \colon \mathbf{1} \to \mathbf{1} \otimes \mathbf{1}; x \mapsto x \otimes x$$

that are mutually inverse.

Proof. The identity element of the monoid $\mathbf{1}$ is x. For unary words xu and xv, entropicity gives $xuv = xvu$, so $(xu \otimes xv)\nabla = xuv = xvu = (xv \otimes xu)\nabla$. $\square$
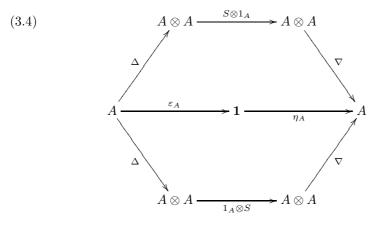
3. Diagrams and definitions

Let A be an algebra in an entropic variety $\mathbf{V}$. Consider the *algebra diagrams*

(3.1)
$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{1_A \otimes \nabla} & A \otimes A \\
{\scriptstyle \nabla \otimes 1_A} \downarrow & & \downarrow {\scriptstyle \nabla} \\
A \otimes A & \xrightarrow{\nabla} & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A \otimes A & \xleftarrow{1_A \otimes \eta} & A \otimes \mathbf{1} \\
{\scriptstyle \eta \otimes 1_A} \downarrow & \searrow {\scriptstyle \nabla} & \downarrow {\scriptstyle \rho_A} \\
\mathbf{1} \otimes A & \xrightarrow{\lambda_A} & A
\end{array}
$$

in the category $\mathbf{V}$, the dual *coalgebra diagrams*

(3.2)
$$
\begin{array}{ccc}
A \otimes A \otimes A & \xleftarrow{1_A \otimes \Delta} & A \otimes A \\
{\scriptstyle \Delta \otimes 1_A} \uparrow & & \uparrow {\scriptstyle \Delta} \\
A \otimes A & \xleftarrow{\Delta} & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A \otimes A & \xrightarrow{1_A \otimes \varepsilon} & A \otimes \mathbf{1} \\
{\scriptstyle \varepsilon \otimes 1_A} \downarrow & \nearrow {\scriptstyle \Delta} & \uparrow {\scriptstyle \rho_A^{-1}} \\
\mathbf{1} \otimes A & \xleftarrow{\lambda_A^{-1}} & A
\end{array}
$$

in the category $\mathbf{V}$, the *bi-algebra diagram*

(3.3)
$$
\begin{array}{ccccccc}
\mathbf{1} \otimes \mathbf{1} & \xrightarrow{\nabla} & \mathbf{1} & \xleftarrow{1} \quad \mathbf{1} & \xrightarrow{\Delta} & \mathbf{1} \otimes \mathbf{1} \\
{\scriptstyle \varepsilon \otimes \varepsilon} \uparrow & & & & & \downarrow {\scriptstyle \eta \otimes \eta} \\
A \otimes A & \xrightarrow{\nabla} & A & & \xrightarrow{\Delta} & A \otimes A \\
{\scriptstyle \Delta \otimes \Delta} \downarrow & & & & & \uparrow {\scriptstyle \nabla \otimes \nabla} \\
A \otimes A \otimes A \otimes A & & \xrightarrow{1_A \otimes \tau \otimes 1_A} & & & A \otimes A \otimes A \otimes A
\end{array}
$$
with ε and η arrows from $\mathbf{1}$ to A.

in the category $\mathbf{V}$, and the *antipode diagram*

(3.4)

in the category $\mathbf{V}$, all of which are commutative diagrams. The bottom arrow in the bi-algebra diagram makes use of the "twist" isomorphism τ of (2.2), while the top row of the diagram uses the multiplication and comultiplication from Proposition 2.8(b).

3.1. Basic definitions

Definition 3.1. Let $\mathbf{V}$ be an entropic variety.

(a.1) A *monoid* in $\mathbf{V}$ is a $\mathbf{V}$-algebra A with a $\mathbf{V}$-homomorphism

$$\nabla \colon A \otimes A \to A$$

known as *multiplication*, and a $\mathbf{V}$-homomorphism $\eta \colon \mathbf{1} \to A$ known as the *unit*, such that the algebra diagrams (3.1) commute.

(a.2) Let A and B be monoids in $\mathbf{V}$. Then a *monoid homomorphism* $f \colon A \to B$ is a $\mathbf{V}$-homomorphism such that the diagrams
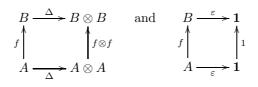

commute.

(b.1) A *comonoid* in $\mathbf{V}$ is a $\mathbf{V}$-algebra A with a $\mathbf{V}$-homomorphism

(3.5) $\Delta \colon A \to A \otimes A; a \mapsto \big((a^{L_1} \otimes a^{R_1}) \cdots (a^{L_{n_a}} \otimes a^{R_{n_a}}) \big) w_a$

known as *comultiplication*, and a $\mathbf{V}$-homomorphism $\varepsilon \colon A \to \mathbf{1}$ known as the *counit*, such that the coalgebra diagrams (3.2) commute.

(b.2) Let A and B be comonoids in **V**. A *comonoid homomorphism* $f\colon A \to B$ is a **V**-homomorphism such that the diagrams

$$
\begin{array}{ccc}
B & \xrightarrow{\ \Delta\ } & B \otimes B \\
\big\uparrow f & & \big\uparrow f \otimes f \\
A & \xrightarrow[\Delta]{} & A \otimes A
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
B & \xrightarrow{\ \varepsilon\ } & \mathbf{1} \\
\big\uparrow f & & \big\uparrow 1 \\
A & \xrightarrow[\varepsilon]{} & \mathbf{1}
\end{array}
$$

commute.

(c) A *bi-algebra* in **V** is a monoid and comonoid A in **V** such that the bi-algebra diagram (3.3) commutes. A *bi-algebra homomorphism* $f\colon A \to B$ is a monoid and comonoid homomorphism between bi-algebras A and B.

(d) A *Hopf algebra* in **V** is a bi-algebra A in **V** with a **V**-homomorphism $S\colon A \to A$ known as the *antipode*, such that the antipode diagram (3.4) commutes.

Remark 3.2. (a) Monoids in entropic varieties were studied in [1, §4].

(b) The image in (3.5) is written in a universal-algebraic version of the well-known *Sweedler notation* for Hopf algebra comultiplications. A more compact but rather less explicit version of this notation is

$$(3.6) \qquad\qquad\qquad a\Delta = a^L \otimes a^R\,,$$

with the understanding that the tensor rank of the image is not implied to be 1. In contrast to the usual Sweedler notation, the notation (3.6) is also appropriate for noncoassociative comultiplications.

Definition 3.3. Let A or $(A, \nabla, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra in an entropic variety **V**. Then A^{op} denotes the structure $(A, \tau\nabla, \eta, \Delta\tau, \varepsilon, S)$ on the underlying set A.

3.2. Basic properties

The primary example for Definition 3.1(d) is the following.

Example 3.4. Suppose that **V** is the category $\underline{\underline{K}}$ of unital modules over a commutative, unital ring K. Then the object $\mathbf{1}$ is the free module K. A *Hopf algebra* is a K-module A equipped with structure making the diagrams (3.1)–(3.4) commute.

Definition 3.5. Let **V** be an entropic variety.

(a) A monoid, bi-algebra, or Hopf algebra A in **V** is *commutative* if $\tau\nabla = \nabla$.

(b) A comonoid, bi-algebra, or Hopf algebra A in **V** is called *cocommutative* if $\Delta\tau = \Delta$.

Proposition 3.6. *Let A be a Hopf algebra in an entropic variety* **V***, with antipode S.*

(a) *The map $S\colon A \to A^{\mathsf{op}}$ is a bi-algebra homomorphism.*
(b) *If A is commutative or cocommutative, then $S^2 = 1_A$.*

Proof. The equivalent statements for Hopf algebras (as in Example 3.4) may be proved by commuting diagrams that are equally valid in a general entropic variety $\mathbf{V}$ (compare, e.g., [11, Proposition 9.1]). $\square$

3.3. Tensor products of monoids

Definition 3.7. Let $\mathbf{V}$ be an entropic variety, with free algebra $\mathbf{1}$ on a singleton $\{x\}$.

(a) Given monoids A and B in $\mathbf{V}$, with respective multiplications ∇_A and ∇_B, the composite $\nabla_{A \otimes B}$ of the following

$$A \otimes B \otimes A \otimes B \xrightarrow{1_A \otimes \tau_{B,A} \otimes 1_B} A \otimes A \otimes B \otimes B \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B$$

defines a *tensor product multiplication.*

(b) Given monoids A and B in $\mathbf{V}$, with respective units η_A and η_B, the composite $\eta_{A \otimes B}$ of

$$\mathbf{1} \xrightarrow{\Delta} \mathbf{1} \otimes \mathbf{1} \xrightarrow{\eta_A \otimes \eta_B} A \otimes B \,,$$

where the first map is given by Proposition 2.8(b), defines a *tensor product unit.*

For the following result, see [1, Theorem 4.2].

Proposition 3.8. *For monoids A and B in an entropic variety $\mathbf{V}$, the tensor product $A \otimes B$ of A and B in $\mathbf{V}$ forms a monoid in $\mathbf{V}$ under the tensor product multiplication and unit.*

The tensor product of monoids serves to give an interpretation of the bi-algebra diagram (3.3).

Lemma 3.9. *Let A be a bi-algebra in an entropic variety $\mathbf{V}$. Then the comultiplication and counit are monoid homomorphisms.*

Proof. The bi-algebra diagram (3.3) for A, and more specifically the quadrangle above the dotted arrow from $A \otimes A \otimes A \otimes A$ to $A \otimes A$, with the top right quadrangle, show that in a bi-algebra, the comultiplication is a monoid homomorphism. Note that the dotted arrow mentioned gives the multiplication on $A \otimes A$. Similarly, the top left quadrangle and the triangle in the diagram show that the counit is a monoid homomorphism. $\square$

In the following sense, the converse of Lemma 3.9 also holds (compare [11, Proposition 7.5]).

Corollary 3.10. *Let A be a monoid and comonoid in an entropic variety $\mathbf{V}$. Then A is a bi-algebra if and only if the comultiplication and counit are monoid homomorphisms.*

3.4. Setlike elements of comonoids

Definition 3.11 ([11, p. 40])**.** Let A be a comonoid in an entropic variety $\mathbf{V}$. Let $\mathbf{1}$ be the free $\mathbf{V}$-algebra on $\{x\}$. Then an element a of A is said to be *setlike* if $\Delta\colon a \mapsto a \otimes a$ and $a\varepsilon = x$.

Remark 3.12. The term "grouplike" is also used (compare [5, p. 8]).

The following results about setlike elements serve as a foil for the main results concerning primitive elements in §4 below (compare §4.5). The current results are well-known when the entropic variety $\mathbf{V}$ is the variety $\underline{K}$ of unital modules over a commutative, unital ring K, but their proof does warrant some care in the general universal-algebraic situation.

Proposition 3.13. *Let A_1 be the set of setlike elements of a bi-algebra A in an entropic variety $\mathbf{V}$, with unit $\eta\colon \mathbf{1} \to A; x \mapsto 1$. Then A_1 is closed under the multiplication $\cdot$ given by*

$$A_1 \times A_1 \to A_1; (a, b) \mapsto (a \otimes b)\nabla\,,$$

and $(A_1, \cdot, 1)$ forms a monoid.

Proof. Consider setlike elements a, b of A. Then by the commuting of the lower pentagon in (3.3), one has

$$
\begin{aligned}
(a \cdot b)\Delta = (a \otimes b)\nabla\Delta &= (a \otimes b)(\Delta \otimes \Delta)(1_A \otimes \tau \otimes 1_A)(\nabla \otimes \nabla) \\
&= \big((a \otimes a) \otimes (b \otimes b)\big)(1_A \otimes \tau \otimes 1_A)(\nabla \otimes \nabla) \\
&= \big((a \otimes b) \otimes (a \otimes b)\big)(\nabla \otimes \nabla) = (a \cdot b) \otimes (a \cdot b)\,.
\end{aligned}
$$

Furthermore, one has $(a \cdot b)\varepsilon = (a \otimes b)\nabla\varepsilon = (a\varepsilon \otimes b\varepsilon)\nabla = (x \otimes x)\nabla = x$ by Proposition 2.8(b) and the commuting of the top left quadrangle of (3.3), so $(A_1, \cdot)$ is a semigroup.

Proposition 2.8(b) and the commuting of the top right quadrangle of (3.3) yield $1\Delta = x\eta\Delta = x\Delta(\eta \otimes \eta) = (x \otimes x)(\eta \otimes \eta) = x\eta \otimes x\eta = 1 \otimes 1$, while the commuting of the top central triangle of (3.3) gives $1\varepsilon = x\eta\varepsilon = x$, so $(A_1, \cdot, 1)$ becomes a monoid. $\qquad\square$

Corollary 3.14. *If A is a Hopf algebra in $\mathbf{V}$, then $(A_1, \cdot, 1)$ is a group, with inversion given by the antipode S.*

Proof. Let a be a setlike element of A. By Proposition 3.6, one has

$$aS\Delta\tau = a\Delta(S \otimes S) = (a \otimes a)(S \otimes S) = aS \otimes aS\,.$$

Thus $aS\Delta = aS \otimes aS$. Again by Proposition 3.6, one has $aS\varepsilon = a\varepsilon = 1$, so $aS \in A_1$. The commuting of (3.4) shows that $aS \cdot a = a\Delta(S \otimes 1_A)\nabla = a\varepsilon\eta = x\eta = 1$, and similarly $a \cdot aS = a\Delta(1_A \otimes S)\nabla = 1$, so the element a is invertible in $(A_1, \cdot, 1)$, with inverse aS. $\qquad\square$

4. Primitive elements of comonoids in Jónsson-Tarski varieties

4.1. Jónsson-Tarski varieties

A variety of universal algebras is a *Jónsson-Tarski variety* if there is a derived binary operation $+$ and nullary operation (constant) selecting a subalgebra $\{0\}$ such that the identities

$$0 + x = x = x + 0$$

hold [2, 8]. In this section, the Jónsson-Tarski varieties under consideration are entropic.

Example 4.1. (a) If K is a commutative, unital ring, then the variety $\underline{\underline{K}}$ of unital modules over K is an entropic Jónsson-Tarski variety, with the usual interpretation of $+$ and 0.

(b) The variety of commutative monoids $(N, +, 0)$ forms an entropic Jónsson-Tarski variety.

(c) The variety of lower-bounded join semilattices forms an entropic Jónsson-Tarski variety. Tensor products in this variety, there denoted by $\mathbf{S}_0$, were studied in [1, §6].

Example 4.1(b) is typical of entropic Jónsson-Tarski varieties, in the sense of the following (well-known) argument.

Lemma 4.2. *Let A be an algebra in an entropic Jónsson-Tarski variety $\mathbf{V}$. Then $(A, +, 0)$ is a commutative monoid.*

Proof. By entropicity, the identity

$$(x + y) + (z + t) = (x + z) + (y + t)$$

holds in A. Setting $y = 0$ gives the associativity of $(A, +, 0)$, while setting $x = t = 0$ gives the commutativity. $\qquad\square$

4.2. Invertibility

For an algebra A in an entropic Jónsson-Tarski variety $\mathbf{V}$, the question of invertibility of elements of the commutative monoid $(A, +, 0)$ of Lemma 4.2 is a fundamental issue, for example in connection with the Lie structure introduced below. Of course, this question is trivial in the classical context of Example 4.1(a). If u is an invertible element of the commutative monoid $(A, +, 0)$, with inverse v, then the expression $a - u$ for a general element a of A will denote $a + v$ in A.

Lemma 4.3. *Let A be an algebra in an entropic Jónsson-Tarski variety $\mathbf{V}$.*

  (a) *Suppose that u is an invertible element of $(A, +, 0)$. Then for each element a of A, the tensor rank 1 element $u \otimes a$ is invertible in $(A \otimes A, +, 0)$.*

  (b) *Suppose that (A, ∇, η) is a monoid in $\mathbf{V}$. Let u be an invertible element of $(A \otimes A, +, 0)$. Then $u\nabla$ is invertible in $(A, +, 0)$.*

Proof. (a) If $u + v = 0$ in A, then $u \otimes a + v \otimes a = (u + v) \otimes a = 0 \otimes a = 0$ in $A \otimes A$.

(b) If $u + v = 0$ in $A \otimes A$, then $u\nabla + v\nabla = (u + v)\nabla = 0\nabla = 0$ in A.   □

4.3. Primitive elements

Lemma 4.4. *Let A be an algebra in an entropic Jónsson-Tarski variety* **V**. *Let* **1** *be the free* **V***-algebra on* $\{x\}$. *Suppose that* $\eta\colon \mathbf{1} \to A; x \mapsto 1$ *is a* **V***-homomorphism. Then the map*

$$\Delta\colon A \to A \otimes A; a \mapsto a \otimes 1 + 1 \otimes a$$

is a **V***-homomorphism.*

Proof. Consider an n-ary operator ω in the type of **V**, and elements $a_1, \ldots, a_n$ of A. Then

$$
\begin{aligned}
a_1 \cdots a_n \omega \Delta &= a_1 \cdots a_n \omega \otimes 1 + 1 \otimes a_1 \cdots a_n \omega \\
&= [(a_1 \otimes 1) \cdots (a_n \otimes 1)\omega] + [(1 \otimes a_1) \cdots (1 \otimes a_n)\omega] \\
&= [(a_1 \otimes 1) + (1 \otimes a_1)] \cdots [(a_n \otimes 1) + (1 \otimes a_n)]\omega \\
&= a_1\Delta \cdots a_n\Delta \omega
\end{aligned}
$$

as required, the penultimate equality holding by entropicity.   □

Definition 4.5. Let A be a comonoid in an entropic Jónsson-Tarski variety **V**. Let **1** be the free **V**-algebra on $\{x\}$, and let $\eta\colon \mathbf{1} \to A; x \mapsto 1$ be a **V**-morphism. Then an element a of A is said to be *primitive* if $\Delta\colon a \mapsto a \otimes 1 + 1 \otimes a$ and $a\varepsilon = 0$.

Remark 4.6. The secondary condition $a\varepsilon = 0$ in Definition 4.5 follows directly from the primary condition $a\Delta = a \otimes 1 + 1 \otimes a$ when the monoid $(A, +, 0)$ of Lemma 4.2 is an abelian group (cf. [3, Prop. III.2.6]).

The following result may be viewed as an analogue of Proposition 3.13.

Proposition 4.7. *Let A be a comonoid in an entropic Jónsson-Tarski variety* **V**. *Let* **1** *be the free* **V***-algebra on* $\{x\}$, *and let* $\eta\colon \mathbf{1} \to A; x \mapsto 1$ *be a* **V***-morphism. Let A_0 be the set of primitive elements of A. Then A_0 is a subalgebra of A, and in particular a submonoid of $(A, +, 0)$.*

Proof. Consider an n-ary operation ω of **V**, and primitive elements $a_1, \ldots, a_n$ of A. Since the comultiplication Δ is a **V**-homomorphism, one has

$$
\begin{aligned}
a_1 \cdots a_n \omega \Delta &= a_1\Delta \cdots a_n\Delta \omega \\
&= [(a_1 \otimes 1) + (1 \otimes a_1)] \cdots [(a_n \otimes 1) + (1 \otimes a_n)]\omega \\
&= [(a_1 \otimes 1) \cdots (a_n \otimes 1)\omega] + [(1 \otimes a_1) \cdots (1 \otimes a_n)\omega] \\
&= a_1 \cdots a_n \omega \otimes 1 + 1 \otimes a_1 \cdots a_n \omega.
\end{aligned}
$$

Also, since the counit ε is a **V**-homomorphism, one has

$$a_1 \cdots a_n \omega \varepsilon = a_1\varepsilon \cdots a_n\varepsilon \omega = 0 \cdots 0\omega = 0,$$

so $a_1 \cdots a_n \omega$ is primitive. Finally, $0\Delta = 0 = 0 + 0 = 0 \otimes 1 + 1 \otimes 0$ and $0\varepsilon = 0$, so 0 is primitive. $\qquad \square$

Corollary 4.8. *Suppose that A is a bi-algebra in $\mathbf{V}$. Suppose that a and b are primitive elements of A, such that $(a \otimes b)\nabla$ and $(b \otimes a)\nabla$ are invertible in $(A, +, 0)$. Then the Lie bracket*

$$(4.1) \qquad\qquad [a, b] = (a \otimes b)\nabla - (b \otimes a)\nabla$$

is a primitive element of the comonoid reduct (A, Δ, ε), and an invertible element of the monoid $(A, +, 0)$.

Proof. First, note that $[a, b]$ is invertible in $(A, +, 0)$, with inverse $[b, a]$. The comultiplication Δ is a ∇-homomorphism, and the multiplication ∇ is a $\mathbf{V}$-homomorphism, so

$$
\begin{aligned}
(a \otimes b)\nabla\Delta &= (a\Delta \otimes b\Delta)\nabla \\
&= \big((a \otimes 1 + 1 \otimes a) \otimes (b \otimes 1 + 1 \otimes b)\big)\nabla \\
&= \big(a \otimes 1 \otimes b \otimes 1 + a \otimes 1 \otimes 1 \otimes b + 1 \otimes a \otimes b \otimes 1 + 1 \otimes a \otimes 1 \otimes b\big)\nabla \\
&= (a \otimes b)\nabla \otimes 1 + a \otimes b + b \otimes a + 1 \otimes (a \otimes b)\nabla \,.
\end{aligned}
$$

Now since $(a \otimes b)\nabla$ is invertible in $(A, +, 0)$, Lemma 4.3(a) implies that the elements $(a \otimes b)\nabla \otimes 1$ and $1 \otimes (a \otimes b)\nabla$ are invertible in $(A \otimes A, +, 0)$. Thus

$$
\begin{aligned}
(a \otimes b)&\nabla\Delta - (a \otimes b)\nabla \otimes 1 - 1 \otimes (a \otimes b)\nabla \\
&= a \otimes b + b \otimes a \\
&= (b \otimes a)\nabla\Delta - (b \otimes a)\nabla \otimes 1 - 1 \otimes (b \otimes a)\nabla \,,
\end{aligned}
$$

the latter equality following as above. Since Δ is a $\mathbf{V}$-homomorphism, one obtains

$$
\begin{aligned}
[a, b]\Delta &= \big((a \otimes b)\nabla - (b \otimes a)\nabla\big)\Delta \\
&= (a \otimes b)\nabla\Delta - (b \otimes a)\nabla\Delta \\
&= (a \otimes b)\nabla \otimes 1 - (b \otimes a)\nabla \otimes 1 + 1 \otimes (a \otimes b)\nabla - 1 \otimes (b \otimes a)\nabla \\
&= [a, b] \otimes 1 + 1 \otimes [a, b] \,.
\end{aligned}
$$

Finally, since the counit ε is simultaneously both a $\mathbf{V}$-homomorphism and a ∇-homomorphism, one has

$$
\begin{aligned}
[a, b]\varepsilon &= \big((a \otimes b)\nabla - (b \otimes a)\nabla\big)\varepsilon \\
&= (a \otimes b)\nabla\varepsilon - (b \otimes a)\nabla\varepsilon \\
&= (a\varepsilon \otimes b\varepsilon)\nabla - (b\varepsilon \otimes a\varepsilon)\nabla = 0 \,,
\end{aligned}
$$

so $[a, b]$ is primitive. $\qquad \square$

4.4. Lie algebras in Hopf algebras

The following result may be viewed as an analogue of Corollary 3.14.

Theorem 4.9. *If A is a Hopf algebra in a Jónsson-Tarski variety $\mathbf{V}$, the primitive elements form a Lie algebra $\big(A_0, +, [\,,\,]\big)$ in $\mathbf{V}$, with inversion given by the antipode S, and with the Lie bracket* (4.1).
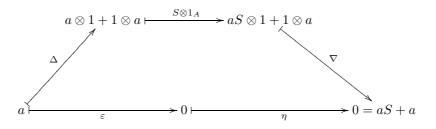
Proof. Proposition 4.7 shows that A_0 forms an algebra in $\mathbf{V}$. Consider a primitive element a of A. By Proposition 3.6, one has

$$
\begin{aligned}
aS\Delta\tau &= a\Delta(S \otimes S) \\
&= (a \otimes 1 + 1 \otimes a)(S \otimes S) \\
&= aS \otimes 1S + 1S \otimes aS = aS \otimes 1 + 1 \otimes aS \,,
\end{aligned}
$$

the last equality holding by Corollary 3.14. Thus $aS\Delta = aS \otimes 1 + 1 \otimes aS$. Again by Proposition 3.6, one has $aS\varepsilon = a\varepsilon = 0$, so $aS \in A_0$.

Now consider the antipode diagram (3.4) for the Hopf algebra A in $\mathbf{V}$. Chase the element a around the top pentagon, again recalling that $1S = 1$:

One obtains $aS + a = 0$: Each element a of A_0 is invertible in the commutative monoid $(A_0, +, 0)$.

Lemma 4.3(a) now shows that for primitive elements a and b, the elements $a \otimes b$ and $b \otimes a$ are invertible in $(A \otimes A, +, 0)$. By Lemma 4.3(b), it follows that $(a \otimes b)\nabla$ and $(b \otimes a)\nabla$ are invertible in $(A, +, 0)$. By Corollary 4.8, the Lie bracket $[a, b]$ of (4.1) is a well-defined element of A_0, with $[b, a] = -[a, b]$.

For elements a_1, a_2, a_3 of A_0, and $\{i, j, k\} = \{1, 2, 3\}$, Lemma 4.3 shows that all the elements $((a_i \otimes a_j)\nabla \otimes a_k)\nabla$ and $(a_i \otimes (a_j \otimes a_k)\nabla)\nabla$ of A are invertible. Verification of the Jacobi identity in $\big(A_0, +, [\,,\,]\big)$ then becomes routine. $\qquad\square$

4.5. Setlike and primitive elements

For summary and comparison, the following table juxtaposes the properties of setlike elements a in a comonoid (A, Δ, ε) in a general entropic variety $\mathbf{V}$, and primitive elements a in a comonoid (A, Δ, ε) in a Jónsson-Tarski variety $\mathbf{V}$.

Setlike (all $\mathbf{V}$, with $x\eta = 1$)	Primitive (Jónsson-Tarski $\mathbf{V}$, with $x\eta = 1$)
$a\Delta = a \otimes a$ and $a\varepsilon = x$	$a\Delta = a \otimes 1 + 1 \otimes a$ and $a\varepsilon = 0$
set A_1 of setlikes	set A_0 of primitives
Prop. 3.13: monoid $(A_1, \cdot, 1)$	Prop. 4.7: commutative monoid $(A_0, +, 0)$
Cor. 3.14: group $(A_1, \cdot, 1)$ in Hopf	Th. 4.9: Lie algebra $\big(A_0, +, [\,,\,]\big)$ in Hopf

5. Tensor algebras

Let $\mathbf{V}$ be an entropic variety, construed as a category. There is a forgetful functor to $\mathbf{V}$ from the category of monoids in $\mathbf{V}$ and their homomorphisms. This forgetful functor has a left adjoint, which builds the free monoid in $\mathbf{V}$ over a given $\mathbf{V}$-algebra.

Definition 5.1. Let A be an algebra in an entropic variety $\mathbf{V}$. Then the *tensor algebra AT* over A is the free monoid in $\mathbf{V}$ over A.

5.1. Tensor powers

For an algebra A in an entropic variety $\mathbf{V}$ with free algebra $\mathbf{1}$ on $\{x\}$, define the successive tensor powers

$$A^{(0)} = \mathbf{1} \quad \text{and} \quad A^{(n+1)} = A^{(n)} \otimes A$$

for natural numbers n. Recall that the category $\mathbf{V}$ is cocomplete [10, §IV.2.2]. Then the tensor algebra is constructed as the coproduct

$$(5.1) \qquad\qquad AT = \coprod_{n \in \mathbb{N}} A^{(n)}$$

of the tensor powers with

$$\nabla \colon (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_{k+l}) \mapsto a_1 \otimes \cdots \otimes a_k \otimes a_{k+1} \otimes \cdots \otimes a_{k+l}$$

and unit η inserting $A^{(0)} = \mathbf{1}$ into the coproduct (5.1). In [1, p. 80], the tensor algebra AT was described as the "free monoid generated by A".

5.2. The tensor bi-algebra

Theorem 5.2. *Suppose that A is an algebra in an entropic Jónsson-Tarski variety $\mathbf{V}$. Then the tensor algebra AT over A carries a uniquely defined bi-algebra structure in which each element of A is primitive.*

Proof. By Lemma 3.9, the comultiplication of a bi-algebra structure on AT would be a monoid homomorphism with domain AT. Since the tensor algebra AT is the free monoid in $\mathbf{V}$ over A, the desired comultiplication on AT is uniquely specified as a monoid homomorphism by whatever $\mathbf{V}$-homomorphic effect it is declared to have on the generating algebra A. By Lemma 4.4, the map

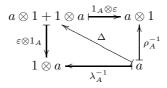$$\Delta \colon A \to A \otimes A; a \mapsto a \otimes 1 + 1 \otimes a$$

is a $\mathbf{V}$-homomorphism. Thus the comultiplication on AT is uniquely defined by declaring the elements of A to be primitive.

The desired counit on AT is similarly specified as a monoid homomorphism by whatever $\mathbf{V}$-homomorphic effect it is declared to have on the generating algebra A. Observe that the map $\varepsilon\colon A \to \mathbf{1}; a \mapsto 0$ is a $\mathbf{V}$-homomorphism, since $\{0\}$ is a subalgebra of $\mathbf{1}$. Then the counit on AT is also defined by declaring the elements of A to be primitive.

By Corollary 3.10, it now remains to show that AT is a comonoid with the given comultiplication and counit, i.e., that the coalgebra diagrams (3.2) for AT commute. By the freeness of AT over A, it suffices to chase elements of A around these diagrams. One obtains

$$
\begin{array}{ccc}
a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a & \xleftarrow{\;1_A \otimes \Delta\;} & a \otimes 1 + 1 \otimes a \\[4pt]
{\scriptstyle \Delta \otimes 1_A}\big\uparrow & & \big\uparrow{\scriptstyle \Delta} \\[4pt]
a \otimes 1 + 1 \otimes a & \xleftarrow[\;\;\Delta\;\;]{} & a
\end{array}
$$

for the coassociativity, and

$$
\begin{array}{ccc}
a \otimes 1 + 1 \otimes a & \xmapsto{\;1_A \otimes \varepsilon\;} & a \otimes 1 \\[4pt]
{\scriptstyle \varepsilon \otimes 1_A}\big\downarrow & {\scriptstyle \Delta}\nwarrow & \big\uparrow{\scriptstyle \rho_A^{-1}} \\[4pt]
1 \otimes a & \xleftarrow[\;\lambda_A^{-1}\;]{} & a
\end{array}
$$

for the counitality. $\qquad\square$

5.3. Tensor algebras as Hopf algebras

Corollary 5.3. *If the commutative monoid $(A, +, 0)$ is an abelian group, then the bi-algebra AT is a Hopf algebra in $\mathbf{V}$.*

Proof. Define $S\colon a_1 \otimes \cdots \otimes a_k \mapsto (-1)^k a_k \otimes \cdots \otimes a_1$. The standard proof that AT is a Hopf algebra when $\mathbf{V} = \underline{\underline{K}}$ for a commutative, unital ring K carries over to the general case. $\qquad\square$

Corollary 5.4. *If AT is a Hopf algebra in $\mathbf{V}$, then the commutative monoid $(A, +, 0)$ is cancellative.*

Proof. By Theorem 4.9, the set AT_0 of primitive elements of AT forms an abelian group $(AT_0, +, 0)$. Since the elements of A are primitive elements of AT, the commutative monoid $(A, +, 0)$ is a submonoid of the group $(AT_0, +, 0)$. $\quad\square$

Problem 5.5. In the context of Corollary 5.4, under what conditions is $(A, +, 0)$ an abelian group?

6. Quantum couples in entropic Jónsson-Tarski varieties

The goal of this section is Theorem 6.1, giving a construction of group quantum couple Hopf algebras within a general entropic Jónsson-Tarski variety. These algebras embrace quantum doubles (compare [13]), group algebras, and dual group algebras, among others. We begin by establishing two preliminary conventions.

6.1. Sigma notation

In an entropic Jónsson-Tarski algebra A, use

$$\sum_{a \in \varnothing} a = 0 \quad \text{and} \quad \sum_{a \in X \cup \{b\}} a = \left(\sum_{a \in X} a\right) + b$$

for a recursive definition of $\sum_{a \in Y} a$ in A over finite subsets Y of A. Compare [2, Defn. 1.2(ii)].

6.2. Automorphic action

Consider a group G. Suppose that h is an element of a group H that acts automorphically (but not necessarily faithfully) on G. For $h \in H$, write the action of h on G as

$$h \colon G \to G; g \mapsto g^h \,.$$

Thus $(fg)^h = f^h g^h$ for $f, g \in G$. Then write g^{-h} for $(g^{-1})^h$ (or equivalently, for $(g^h)^{-1}$).

6.3. The quantum couple

For a finite group G, and a group H that acts automorphically (but not necessarily faithfully) on G, the Hopf algebra D that is constructed in the following theorem is known as the *quantum couple* of H and G. If H is not commutative, then D is not commutative. If G is not commutative, then D is not cocommutative.

Theorem 6.1. *Let* **V** *be an entropic Jónsson-Tarski variety. Let G be a finite group with identity element e. Let GV be the free **V**-algebra over the set G. Let H be a (not necessarily finite) group with identity element i. Suppose that H acts automorphically (but not necessarily faithfully) on G. Let HV be the free **V**-algebra over the set H. Write $D = HV \otimes GV$, and write $h|g$ for $h \otimes g$ with elements g of G and h of H. Define a multiplication $\nabla \colon D \times D \to D$ by*

(6.1)
$$(h|f \otimes k|g)\nabla = \begin{cases} hk|g & \text{if } f^k = g; \\ 0 & \text{otherwise} \end{cases}$$

for $f, g \in G$ and $h, k \in H$. Define a unit $\eta \colon \mathbf{1} \to D; x \mapsto \sum_{g \in G} i|g$. Define

$$S \colon D \to D; h|g \mapsto h^{-1}|g^{-h^{-1}}$$

as the antipode. Define a comultiplication

$$\Delta \colon D \to D \otimes D; h|g \mapsto \sum_{g^L g^R = g} h|g^L \otimes h|g^R \,.$$

Define a counit $\varepsilon \colon D \to \mathbf{1}$ *by*

(6.2)
$$(h|g)\varepsilon = \begin{cases} x & \text{if } g = e, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(D, \nabla, \eta, \Delta, \varepsilon, S)$ *is a Hopf algebra in* **V**.

Proof. The multiplication is associative, since both $(\nabla \otimes 1)\nabla$ and $(1 \otimes \nabla)\nabla$ map $h_1|f_1 \otimes h_2|f_2 \otimes h_3|f_3$ to $h_1 h_2 h_3|f_3$ if $f_1^{h_2} = f_2$ and $f_2^{h_3} = f_3$, and to 0 otherwise. (Note that $f_1^{h_2 h_3} = f_3$ if and only if $f_1^{h_2} = f_2$, when $f_2^{h_3} = f_3$.) Also,

$$(h|f)(x\eta)\nabla = (h|f) \otimes \left(\sum_{g \in G} i|g \right)\nabla = \sum_{g \in G}(h|f) \otimes (i|g)\nabla = h|f$$

$$= \sum_{g \in G}(i|g) \otimes (h|f)\nabla = \left(\sum_{g \in G} i|g \right) \otimes (h|f)\nabla = (x\eta)(h|f)\nabla$$

verifies the unit laws. Thus (D, ∇, η) is a monoid in **V**.

Consider an element $h|g$ of D, and verifying the coassociativity at $h|g$. One has

$$(h|g)\Delta(\Delta \otimes 1) = \sum_{g^L g^R = g} \left[\sum_{g^{LL} g^{LR} = g^L} h|g^{LL} \otimes h|g^{LR} \otimes h|g^R \right]$$

and

$$(h|g)\Delta(1 \otimes \Delta) = \sum_{g^L g^R = g} \left[\sum_{g^{RL} g^{RR} = g^R} h|g^L \otimes h|g^{RL} \otimes h|g^{RR} \right] \,.$$

Both of these sums equal

$$\sum_{g_1 g_2 g_3 = g} h|g_1 \otimes h|g_2 \otimes h|g_3 \,,$$

completing the verification. For counitality, one has

$$(h|g)\Delta(\varepsilon \otimes 1) = \sum_{g^L g^R = g} (h|g^L)\varepsilon \otimes h|g^R = x \otimes h|g = (h|g)\lambda_D^{-1}$$

and

$$(h|g)\Delta(1 \otimes \varepsilon) = \sum_{g^L g^R = g} h|g^L \otimes (h|g^R)\varepsilon = (h|g) \otimes x = (h|g)\rho_D^{-1}$$

as required. Thus (D, Δ, ε) is a comonoid in **V**.

For the antipode diagram (3.3), there are two separate cases to consider. Chasing an element $h|e$ (for $h \in H$) around the upper pentagon, one has

$$\sum_{g^L g^R = e} h|g^L \otimes h|g^R \xrightarrow{\;S \otimes 1_{GV}\;} \sum_{g^L g^R = e} h^{-1}|(g^L)^{-h^{-1}} \otimes h|g^R$$

with Δ up on the left, ∇ down on the right, and

$$h|e \xmapsto{\;\;\varepsilon\;\;} x \xmapsto{\;\;\eta\;\;} \sum_{g^R \in G} i|g^R$$

since $(g^L)^{-1} = g^R$ if and only if $g^L g^R = e$. Chasing an element $h|g$ with $e \neq g \in G$ around the upper pentagon, one has

$$\sum_{g^L g^R = g} h|g^L \otimes h|g^R \xrightarrow{\;S \otimes 1_{GV}\;} \sum_{g^L g^R = g} h^{-1}|(g^L)^{-h^{-1}} \otimes h|g^R$$

with Δ up on the left, ∇ down on the right, and

$$h|g \xmapsto{\;\;\varepsilon\;\;} 0 \xmapsto{\;\;\eta\;\;} 0$$

since $(g^L)^{-1} \neq g^R$ if $g^L g^R = g \neq e$. The commuting of the lower pentagon is similar.

It remains to be shown that $(D, \nabla, \eta, \Delta, \varepsilon, S)$ is a bi-algebra. To this end, it is convenient to abbreviate the multiplication definition (6.1) as $(h|f \otimes k|g)\nabla = \delta_{f^k g} hk|g$, and the counit definition (6.2) as $(h|g)\varepsilon = \delta_{ge} x$, using the Kronecker delta. Now consider the bi-algebra diagram (3.3). For the commuting of the lower pentagon, one has
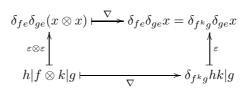
$$h|f \otimes k|g \xrightarrow{\;\nabla\;} \delta_{f^k g} hk|g \xrightarrow{\;\Delta\;} \delta_{f^k g} hk|g^L \otimes hk|g^R$$

with $\Delta \otimes \Delta$ down on the left, $\nabla \otimes \nabla$ up on the right, and

$$h|f^L \otimes h|f^R \otimes k|g^L \otimes k|g^R \xrightarrow{\;1_A \otimes \tau \otimes 1_A\;} h|f^L \otimes k|g^L \otimes h|f^R \otimes k|g^R$$

for $g \in G$ and $h \in H$, as required. Note the use of the abbreviated Sweedler notation here, e.g. $k|g^L \otimes k|g^R$ in place of $\sum_{g^L g^R = g} k|g^L \otimes k|g^R$. Furthermore, note that $f^k = g$ and $f^L f^R = f$ imply $f^{Lk} f^{Rk} = f^k$, so $f^{Lk} = g^L$ and $f^{Rk} = g^R$ with $g^L g^R = g$.

For the commuting of the top right quadrangle in (3.3), one has

$$x\eta\Delta = \sum_{g \in G} i|g\Delta = \sum_{g \in G} \sum_{g^L g^R = g} i|g^L \otimes i|g^R = \sum_{f, g \in G} i|f \otimes i|g$$

$$= \sum_{f \in G} i|f \otimes \sum_{g \in G} i|g = (x \otimes x)(\eta \otimes \eta) = x\Delta(\eta \otimes \eta),$$

as required. For the commuting of the top left quadrangle in (3.3), one has

$$
\begin{array}{ccc}
\delta_{fe}\delta_{ge}(x \otimes x) & \xrightarrow{\ \nabla\ } & \delta_{fe}\delta_{ge}x = \delta_{f^kg}\delta_{ge}x \\
\scriptstyle{\varepsilon \otimes \varepsilon}\big\uparrow & & \big\uparrow\scriptstyle{\varepsilon} \\
h|f \otimes k|g & \xrightarrow[\nabla]{\quad\qquad} & \delta_{f^kg}hk|g
\end{array}
$$

for $f, g \in G$ and $h, k \in H$, while for the commuting of the top middle triangle in (3.3), one has $x\eta\varepsilon = \sum_{g \in G}(i|g)\varepsilon = \sum_{g \in G}\delta_{ge}x = x$. $\square$

6.4. Special cases

If **V** is the variety of vector spaces over a field, the quantum couple specializes to various well-known Hopf algebras.

Example 6.2. If the group G in Theorem 6.1 is trivial, the quantum couple D specializes to the *group algebra* of the group H (compare [5, Ex. 1.6]).

Example 6.3. If the automorphism group H in Theorem 6.1 is trivial, the quantum couple D specializes to the *dual group algebra* of the finite group G (compare [5, Ex. 2.1]).

Example 6.4. If the automorphism group H in Theorem 6.1 is the finite group G acting on itself by conjugation, then the quantum couple D specializes to the *quantum double* of the group G (compare [13]).

Thus the quantum couple construction serves to implement versions of these algebras in a general entropic Jónsson-Tarski variety.

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