Fields Institute Communications
Volume 43, 517-522

Extension Theory in Mal'tsev Varieties

Jonathan D.H. Smith
Department of Mathematics
Iowa State University
Ames, Iowa 50011, U.S.A.
jdhsmith@math.iastate.edu

Abstract. The paper provides a brief survey of extension theory for Mal'tsev varieties based on centrality and monadic cohomology. Extension data are encoded in the form of a *seeded simplicial map.* Such a map yields an extension if and only if it is unobstructed. Second cohomology groups classify extensions, and third cohomology groups classify obstructions.

1 Introduction

Extension theory for Mal'tsev varieties was developed in [9, Chapter 6], generalising earlier treatment of special cases such as groups (cf. [3], [7]), commutative algebras (cf. [1]), loops [6], and other "categories of interest" [8]. Because of renewed attention being paid to the topic, as evidenced by recent publications such as [4], a brief survey of the theory appears timely. With the exception of parts of Section 2, the context throughout the paper is that of a Mal'tsev variety $\mathfrak{V}$. An *extension* of a $\mathfrak{V}$-algebra R is considered as a $\mathfrak{V}$-algebra T equipped with a congruence α such that R is isomorphic to the quotient T^α of T by the congruence α.

The essential properties of centrality in Mal'tsev varieties are recalled in Section 2. Section 3 describes the "seeded simplicial maps" which provide a concise encoding of the raw material required for constructing an extension (analogous to the "abstract kernels" of [7]). Section 4 gives a brief, algebraic description of the rudiments of monadic cohomology, culminating in the Definition 4.2 of the obstruction of a seeded simplicial map as a cohomology class. Theorem 5.2 then shows that a seeded simplicial map yields an extension if and only if it is unobstructed. The final section discusses the classification of extensions by second cohomology groups, and of obstructions by third cohomology groups. Against this background, the pessimism expressed in [5] ("To classify the extensions ... is too big a project to admit of a reasonable answer") appears unwarranted.

1991 *Mathematics Subject Classification.* Primary 18G50; Secondary 08B10.

For concepts and conventions not otherwise explained in the paper, readers are referred to [10]. In particular, note the general use of postfix notation, so that composites are read in natural order from left to right.

2 Centrality in Mal'tsev varieties

Recall that a variety $\mathfrak{V}$ of universal algebras is a *Mal'tsev variety* if there is a derived ternary *parallelogram* operation P such that the identities

$$(x,x,y)P = y = (y,x,x)P$$

are satisfied. Equivalently, the relation product of two congruences is commutative, and thus agrees with their join. Moreover, reflexive subalgebras of direct squares are congruences [9, Proposition 143].

Consider two congruences γ and β on a general algebra, not necessarily in a Mal'tsev variety. Then γ is said *to centralise* β if there is a congruence $(\gamma|\beta)$ on β, called a *centreing congruence*, such that the following conditions are satisfied [9, Definition 211]:

(C0): $(x,y)\ (\gamma|\beta)\ (x',y') \Rightarrow x\ \gamma\ x'$;
(C1): $\forall (x,y) \in \beta,\ \pi^0 : (x,y)^{(\gamma|\beta)} \to x^\gamma; (x',y') \mapsto x'$ bijects;
(C2): **(RR):** $\forall (x,y) \in \gamma,\ (x,x)\ (\gamma|\beta)\ (y,y)$;
(RS): $(x,y)\ (\gamma|\beta)\ (x',y') \Rightarrow (y,x)\ (\gamma|\beta)\ (y',x')$;
(RT): $(x,y)\ (\gamma|\beta)\ (x',y')$ and $(y,z)\ (\gamma|\beta)\ (y',z') \Rightarrow (x,z)\ (\gamma|\beta)\ (x',z')$.

In a Mal'tsev variety $\mathfrak{V}$, centreing congruences are unique [9, Proposition 221]. Moreover, for each congruence α on an algebra A in $\mathfrak{V}$, there is a unique largest congruence $\eta(\alpha)$, called the *centraliser* of α, which centralises α [9, 228]. Note that $\alpha \circ \eta(\alpha)$ centralises $\alpha \cap \eta(\alpha)$ [9, Corollary 227].

If R is a member of a variety $\mathfrak{V}$ of universal algebras, then the *category of R-modules* is the category of abelian groups in the slice category $\mathfrak{V}/R$. For example, if A is an algebra in a Mal'tsev variety $\mathfrak{V}$ having nested congruences $\beta \le \gamma$ such that γ centralises β, then $\beta^{(\gamma|\beta)} \to A^\gamma$ is an A^γ-module. Indeed, given $a_1\,\beta\,a_0\,\gamma\,b_0\,\beta\,b_2$ in A, one has

$$(a_0,a_1)^{(\gamma|\beta)} + (b_0,b_1)^{(\gamma|\beta)} = (a_0,a_3)^{(\gamma|\beta)}$$

for a_2 given by $(a_0,a_2)(\gamma|\beta)(b_0,b_2)$ using (C1) and then for a_3 given similarly by $(a_0,a_1)(\gamma|\beta)(a_2,a_3)$.

3 Seeded simplicial maps

The data used for the construction of extensions are most succinctly expressed in terms of simplicial maps. These are described using the direct algebraic approach of [9], to which the reader is referred for fuller detail. Compare also [2].

Let ε_n^i be the operation which deletes the $(i+1)$-th letter from a non-empty word of length n. Let δ_n^i be the operation which repeats the $(i+1)$-th letter in a non-empty word of length n. These operations, for all positive integers n and natural numbers $i < n$, generate (the morphisms of) a category Δ called the *simplicial category.* A *simplicial object* B^* in $\mathfrak{V}$ is (the image of) a functor from Δ to $\mathfrak{V}$. A *simplicial map* is (the set of components of) a natural transformation between such functors. Generically, the morphisms of a simplicial object B^* are denoted by their preimages in Δ, namely as $\varepsilon_n^i : B^n \to B^{n-1}$ and $\delta_n^i : B^n \to B^{n+1}$. (In other words, one treats simplicial objects as heterogeneous algebras in $\mathfrak{V}$.)

Given $(\theta^0, \ldots, \theta^{n-1}) \in \mathfrak{V}(X,Y)^n$, the *simplicial kernel* $\ker(\theta^0, \ldots, \theta^{n-1})$ is the largest subalgebra K of the power X^{n+1} for which the θ^i and the restrictions of the projections from the power model the identities satisfied by the simplicial ε_n^i and ε_{n+1}^i. For example, the simplicial kernel of a single $\mathfrak{V}$-morphism $\theta^0 : X \to Y$ is $K = \{(x_0, x_1) \in X^2 \mid x_0\theta^0 = x_1\theta^0\}$, the usual kernel of θ^0, modelling the single simplicial identity $\varepsilon_2^0\varepsilon_1^0 = \varepsilon_2^1\varepsilon_1^0$ by $\pi^0\theta^0 = \pi^1\theta^0$ for $\pi^i : K \to X; (x_0, x_1) \mapsto x_i$.

For each positive integer n, removing all operations from Δ that involve words of length greater than n leaves the *simplicial category* Δ_n *truncated at* n. Functors from Δ_n are called *simplicial objects truncated at dimension* n. Truncated simplicial objects may be extended to full simplicial objects by successively tacking on simplicial kernels. In such cases one may omit the epithet "truncated," speaking merely of simplicial objects, even when one has only specified the lower-dimensional part.

Definition 3.1 A simplicial object B^* is said to be *seeded* if:

1. It is truncated at dimension 2;
2. $(\varepsilon_2^0, \varepsilon_2^1) : B^2 \to \ker(\varepsilon_1^0)$ surjects;
3. $\varepsilon_1^0 : B^1 \to B^0$ surjects;
4. $\ker(\varepsilon_2^0 : B^2 \to B^1) = \eta(\ker(\varepsilon_2^1 : B^2 \to B^1))$.

Lemma 3.2 *In a seeded simplicial object B, let C be the equalizer of the pair $(\varepsilon_2^0, \varepsilon_2^1)$. Define V on C by*

$$c\ V\ c' \Leftrightarrow ((c\varepsilon_2^0\delta_1^0, c), (c'\varepsilon_2^0\delta_1^0, c')) \in (\ker\varepsilon_2^0 \circ \ker\varepsilon_2^1 | \ker\varepsilon_2^0 \cap \ker\varepsilon_2^1).$$

Then

$$C^V \to B^0; c^V \mapsto c\varepsilon_2^0\varepsilon_1^0 \tag{3.1}$$

is a module over B^0, isomorphic to $(\ker\varepsilon_2^0 \cap \ker\varepsilon_2^1)^{(\ker\varepsilon_2^0 \circ \ker\varepsilon_2^1 | \ker\varepsilon_2^0 \cap \ker\varepsilon_2^1)}$.

The module (3.1) of Lemma 3.2 is called the module *grown* by the seeded simplicial object B^*. If α is a congruence on a $\mathfrak{V}$-algebra T, then

$$\alpha^{(\eta(\alpha)|\alpha)} \rightrightarrows T^{\eta(\alpha)} \to T^{\alpha\circ\eta(\alpha)} \tag{3.2}$$

is a seeded simplicial object with $\varepsilon_2^i : (t_0, t_1)^{(\eta(\alpha)|\alpha)} \mapsto t_i^{\eta(\alpha)}$, growing the module

$$(\alpha \cap \eta(\alpha))^{(\alpha\circ\eta(\alpha)|\alpha\cap\eta(\alpha))} \to T^{\alpha\circ\eta(\alpha)}; (t_0, t_1)^{(\alpha\circ\eta(\alpha)|\alpha\cap\eta(\alpha))} \mapsto t_0^{\alpha\circ\eta(\alpha)}.$$

The seeded simplicial object (3.2) is said to be *planted* by the congruence α on the algebra T.

Definition 3.3 A simplicial map $p^* : A^* \to B^*$ is said to be *seeded* if the codomain object B^* is seeded in the sense of Definition 3.1, and if $p^0 : A^0 \to B^0$ surjects.

4 Obstructions

Along with the simplicial theory outlined in Section 3, the second tool used for studying extensions of Mal'tsev algebras is monadic cohomology. Once again, full details may be found in [2] and [9]; the summary given here follows the direct approach of the latter reference.

For each $\mathfrak{V}$-algebra A, let AG denote the free $\mathfrak{V}$-algebra over the generating set $\{\{a\} \mid a \in A\}$. Given a $\mathfrak{V}$-algebra R, let $\varepsilon_n^j : RG^n \to RG^{n-1}$ denote the uniquely defined $\mathfrak{V}$-morphism deleting the j-th layer of braces, where $j = 0$ corresponds to the inside layer and $j = n-1$ to the outside. Let $\delta_n^j : RG^n \to RG^{n+1}$ insert the j-th layer of braces. One obtains a simplicial object RG^*, known as the *free resolution* of A. Each RG^n projects to R by a composition

$$\varepsilon_n^0 \dots \varepsilon_1^0 : RG^n \to R. \tag{4.1}$$

An R-module $E \to R$ becomes an RG^n-module by pullback along (4.1). Write $\mathrm{Der}(RG^n, E)$ for the abelian group $\mathfrak{V}/R(RG^n \to R, E \to R)$ of *derivations*. Define *coboundary homomorphisms*

$$d_n : \mathrm{Der}(RG^n, E) \to \mathrm{Der}(RG^{n+1}, E); f \mapsto \sum_{i=0}^{n} (-)^i \varepsilon_{n+1}^i f$$

for each natural number n. For each positive integer n, define

$$\mathrm{H}^n(R, E) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1}), \tag{4.2}$$

the so-called n-th *monadic cohomology group of R with coefficients in E*. [Note that [2] and [9] use $\mathrm{H}^{n-1}(R, E)$ for (4.2).] The cosets forming (4.2) are known as *cohomology classes*. Elements of $\mathrm{Ker}(d_n)$ are known as *cocycles*, and elements of $\mathrm{Im}(d_{n-1})$ are *coboundaries*.

Lemma 4.1 *Let $p^* : RG^* \to B^*$ be a seeded simplicial map whose codomain grows module M. Pull M from B^0 back to R along p^0. Then*

$$p^3(\varepsilon_3^0, \varepsilon_3^1, \varepsilon_3^2)P^V : RG^3 \to M \tag{4.3}$$

is a cocycle in $\mathrm{Der}(B^3, M)$.

Definition 4.2 The cohomology class of (4.3) is called the *obstruction* of the seeded simplicial map p^*. The simplicial map is said to be *unobstructed* if this class is zero.

Lemma 4.3 *The obstruction of a seeded simplicial map $p^* : RG^* \to B^*$ is uniquely determined by its bottom component $p^0 : R \to B^0$.*

The diagram-chasing proofs of Lemmas 4.1 and 4.3 are given in [9, pp.124–7].

5 Constructing extensions

Definition 5.1 A seeded simplicial map $p^* : RG^* \to B^*$ is said to be *realised* by an algebra T if there is a congruence α on T planting B^* such that p^0 is the natural projection $T^\alpha \to T^{\alpha\circ\eta(\alpha)}$.

Theorem 5.2 *A seeded simplicial map $p^* : RG^* \to B^*$ is unobstructed iff it is realised by an algebra T.*

Proof (Sketch.) "**If:**" Consider the diagram

$$\begin{array}{ccccccc}
\Rrightarrow & RG^2 & \Rrightarrow & RG & \to & R \\
 & \downarrow \sigma^2 & & \downarrow \sigma^1 & & \downarrow \sigma^0 \\
\Rrightarrow & \alpha & \Rrightarrow & T & \to & R \\
 & \downarrow & & \downarrow & & \downarrow p^0 \\
\Rrightarrow & \alpha^{(\eta(\alpha)|\alpha)} & \Rrightarrow & T^{\eta(\alpha)} & \to & T^{\alpha\circ\eta(\alpha)}
\end{array} \tag{5.1}$$

in which σ^0 is the identity on $R = T^\alpha$, σ^1 is given by the freeness of RG, and σ^2 exists since $\alpha = \ker(T \to R)$. Take p^2, p^1, p^0 to be the composites down the respective columns of (5.1), the second factors of these composites all being natural projections. Writing $\pi^i : \alpha \to T; (t_0, t_1) \mapsto t_i$, one has

$$\begin{aligned}(\varepsilon_3^0\sigma^2, \varepsilon_3^1\sigma^2, \varepsilon_3^2\sigma^2)P\pi^0 &= (\varepsilon_3^0\varepsilon_2^0, \varepsilon_3^1\varepsilon_2^0, \varepsilon_3^2\varepsilon_2^0)P\sigma^1 = \varepsilon_3^2\varepsilon_2^0\sigma^1\\ &= \varepsilon_3^0\varepsilon_2^1\sigma^1 = (\varepsilon_3^0\varepsilon_2^1, \varepsilon_3^1\varepsilon_2^1, \varepsilon_3^2\varepsilon_2^1)P\sigma^1 = (\varepsilon_2^0\sigma^2, \varepsilon_2^1\sigma^2, \varepsilon_2^2\sigma^2)P\pi^1,\end{aligned}$$

so the obstruction of p^* is the zero element $(\varepsilon_3^0 p^2, \varepsilon_3^1 p^2, \varepsilon_3^2 p^2)P^V$ of the group $\mathrm{Der}(RG^3, (\alpha \cap \eta(\alpha))^{(\alpha\circ\eta(\alpha)|\alpha\cap\eta(\alpha))})$, as required.

"**Only if:**" If p^* is unobstructed, then as shown in [9, p.129], one may assume without loss of generality that (4.3) itself is zero, and not just in the zero cohomology class. Let Q be a pullback in

$$\begin{array}{ccc} Q & \rightarrow & RG \\ \downarrow & & \downarrow {\scriptstyle p_1} \\ B^2 & \underset{\varepsilon_2^1}{\rightarrow} & B^1 \end{array}$$

realised, say, by $Q = \{(b, w) \in B^2 \times RG \mid wp^1 = b\varepsilon_2^1\}$. Define a congruence W on Q by $(b, w)\ W\ (b', w')$ iff $w\varepsilon_1^0 = w'\varepsilon_1^0$ and

$$(b, b')\ (\ker \varepsilon_2^1 | \ker \varepsilon_2^0)\ (\{w\}p^2, \{w'\}p^2).$$

Set $T = Q^W$, and take α on T to be the kernel of $T \to R; (b, w)^W \mapsto w\varepsilon_1^0$. For the details of the verification that T realises p^*, with α planting B^*, see [9, pp.129–132]. In particular, note that $\eta(\alpha)$ is the kernel of $T \to B^1; (b, w)^W \mapsto b\varepsilon_2^0$.

□

6 Classifying extensions and obstructions

Let $p^* : RG^* \to B^*$ be a seeded simplicial map whose codomain grows a module $M \to B^0$. Pull M back along $p^0 : R \to B^0$ to an R-module. An extension $\alpha \rightrightarrows T \to R$ is said to be *singular for* p^* if its kernel α is self-centralising, with an R-module isomorphism $\alpha^{(\alpha|\alpha)} \to M$. (Note that the central extensions of [4] form a special case of the singular extensions, in which α centralises all of $T \times T$.) Let p^*S be the set of $\mathfrak{V}/R$-isomorphism classes of extensions that are singular for p^*. This set becomes an abelian group, with the class of the split extension $M \to R$ as zero. The addition operation on p^*S is known as the *Baer sum*. To obtain a representative of the Baer sum of the isomorphism classes of two extensions $\alpha_i \rightrightarrows T_i \to R$, with module isomorphism $\theta : \alpha_1^{(\alpha_1|\alpha_1)} \to \alpha_2^{(\alpha_2|\alpha_2)}$, take the quotient of the pullback $T_1 \times_R T_2$ by the congruence

$$\{((t_1, t_2), (t_1', t_2')) \mid (t_i, t_i') \in \alpha_i,\ (t_1, t_1')^{(\alpha_1|\alpha_1)}\theta = (t_2, t_2')^{(\alpha_2|\alpha_2)}\}.$$

Singular extensions are then classified as follows [9, Theorem 632].

Theorem 6.1 *The groups p^*S and $\mathrm{H}^2(R, M)$ are isomorphic.*

Now assume additionally that the seeded simplicial map $p^* : RG^* \to B^*$ is unobstructed. An extension $\alpha \rightrightarrows T \to R$ is said to be *non-singular for* p^* if T realises p^*. Let p^*N denote the set of $\mathfrak{V}/R$-isomorphism classes of extensions that are non-singular for p^*. By Theorem 5.2, p^*N is non-empty. Non-singular extensions are then classified as follows [9, Theorem 634].

Theorem 6.2 *The abelian group p^*S acts regularly on p^*N, so the sets p^*N and $\mathrm{H}^2(R, M)$ are isomorphic.*

Let $\beta \rightrightarrows S \to R$ be singular for p^*, and let $\alpha \rightrightarrows T \to R$ be non-singular for p^*. To obtain a representative for the image of the class of α under the action of the class of β, assuming an R-module isomorphism $\theta : (\alpha \cap \eta(\alpha))^{(\alpha \circ \eta(\alpha)|\alpha \cap \eta(\alpha))} \to \beta^{(\beta|\beta)}$, take the quotient of the pullback $T \times_R S$ by the congruence

$$\{((t, s), (t', s')) \mid (t, t') \in \alpha \cap \eta(\alpha),\ (s', s) \in \beta,\ (t, t')^{(\alpha|\alpha\cap\eta(\alpha))}\theta = (s', s)^{(\beta|\beta)}\}.$$

The final result [9, Theorem 641] shows how obstructions may be classified by elements of the third monadic cohomology groups. Note that for non-trivial R, the hypothesis on R is always satisfied in varieties $\mathfrak{V}$, such as the variety of all groups, where free algebras have little centrality. On the other hand, it is not satisfied, for example, by the three-element group in the variety of commutative Moufang loops.

Theorem 6.3 *Let R be a $\mathfrak{V}$-algebra for which $\eta(\ker(\varepsilon_1^0 : RG \to R)) = \widehat{RG}$. Let $M \to R$ be an R-module, and let $\xi \in \mathrm{H}^3(R, M)$. Then ξ is the obstruction to a seeded simplicial map $p^* : RG^* \to B^*$ whose codomain grows a module that pulls back to $M \to R$ along p^0.*

References

[1] Barr, M. *Cohomology and obstructions: commutative algebras*, Springer Lecture Notes in Mathematics No. 80, (ed. B. Eckmann), Springer-Verlag, Berlin, 1969, pp. 357–375.

[2] Duskin, J. *Simplicial Methods and the Interpretation of "Triple" Cohomology*, Memoirs of the American Mathematical Society No. 163, American Mathematical Society, Providence, RI, 1975.

[3] Gruenberg, K. *Cohomological Topics in Group Theory*, Springer Lecture Notes in Mathematics No. 143, Springer-Verlag, Berlin, 1970.

[4] Janelidze, G. and Kelly, G.M. *Central extensions in Mal'tsev varieties*, Th. and Appl. of Categ. **7** (2000), 219–226.

[5] Janelidze, G. and Kelly, G.M. *Central extensions in universal algebra: a unification of three notions*, Alg. Univ. **44** (2000), 123–128.

[6] Johnson, K.W. and Leedham-Green, C.R. *Loop cohomology*, Czech. Math. J. **40** (**115**) (1990), 182–195.

[7] Mac Lane, S. *Homology*, Springer-Verlag, Berlin, 1963.

[8] Orzech, G.G. *Obstruction Theory in Algebraic Categories*, Ph.D. thesis, University of Illinois, Urbana, IL, 1970.

[9] Smith, J.D.H. *Mal'cev Varieties*, Springer Lecture Notes in Mathematics No. 554, Springer-Verlag, Berlin, 1976.

[10] Smith, J.D.H. and Romanowska, A.B. *Post-Modern Algebra*, Wiley, New York, NY, 1999.