MODES AND MODALS

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Abstract. The paper gives a self-contained introduction to modes (idempotent entropic algebras) and modals (modes distributive over semilattices). Constructions of modals from modes are discussed, including a new contravariant submode functor to a category of concave functions. Some characteristic applications of modal theory are presented, including a new identification of the modal structure of multiplication tables of central quasigroups.

Contents

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This paper is intended as an introduction to modal theory, offering a representative cross-section of its internal techniques and external applications. Modal theory traces its origins back to the paper [RS1], recognizing the import of the conjunction of idempotence with entropicity. An initial exploration of the theory appeared in [RS2]. The subsequent fifteen years have seen considerable development. Romanowska gave an encyclopedic survey at the end of the 1980s [Ro]. The current paper may be read as a self-contained approach to the subject as it now stands. The paper includes some new theory (the contravariant subalgebra construction of §7) and a new application (to multiplication tables of central quasigroups in §12).

Since modal theory is above all an algebraic theory, the first two sections offer a thousand-word course in universal algebra. This is not meant in the spirit of a fifteen-second recital of Hamlet's "To be, or not to be" soliloquy, but rather as a guide for otherwise sophisticated readers who might not have had the benefit of a more conventional hundred-hour introduction to the subject. Experts should note the conventions adopted, such as admittance of empty algebras but exclusion of fictitious variables.

Modes, or idempotent entropic algebras, are introduced in §3. Section 4 presents the key examples: sets, left-zero semigroups, semilattices, affine spaces, convex sets and barycentric algebras. Modals, or modes distributing over semilattices, are introduced in §5, along with the three fundamental lemmas. Section 6 shows how a mode yields two modals, the modal of "polytopes" or finitely-generated non-empty submodes and the modal of all non-empty submodes. The mode reducts of these modals exhibit the self-replicating property that is characteristic for the conjunction of idempotence and entropicity.

Section 7 introduces a new modal construction: the contravariant subalgebra functor assigning the modal of all (possibly empty) submodes to a mode. The codomain category of this functor has one curious feature. Most of the concrete categories encountered in mathematics have morphism properties that are naturally preserved under composition, typically homomorphism between algebras or continuity between topological spaces. Besides having both these properties, the morphisms of the codomain category are concave functions, even though in general the composite of concave functions is not necessarily concave.

Section 8 gives a brief sampling of some structure theory for modes, connecting Płonka sums with embedding into powers of cogenerators. The final four sections offer a range of applications of modal theory to other parts of mathematics: to geometry in §9, to differential geometry in §10, to analysis in §11, and to combinatorial algebra in §12.

In entering its topics to some depth, this paper has had to sacrifice any attempt at completeness. Probably the major omission has been the extension of the affine space examples to idempotent reducts of semimodules over commutative semirings. The reader is referred to Kearnes' paper [Ke] for specific details, and to Golan's book [Go], particularly Chapter 13, for general background.

1. Combinatorial universal algebra.

A *type* $\tau : \Omega \to \mathbb{N}$ is function whose codomain is the set of natural numbers. Elements of the domain of τ are called the *basic operators* of the type. An *algebra of type* τ or τ-*algebra* A or (A, Ω) is a set A equipped with *basic operations*

$$\omega : A^{\omega\tau} \to A; (a_1, \ldots, a_{\omega\tau}) \mapsto a_1 \ldots a_{\omega\tau}\omega \tag{1.1}$$

for each basic operator. The class of all such algebras is denoted by $\underline{\underline{\tau}}$. A subset B of A is a *subalgebra* of (A, Ω) if

$$\forall\ \omega \in \Omega, (\forall\ 1 \leq i \leq \omega\tau, x_i \in B) \Rightarrow x_1 \ldots x_{\omega\tau}\omega \in B. \tag{1.2}$$

The subset B is a *wall* of (A, Ω) if

$$\forall\ \omega \in \Omega, (\forall\ 1 \leq i \leq \omega\tau, x_i \in B) \Leftrightarrow x_1 \ldots x_{\omega\tau}\omega \in B. \tag{1.3}$$

The subset B is *sink* of (A, Ω) if

$$\forall\ \omega \in \Omega, (\exists 1 \leq i \leq \omega\tau.\ x_i \in B) \Rightarrow x_1 \ldots x_{\omega\tau} \in B. \tag{1.4}$$

Since intersections of subalgebras are subalgebras, each subset X of A determines a smallest subalgebra $\langle X \rangle$ of A containing X, known as the *subalgebra generated by* X. Given a family of algebras (A_i, Ω), their product ΠA_i becomes an algebra $\Pi(A_i, \Omega)$ or $(\Pi A_i, \Omega)$, the *product algebra*, under componentwise operations. A function $f : (A, \Omega) \to (B, \Omega)$ between algebras is a *homomorphism* if its *graph*

$$\{(a, b) \in A \times B | af = b\} \tag{1.5}$$

is a subalgebra of $(A \times B, \Omega)$. (Note that it is often convenient to identify a function with its graph.) An equivalence relation α on an algebra A is a *congruence* if it is a subalgebra of $A \times A$. This implies that the *natural projection*

$$\text{nat } \alpha : A \to A^\alpha; a \mapsto a^\alpha, \tag{1.6}$$

mapping an element a of A to its equivalence class $a^\alpha = \{b \in A | a\alpha b\}$ in the *quotient* $A^\alpha = \{a^\alpha | a \in A\}$, is a homomorphism. Conversely, the kernel of a homomorphism is a congruence on the domain of the homomorphism.

Given a set L, the *free monoid* L^* over L is the set of all *words* $l_1 l_2 \dots l_n$ with l_i in L and n in $\mathbb{N}$. Words are multiplied by concatenation; the unit element is the empty word $(n = 0)$. Given a set X, the free monoid $(X \cup \Omega)^*$ becomes a τ-algebra under

$$\omega : (w_1, \dots, w_{\omega\tau}) \mapsto w_1 \dots w_{\omega\tau}\omega \tag{1.7}$$

for each basic operator ω. Define $X\Omega$, the *algebra of τ-words in X*, to be the subalgebra of $((X \cup \Omega)^*, \Omega)$ generated by X.

Proposition 1.1. *Each function $f : X \to A$ to the underlying set of a τ-algebra (A, Ω) extends to a unique homomorphism $\bar{f} : (X\Omega, \Omega) \to (A, \Omega)$.*

Proof. The graph of $\bar{f}$ is the subalgebra of $(X\Omega \times A, \Omega)$ generated by the graph of f. $\square$

Now fix a set P of *variables* or *arguments* by a bijection

$$\beta : \mathbb{Z}^+ \to P; n \mapsto x_n. \tag{1.8}$$

Make the power set 2^P a τ-algebra by

$$\omega : (A_1, \dots, A_{\omega\tau}) \mapsto A_1 \cup \dots \cup A_{\omega\tau} \tag{1.9}$$

for ω in Ω. By Proposition 1.1, there is a homomorphism

$$\arg\ : P\Omega \to 2^P; x_n \mapsto \{x_n\}, \tag{1.10}$$

called the *argument map*. Note $\arg(x_1 \dots x_{\omega\tau}\omega) = \{x_1, \dots, x_{\omega\tau}\}$. Since the argument map sends each element of $P\Omega$ to a finite subset of P, there is a well-defined function

$$\tau' : P\Omega \to \mathbb{N}; w \mapsto \max(\beta^{-1}(\arg w)) \tag{1.11}$$

called the *derived type* of τ. Elements of $P\Omega$ are called *derived operators* of τ. Given a τ-algebra (A, Ω), one obtains a τ'-algebra $(A, P\Omega)$ with *derived operations*

$$u : A^{u\tau'} \to A;\ (a_1, \dots, a_{u\tau'}) \mapsto \overline{u(x_i \mapsto a_i)}. \tag{1.12}$$

In other words, u acts on A by sending $(a_1, \dots, a_{u\tau'})$ to the image of u under the homomorphic extension $\bar{f} : P\Omega \to A$ of the function $f : P \to A;\ x_i \mapsto a_i$. It is convenient to write a derived operator u in the form $x_1 \dots x_{u\tau'}u$, so the (1.8) becomes

$$u : A^{u\tau'} \to A;\ (a_1, \dots, a_{u\tau'}) \mapsto a_1 \dots a_{u\tau'}u. \tag{1.13}$$

One may identify a basic operator ω with the derived operator $x_1 \dots x_{\omega\tau}\omega$.

Finally, fix a τ-algebra (A, Ω). For a subset Ψ of $P\Omega$, and restriction $\sigma : \Psi \to \mathbb{N}$ of $\tau' : P\Omega \to \mathbb{N}$, the σ-algebra (A, Ψ) is called a *reduct* of the τ-algebra (A, Ω). Subalgebras of such reducts (A, Ψ) are called *subreducts* of the original algebra (A, Ω). In the other direction, one may extend the type $\tau : \Omega \to \mathbb{N}$ to the disjoint union $\tau_A = (A \to \{0\}) \mathbin{\dot\cup} (\tau : \Omega \to \mathbb{N})$. For an element a of A, the nullary operator a yields a *constant* nullary operation

$$a : A^0 \to A \tag{1.14}$$

with image $\{a\}$. Derived operators of the τ_A-algebra $(A, A \mathbin{\dot\cup} \Omega)$ are called *polynomials* of the original τ-algebra (A, Ω). The corresponding derived operations on A are called the *polynomial functions* of the τ-algebra (A, Ω).

2. Categorical universal algebra.

Given a class $\underline{\underline{V}}$ of algebras of type τ, the category $\underline{\underline{V}}$ (same symbol) will denote the category whose object class is the class $\underline{\underline{V}}$ and such that, for $\underline{\underline{V}}$-algebras A and B, the set $\underline{\underline{V}}(A, B)$ of morphisms from A to B is the set of all homomorphisms from A to B. The category $\underline{\underline{V}}$ is the domain of two forgetful functors, the *inclusion*

$$G : \underline{\underline{V}} \hookrightarrow \underline{\underline{\tau}} \tag{2.1}$$

and the *underlying set functor*

$$U : \underline{\underline{V}} \to \underline{\underline{\mathrm{Set}}}; (f : (A, \Omega) \to (B, \Omega)) \mapsto (f : A \to B) \tag{2.2}$$

to the category $\underline{\underline{\mathrm{Set}}}$ of sets and functions. Proposition 1.1 shows that $U : \underline{\underline{\tau}} \to \underline{\underline{\mathrm{Set}}}$ has a left adjoint $\Omega : \underline{\underline{\mathrm{Set}}} \to \underline{\underline{\tau}}$. For a set X, the unit $\eta_X : X \to X\Omega U$ just construes an element x of X as a one-letter word. For a τ-algebra (A, Ω), the counit $\varepsilon_A : AU\Omega \to A$ is the homomorphic extension of the identity function $AU \to A; a \mapsto a$ given by Proposition 1.1. In particular, for each basic operation ω, the counit ε_A maps a word $a_1 \dots a_{\omega\tau}\omega$ in $AU\Omega$ to the corresponding element $a_1 \dots a_{\omega\tau}\omega$ of A given by the image of (1.1).

A class $\underline{\underline{V}}$ of algebras of type τ is a *prevariety* if isomorphic copies, subalgebras and products of $\underline{\underline{V}}$-algebras are again $\underline{\underline{V}}$-algebras. If $\underline{\underline{V}}$ is a prevariety, then the forgetful functors (2.1) and (2.2) each have left adjoints. The left adjoint of the inclusion $G : \underline{\underline{V}} \hookrightarrow \underline{\underline{\tau}}$ is the *replication functor* R_V or $R : \underline{\underline{\tau}} \to \underline{\underline{V}}$. The unit of the adjunction, for a τ-algebra A, is the surjective corestriction $\eta_A : A \to ARG$ of the product of the set of natural projections $\mathrm{nat}\,\alpha : A \to A^\alpha$ of congruences α on A whose quotient lies in $\underline{\underline{V}}$. In other words, η_A is the natural projection of the smallest congruence ρ_V or ρ on A whose quotient A^ρ lies in $\underline{\underline{V}}$. The congruence ρ is called the $\underline{\underline{V}}$*-replica congruence* of A, and the corresponding quotient A^ρ in $\underline{\underline{V}}$ is called the $\underline{\underline{V}}$*-replica* of A. The left adjoint of $U : \underline{\underline{V}} \to \underline{\underline{\mathrm{Set}}}$ is the *free* $\underline{\underline{V}}$*-algebra functor* $V : \underline{\underline{\mathrm{Set}}} \to \underline{\underline{V}}$. The unit $\eta_X : X \to XVU$ of the adjunction, for a set X, is the composite of the unit $\eta_X : X \to X\Omega U$ of the adjunction $(\Omega : \underline{\underline{\mathrm{Set}}} \to \underline{\underline{\tau}}\ , U : \underline{\underline{\tau}} \to \underline{\underline{\mathrm{Set}}}\ , \eta, \varepsilon)$ and (the image under $U : \underline{\underline{V}} \to \underline{\underline{\mathrm{Set}}}$ of) the unit $\eta_{X\Omega} : X\Omega \to X\Omega RG$ of the adjunction

$(R : \underline{\underline{\tau}} \to \underline{\underline{V}}, G : \underline{\underline{V}} \to \underline{\underline{\tau}}, \eta, \varepsilon)$. In other words, the adjunction $(V : \underline{\underline{\mathrm{Set}}} \to \underline{\underline{V}}$, $U : \underline{\underline{V}} \to \underline{\underline{\mathrm{Set}}}, \eta, \varepsilon)$ is the composite of these two given adjunctions. Two τ-words in X are said to be $\underline{\underline{V}}$*-synonymous* if they are related by the replica congruence ρ_V on $X\Omega$. Thus the free $\underline{\underline{V}}$-algebra on a set X is the algebra of $\underline{\underline{V}}$-synonymy classes.

Fix a type $\tau : \Omega \to \mathbb{N}$. An *identity* in the type τ is a pair (u, v) of derived operators of τ. (Cross-Channel cultural note: Anglophones may refer to an identity as a "law", whereas Francophones may refer to a basic operation as a "loi de composition".) It is often convenient to write the identity (u, v) in the form $x_1 \dots x_{u\tau'} u = x_1 \dots x_{v\tau'} v$. For example, in the type $\{(\mu, 2)\}$ of a binary "multiplication" μ, the *reductive law* is

$$x_3 x_2 \mu = x_3 x_2 x_1 \mu\mu, \tag{2.3}$$

or $x_3 \cdot x_2 = x_3 \cdot (x_2 \cdot x_1)$ using infix notation for the multiplication. A τ-algebra (A, Ω) is said *to satisfy* the identity (u, v) if the derived operations u and v coincide on A.

Birkhoff's Theorem 2.1 [Bi]. *A class $\underline{\underline{V}}$ of τ-algebras is closed under homomorphic images, subalgebras and products iff it is the class of all τ-algebras satisfying a given set of identities.* $\square$

A *variety* is a class $\underline{\underline{V}}$ of τ-algebras satisfying the two equivalent conditions of Birkhoff's Theorem. Note that a variety is a prevariety. The full set of identities satisfied by each algebra from a variety $\underline{\underline{V}}$ is the $\underline{\underline{V}}$-replica congruence ρ_V on the algebra $P\Omega$ of derived operators.

Now let $\sigma : \Psi \to \mathbb{N}$ and $\tau : \Omega \to \mathbb{N}$ be two types. Let $\underline{\underline{V}}$ be a variety of σ-algebras, and let $\underline{\underline{W}}$ be a variety of τ-algebras. Then $\underline{\underline{V}} \otimes \underline{\underline{W}}$ denotes the variety of algebras A of type $\sigma \dot\cup \tau$ whose σ-reducts lie in $\underline{\underline{V}}$, whose τ-reducts lie in $\underline{\underline{W}}$, and which satisfy the identities saying that each basic operation (1.1) from τ is a homomorphism of σ-algebras. These identities may equivalently be viewed as saying that each basic operation from σ on a $\underline{\underline{V}} \otimes \underline{\underline{W}}$-algebra is a homomorphism of τ-algebras. Note that there are forgetful functors $\underline{\underline{V}} \otimes \underline{\underline{W}} \to \underline{\underline{V}}$ and $\underline{\underline{V}} \otimes \underline{\underline{W}} \to \underline{\underline{W}}$.

3. Modes.

Fix a type $\tau : \Omega \to \mathbb{N}$. A *$\tau$-algebra* (A, Ω) is said to be *entropic* if each basic operation ω is a homomorphism $\omega : (A^{\omega\tau}, \Omega) \to (A, \Omega)$. In other words, for each set $\{\omega, \varphi\}$ of basic operations, say with $\omega\tau = n$ and $\varphi\tau = m$, the identity

$$(x_{11} \dots x_{1n}\omega) \dots (x_{m1} \dots x_{mn}\omega)\varphi = (x_{11} \dots x_{m1}\varphi) \dots (x_{1n} \dots x_{mn}\varphi)\omega \tag{3.1}$$

is satisfied. If ω and φ are equal binary operations denoted by an infix $*$, then (3.1) becomes

$$(x_{11} * x_{12}) * (x_{21} * x_{22}) = (x_{11} * x_{21}) * (x_{12} * x_{22}). \tag{3.2}$$

(The identities (3.1) and (3.2) have been given various names in the literature: for a partial list, see [S1, §6]. The word "entropic", in use in this context for over half a century, refers to the "inner turning" of x_{12} and x_{21} in (3.2). For a connection with the information-theoretic concept of entropy, see [S3].) Note that modules over a commutative ring are entropic. Entropic algebras are characterized by the following "folk theorems" (cf. [RS2, 159]).

Proposition 3.1. *A τ-algebra A is entropic iff, for each τ-algebra X, the morphism set $\underline{\underline{\tau}}(X, A)$ is a subalgebra of the product algebra $\underline{\underline{Set}}(X, A) = A^X$.* $\square$

Corollary 3.2. *If $\underline{\underline{V}}$ is a variety of entropic algebras, then for each pair A, B of objects of $\underline{\underline{V}}$, the morphism set $\underline{\underline{V}}(A, B)$ is again an object of $\underline{\underline{V}}$.* $\square$

Now let $\underline{\underline{V}}$ be a variety of entropic algebras. Fix a $\underline{\underline{V}}$-algebra B. Then by Corollary 3.2, there is a functor

$$\underline{\underline{V}}(B, ?) : \underline{\underline{V}} \to \underline{\underline{V}}; (f : X \to Y) \mapsto (\underline{\underline{V}}(B, X) \to \underline{\underline{V}}(B, Y); h \mapsto hf). \tag{3.3}$$

Exactly as in the familiar case where $\underline{\underline{V}}$ is the variety of modules over a commutative ring, the functor (3.4) has a left adjoint

$$? \otimes B : \underline{\underline{V}} \to \underline{\underline{V}}; (f : X \to Y) \mapsto (f \otimes B : X \otimes B \to Y \otimes B), \tag{3.4}$$

yielding an adjunction

$$\underline{\underline{V}}(A \otimes B, C) \cong \underline{\underline{V}}(A, \underline{\underline{V}}(B, C)). \tag{3.5}$$

Writing the unit in the form

$$\eta_A : A \to \underline{\underline{V}}(B, A \otimes B); a \mapsto (b \mapsto a \otimes b) \tag{3.6}$$

for a $\underline{\underline{V}}$-algebra A, the counit is just the evaluation

$$\varepsilon_A : \underline{\underline{V}}(B, A) \otimes B \to A; f \otimes b \mapsto b^f \tag{3.7}$$

of homomorphisms. The image of a $\underline{\underline{V}}$-algebra A under (3.4) is called the *tensor product* $A \otimes B$ of A and B. Using the notation (3.6), the algebra $A \otimes B$ is generated by its set $\{a \otimes b | a \in A, b \in B\}$ of *primitive elements*. Since (3.6) is a $\underline{\underline{V}}$-morphism, the maps $A \to A \otimes B; x \mapsto x \otimes b$ and $B \to A \otimes B; y \mapsto a \otimes y$ are homomorphisms for fixed elements a of A and b of B. (A discussion of tensor products in entropic varieties is given in [DD]. Their good behaviour may be contrasted with the case of semigroups [Gr]. For applications to the study of communicating processes, see [RS5].)

A τ-algebra (A, Ω) is said to be *idempotent* if each singleton subset of A is actually a subalgebra. In other words, for each basic operation ω, the identity

$$xx \ldots x\omega = x \tag{3.8}$$

is satisfied.

Definition 3.3. A *mode* is an idempotent, entropic algebra. $\square$

As an immediate consequence of the definition, one obtains

Proposition 3.4. *Products, quotients, and subreducts of modes are modes.* $\square$

Kearnes has given the following characterization of modes.

Proposition 3.5. *A τ-algebra (A, Ω) is a mode iff each polynomial function of (A, Ω) is a homomorphism.*

Proof. First, suppose that A is a mode. Let a be an element of A. Since $\{a\}$ is a subalgebra of (A, Ω), it follows that the constant nullary operation (1.13) is a homomorphism. Since A is entropic, the basic operations are homomorphisms. Thus each operation derived from $A \cup \Omega$, i.e. each polynomial function, is a homomorphism. Conversely, suppose that each polynomial function of (A, Ω) is a homomorphism. Since the basic operations are polynomial functions, A is entropic. Since (1.13) is a homomorphism for each element a of A, its image $\{a\}$ is a subalgebra of A. Thus A is also idempotent. $\square$

4. Examples of modes.

The aim of this section is to present some of the basic examples of modes.

(A) SETS. Construed as algebras of empty type, sets are modes.

(B) LEFT-ZERO SEMIGROUPS. A *left-zero semigroup* is a semigroup $(S, .)$ satisfying the *left-zero identity*

$$x \cdot y = x, \tag{4.1}$$

i.e. the identity saying that each element x of the semigroup acts as a left zero element. The variety of left-zero semigroups is denoted by $\underline{\underline{\mathrm{Lz}}}$.

(C) SEMILATTICES. A *semilattice* H is an idempotent, commutative semigroup. Semilattices are closely associated with partial orders having either greatest lower bounds, or least upper bounds, of all pairs of elements. Recall that a partial order $(H, \leq)$ may be construed as a (small) category (H) with object set H by taking the morphism set $(H)(x, y)$ to be the singleton $\{x \to y\}$ if $x \leq y$, and empty otherwise. If $(H, \leq)$ has a greatest lower bound $\mathrm{glb}\{x, y\}$ for $\{x, y\} \subseteq H$, then $\mathrm{glb}\{x, y\} = x \cdot y$ is a product in (H). In this case $(H, .)$ becomes a semilattice, a *meet semilattice*. On the other hand, defining

$$x \leq_. y \Leftrightarrow x.y = x \tag{4.2}$$

on a semilattice $(H, .)$ yields an ordered set $(H, \leq_.)$ with greatest lower bounds given by products in the semilattice. Dually, one obtains a *join semilattice* $(H, +)$ with

$$x \leq_+ y \Leftrightarrow x + y = y; \tag{4.3}$$

the "sum" operation $+$ on the semilattice corresponds to least upper bounds in $(H, \leq_+)$ or coproducts in (H). Now semilattices are modes. The variety of all semilattices is denoted by $\underline{\underline{\mathrm{Sl}}}$.

(D) AFFINE SPACES. Let R be a commutative (unital) ring. Let R-$\underline{\underline{\mathrm{Mod}}}$ be the variety of (unital, right) R-modules, construed as algebras $(E, +, 0, R)$ with a binary addition, nullary zero, and unary scalar multiplications. Given a module E, the corresponding *affine space* may be described algebraically as the set E equipped with all the idempotent linear or *affine* operations

$$E^n \to E; (a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i r_i \tag{4.4}$$

for each positive n and each element $(r_1, \ldots, r_n)$ of R^n with $r_1 + \cdots + r_n = 1$. In this sense, affine spaces are modes. It is convenient to write

$$r' = 1 - r \tag{4.5}$$

and

$$r \circ s = r + s - rs = (r's')' \tag{4.6}$$

for r, s in R. Then define a set $\underline{R}$ of binary operations

$$\underline{r} : E^2 \to E; (x, y) \mapsto x(1 - r) + yr \tag{4.7}$$

for r in R. In particular, note

$$xxy\underline{r}\,\underline{s} = xy\underline{rs},\ \ xy\underline{r'} = yx\underline{r} \tag{4.8}$$

for r and s in R. Define the ternary *Mal'cev operation*

$$P : E^3 \to E; (x, y, z) \mapsto x - y + z. \tag{4.9}$$

As noted by Ostermann and Schmidt [OS], cf. [Cs], the affine operations (4.4) are precisely the derived operations of the algebra $(E, \underline{R}, P)$. Moreover, the algebra $(E, \underline{R}, P)$ has the affine group as its group of automorphisms, and may thus be identified with the affine geometry (cf. [Ne], [OS]). Let $\underline{\underline{R}}$ denote the class of affine spaces over the commutative ring R. Then $\underline{\underline{R}}$ may be characterized [RS2, 255] as the variety of modes of type $(R \times \{2\}) \cup \{(P, 3)\}$ satisfying the *parallelogram laws*

$$xxyP = y = yxxP \tag{4.10}$$

together with

$$\begin{cases} (A1) & xyxP = yx\underline{2} \\ (A2) & xy\underline{p}\ xy\underline{q}\underline{r} = xy\underline{pqr} \\ (A3) & xy\underline{p}\ xy\underline{q}\ xy\underline{r}P = xy\underline{pqrP} \\ (A4) & xy\underline{0} = x = yx\underline{1} \end{cases} \tag{4.11}$$

for p, q, r in R. If 2 is invertible in R, then one may derive the Mal'cev operation (4.9) as

$$xyzP = yxz\underline{2}^{-1}\underline{2}. \tag{4.12}$$

In this case, $\underline{\underline{R}}$ may be characterized as the variety of modes of type $R \times \{2\}$ satisfying (A2) and (A4) of (4.11) [OS] [RS, 256]. (More generally, such a characterization obtains iff R has no 2-element quotient [Sz].) Finally, note that tensor products of commutative rings correlate with tensor products of varieties via

$$\underline{\underline{R}} \otimes \underline{\underline{S}} = \underline{\underline{R \otimes S}} \tag{4.13}$$

for commutative rings R and S.

(E) CONVEX SETS. Let I° denote the open unit interval $]0, 1[= \{x | 0 < x < 1\}$ in $\mathbb{R}$. Consider reducts (E, I°) of real affine spaces $(E, \mathbb{R})$. The subalgebras of such reducts are precisely the convex sets. One may thus characterize the class $\underline{\underline{C}}$ of convex sets as the class of algebras (A, I°) of type $I^\circ \times \{2\}$ satisfying *idempotence*

$$xx\underline{p} = x, \tag{4.14}$$

skew-commutativity

$$xy\underline{p} = yx\underline{p}', \tag{4.15}$$

skew-associativity

$$xy\underline{p}z\underline{q} = xyz\underline{q/(p \circ q)}\ \underline{p \circ q}, \tag{4.16}$$

and *cancellativity*

$$xy\underline{p} = xz\underline{p} \Rightarrow y = z \tag{4.17}$$

for p, q in I° [Ne] [RS2, 269]. Note that $\underline{\underline{C}}$ forms a prevariety of modes. Many aspects of convexity theory have direct algebraic interpretations. Here are two elementary samples:

(i) The free $\underline{\underline{C}}$-algebra XC on a set X is the simplex with vertex set X, i.e. the $(|X| - 1)$-dimensional simplex.

(ii) Walls in the algebraic sense (1.3) correspond to walls of convex sets in the geometric sense [Mi].

(F) BARYCENTRIC ALGEBRAS. Because of the implication in (4.17), the class $\underline{\underline{C}}$ of convex sets does not form a variety. However, Birkhoff's Theorem 2.1 shows that the class $\underline{\underline{B}}$ of homomorphic images of convex sets is characterized by a set of identities. In fact, one obtains $\underline{\underline{B}}$ as the variety of *barycentric algebras*, namely I°-algebras satisfying (4.14)–(4.16) [Ne] [RS2, 214]. Note that a semilattice $(H, .)$ is also a barycentric algebra, with

$$xy\underline{p} = x \cdot y \tag{4.18}$$

for all x, y in H and p in I°. Indeed, the uniqueness of left adjoints in §2 shows that the free semilattice on a set X is the semilattice replica of the free $\underline{\underline{C}}$-algebra or set of finite probability distributions on X.

5. Modals.

Fix a type $\tau : \Omega \to \mathbb{N}$, and a class $\underline{\underline{V}}$ of algebras of type τ.

Definition 5.1. A $\underline{\underline{V}}$-*modal* is an algebra $(D, +, \Omega)$ of type $\{(+, 2)\} \cup \tau$ such that:

(a) $(D, +)$ is a (join) semilattice;

(b) (D, Ω) is a $\underline{\underline{V}}$-mode;

(c) (D, Ω) *distributes over* $(D, +)$, i.e.

$$\left\{ \begin{array}{l} \forall \omega \in \Omega,\ \forall 1 \leq j \leq \omega\tau,\ \forall x_1, \ldots, x_j, x_j', \ldots, x_{\omega\tau} \in D, \\ x_1 \ldots (x_j + x_j') \ldots x_{\omega\tau}\omega = x_1 \ldots x_j \ldots x_{\omega\tau}\omega + x_1 \ldots x_j' \ldots x_{\omega\tau}\omega. \end{array} \right. \tag{5.1}$$ $\square$

Note that Definition 5.1(c) may be cast in the form that

$$\omega : (D^{\otimes \omega\tau}, +) \to (D, +); x_1 \otimes \cdots \otimes x_{\omega\tau} \mapsto x_1 \ldots x_{\omega\tau}\omega \tag{5.2}$$

is a semilattice homomorphism for each ω in Ω, the tensor power $D^{\otimes \omega\tau}$ or tensor product $D \otimes \cdots \otimes D$ of $\omega\tau$ copies of $(D, +)$ being taken in the (entropic) variety $\underline{\underline{\mathrm{Sl}}}$ of semilattices. Further, note that each modal $(D, +, \Omega)$ carries the order $(D, \leq_+)$ or $(D, \leq)$ given by the join semilattice structure (4.3) on D.

For a function $f : A \to (B, \leq)$ with ordered codomain, the *epigraph* is the set

$$\{(a, b) \in A \times B | af \leq b\}. \tag{5.3}$$

A function $f : (A, \Omega) \to (D, +, \Omega)$ from a $\underline{\underline{\tau}}$-mode A to a $\underline{\underline{\tau}}$-modal D is said to be τ-*convex* (or just *convex*) if its epigraph is a subalgebra of $(A \times B, \Omega)$. Comparing with (1.5), it is apparent that mode homomorphisms $f : (A, \Omega) \to (D, \Omega)$ are convex.

Example 5.2. For the type $\tau : I^\circ \to \{2\}$ of barycentric algebras, the algebra $(\mathbb{R}, \max, I^\circ)$, with mode reduct the convex set $(\mathbb{R}, I^\circ)$and with join semilattice ordering $(\mathbb{R}, \leq)$ the usual ordering on $\mathbb{R}$, is a modal. Given a convex set (C, I°), a real valued function $f : C \to \mathbb{R}$ is a τ-convex function $f : (C, I^\circ) \to (\mathbb{R}, \max, I^\circ)$ in the modal-theoretic sense iff it is convex (upwards) in the sense of real analysis. As a more combinatorial example, consider the power set $(2^X, \cap)$ of a finite set X as a semilattice under intersection, and then as a barycentric algebra $(2^X, I^\circ)$ via (4.18). The cardinality function

$$(2^X, I^\circ) \to (\mathbb{R}, \max, I^\circ); Y \mapsto |Y| \tag{5.4}$$

is convex. $\square$

The fundamental elementary properties of a modal $(D, +, \Omega)$ are summarized by the following lemmas.

Monotonicity Lemma 5.3. *Each basic operation*

$$\omega : (D^{\omega\tau}, \leq_+) \to (D, \leq_+) \tag{5.5}$$

is monotone. $\square$

Convexity Lemma 5.4. *For each positive integer r, the join*

$$\Sigma_r : (D^r, \Omega) \to (D, +, \Omega); (x_1, \ldots, x_r) \mapsto x_1 + \cdots + x_r \tag{5.6}$$

is convex. $\square$

Sum-Superiority Lemma 5.5. *For each basic operation ω, one has $\omega \leq \Sigma_{\omega\tau}$.* $\square$

6. Modals of subalgebras.

Fix a type $\tau : \Omega \to \mathbb{N}$. Given an algebra (A, Ω) of type τ, let AS denote the set of all non-empty subalgebras of (A, Ω). Extending language used when (A, Ω) is the convex set $(\mathbb{R}^n, I^\circ)$ for some n, define a *polytype* of (A, Ω) to be a finitely generated subalgebra of (A, Ω). Let AP denote the set of all non-empty polytopes of (A, Ω). The power set 2^A, the set of subsets or "complexes" of A, is a τ-algebra under the so-called *complex operations*

$$\omega : (X_1, \ldots, X_{\omega\tau}) \mapsto \{x_1 \ldots x_{\omega\tau}\omega | \forall\ 1 \leq i \leq \omega\tau, x_i \in X_i\} \tag{6.1}$$

for $\omega \in \Omega$. A derived operator u of τ is said to be *linear* if each of its arguments appears only once as a letter in the word u. An identity (u, v) is said to be *linear* if both u and v are linear. Using these concepts, one may formulate the fundamental *self-replicating property* of modes [RS2, §1.4].

Theorem 6.1. *Let (A, Ω) be a mode. Then under the complex operations, (AP, Ω) and (AS, Ω) are modes satisfying all the linear identities satisfied by (A, Ω).* $\square$

Example 6.2. For the convex set $\mathbb{R}$, the mode $\mathbb{R}P$ is isomorphic to the northwest halfplane of $\mathbb{R}^2$ via the map sending a non-empty polytope or closed interval $[a, b]$ to the point (a, b) with $a \leq b$. $\square$

For a mode (A, Ω), the sets AP and AS not only carry the mode structure given by Theorem 6.1, but also the join semilattice order given by containment. In fact, one obtains modals $(AP, +, \Omega)$ and $(AS, +, \Omega)$. Let $\underline{\underline{V}}$ be the variety of τ-modes satisfying a given set of linear identities. Let $\underline{\underline{D}}$ be the variety of $\underline{\underline{V}}$-modals. The polytope construction yields the *covariant polytope functor*

$$P : \underline{\underline{V}} \to \underline{\underline{D}}; (f : A \to B) \mapsto (AP \to BP; X \mapsto Xf). \tag{6.2}$$

There is also a forgetful functor

$$U : \underline{\underline{D}} \to \underline{\underline{V}}; (D, +, \Omega) \mapsto (D, \Omega) \tag{6.3}$$

forgetting the semilattice structure of modals.

Theorem 6.3. *The polytope functor P of (6.2) is left adjoint to the forgetful functor U of (6.3). For a $\underline{\underline{V}}$-mode A, the unit*

$$\eta_A : A \to APU; a \mapsto \{a\} \tag{6.4}$$

embeds A as the algebra of singletons. For a $\underline{\underline{V}}$-modal $(D, +, \Omega)$, the counit

$$\varepsilon_D : DUP \to D; \langle d_1, \ldots, d_n \rangle \mapsto d_1 + \cdots + d_n \tag{6.5}$$

sums the generators of a polytope. $\square$

The proof of Theorem 6.3, using Lemmas 5.3–5, is given in [RS2, §3.5].

A join semilattice $(H,+)$ is said to be *complete* if each non-empty subset X of H has a least upper bound or supremum $\sup X$ in $(H,\leq_+)$. A τ-algebra (D,Ω) is said to be *completely distributive* over a complete semilattice $(D,+)$ if

$$(6.6)\qquad \begin{cases} \forall\omega\in\Omega,\ \forall 1\leq j\leq\omega\tau,\ \forall\varnothing\subset X\subseteq D,\ \forall x_1,\dots,x_{j-1},x_{j+1},\dots,x_{\omega\tau}\in D, \\ \sup\{x_1\dots x_{j-1}xx_{j+1}\dots x_{\omega\tau}\omega|x\in X\} = x_1\dots x_{j-1}(\sup X)x_{j+1}\dots x_{\omega\tau}\omega. \end{cases}$$

A modal $(D,+,\Omega)$ is said to be *complete* if $(D,+)$ is a complete join semilattice and (D,Ω) is completely distributive over $(D,+)$.

Let $\underline{\underline{E}}$ be the category of complete $\underline{\underline{V}}$-modals and homomorphisms. The non-empty submode construction yields the *covariant submode functor*

$$(6.7)\qquad S:\underline{\underline{V}}\to\underline{\underline{E}};(f:A\to B)\mapsto(AS\to BS;X\mapsto Xf)$$

(cf. [RS2, 335]). There is also a forgetful functor

$$(6.8)\qquad U:\underline{\underline{E}}\to\underline{\underline{V}};(f:(D,+,\Omega)\to(E,+,\Omega))\mapsto(f:(D,\Omega)\to(E,\Omega))$$

analogous to (6.3). The corresponding analogue of Theorem 6.3 is then as follows.

Theorem 6.5. *The covariant submode functor S of (6.7) is left adjoint to the forgetful functor U of (6.8). For a $\underline{\underline{V}}$-mode A, the unit*

$$(6.12)\qquad \eta_A;A\to ASU;a\mapsto\{a\}$$

embeds A as the algebra of singletons. For a complete $\underline{\underline{V}}$-modal $(E,+,\Omega)$, the counit is

$$(6.13)\qquad \varepsilon_E:EUS\to E;X\mapsto\sup X.$$

Proof. For a non-empty subalgebra X of a $\underline{\underline{V}}$-mode A, one has $X\eta_A^S\varepsilon_{AS} = \{\{x\}|x\in X\}\varepsilon_{AS}=\sup\{\{x\}|x\in X\}=X$, verifying $\eta_A^S\cdot\varepsilon_{AS}=1_{AS}$. For an element e of a complete modal E, one has $e\eta_{EU}\varepsilon_E^U=\{e\}\varepsilon_E^U=\sup\{e\}=e$, verifying $\eta_{EU}\cdot\varepsilon_E^U=1_{EU}$. (Cf. [Me, Theorem IV.1.2(v)].) $\square$

7. The contravariant subalgebra functor.

Given an algebra (A, Ω) of a type $\tau : \Omega \to \mathbb{N}$, let AT denote the *totality* or set of all subalgebras of (A, Ω), including the empty subalgebra if $0 \notin \Omega\tau$. An identity $u = v$ is said to be *regular* if arg $u =$ arg v. Analogous to Theorem 6.1, one has the following self-replicating property.

Proposition 7.1. *Let (A, Ω) be a mode. Then under the complex operations, (AT, Ω) is a mode satisfying all the regular linear identities satisfied by (A, Ω).* $\square$

A poset $(G, \leq)$ is a *complete lattice* iff each (possibly empty) subset X of G has a supremum $\sup X$ and infimum (or greatest lower bound) $\inf X$ in $(G, \leq)$. In other words, the small category (G) is complete and cocomplete. The least or *zero* element of G is the supremum of the empty subset of G. A non-zero element c of G is *compact* or "finite" if

$$(\exists X \subseteq G.\ c \leq \sup X) \Rightarrow (\exists \text{ finite } F \subseteq G.\ c \leq \sup F). \tag{7.1}$$

For example, the compact elements of the complete lattice of open subsets of a topological space are precisely the non-empty open subsets that are compact as subspaces. The set GQ of compact elements of a complete lattice $(G, +, .)$ forms a join subsemilattice $(GQ, +)$ of $(B, +)$. Recall the definitions of the *down-set*

$$\downarrow x = \{y \in G | y \leq x\} \tag{7.2}$$

and *up-set*

$$\uparrow x = \{y \in G | y \geq x\} \tag{7.3}$$

of an element x of a poset $(G, \leq)$. A complete lattice G is *algebraic* if each element x of G is the supremum

$$x = \sup(GQ \cap \ \downarrow x) \tag{7.4}$$

of the compact elements beneath it. A poset is *bounded* if it has a least element 0 and a greatest element 1. A topological space is *zero-dimensional* if it has a basis of clopen sets. A bounded poset is a $\mathfrak{B}$-*space* if it is a compact, Hausdorff, zero-dimensional topological meet semilattice.

Proposition 7.2. *The following conditions on a bounded poset G are equivalent:*

(a) *G is an algebraic lattice;*

(b) *G is a $\mathfrak{B}$-space;*

(c) *There is a type $\tau : \Omega \to \mathbb{N}$ such that G is isomorphic to the poset $(AT, \subseteq)$ of subalgebras of a τ-algebra (A, Ω).*

Proof. See [Bk, Th. VIII.5.8′] and [Jo, Cor. VI.3.6]. □

In Proposition 7.2(c), the compact elements of AT are precisely the polytopes of (A, Ω), i.e.

$$ATQ = AP \tag{7.5}$$

[Bk, Th. VIII.5.7].

Mimicking the definition of an "arithmetic" lattice [Gi, Defn. I.4.6], consider the following

Definition 7.3. Let $\underline{\underline{V}}$ be a class of algebras of type $\tau : \Omega \to \mathbb{N}$. A complete $\underline{\underline{V}}$-modal $(D, +, \Omega)$ is *arithmetic* if $(D, \leq_+)$ is an algebraic lattice and (DQ, Ω) is a submode of (D, Ω). □

By Proposition 7.2, (7.5), (6.7) and Theorem 6.1, the totality AT of a $\underline{\underline{\tau}}$-mode (A, Ω) forms an arithmetic $\underline{\underline{\tau}}$-modal $(AT, +, \Omega)$. The aim now is to make the assignment $A \mapsto AT$ the object part of a functor like (6.2) and (6.7), so that an analogue of Theorems 6.3 and 6.5 holds. For the rest of this section, fix a type $\tau : \Omega \to \mathbb{N}$.

For a function $f : D \to (E, \leq)$ with ordered codomain, the *hypograph* is the set

$$\{(d, e) \in D \times E | df \geq e\} \tag{7.6}$$

– compare (5.3). A function $f : (D, +, \Omega) \to (E, +, \Omega)$ between $\underline{\underline{\tau}}$-modals is *concave* if the hypograph of f is a subalgebra of $(D \times E, \Omega)$. Although a composite of concave functions need not be concave, note that the composite of two monotone concave functions is again monotone concave.

The class of $\mathfrak{B}$-spaces becomes the object class of a category $\mathfrak{B}$ on taking a $\mathfrak{B}$-morphism $f : (G_1, ., 0, 1) \to (G_2, ., 0, 1)$ between two $\mathfrak{B}$-spaces with meet semilattice operation ., lower bound 0 and upper bound 1 to be a continuous semilattice homomorphism with $0f = 0$ and $1f = 1$. There is then a contravariant functor $Q : \mathfrak{B} \to \underline{\underline{\mathrm{Sl}}}$ with

$$f^Q : (G_2Q, +) \to (G_1Q, +); c \mapsto \inf f^{-1} \uparrow c \tag{7.7}$$

for a $\mathfrak{B}$-morphism $f : G_1 \to G_2$ [HMS, Th. II.3.7 and Prop. II.3.20] [RS10, (2.6)].

Definition 7.4. Let $\underline{\underline{V}}$ be a variety of modes of type $\tau : \Omega \to \mathbb{N}$ satisfying a set of regular linear identities. A concave function $f : (D, +, \Omega) \to (E, +, \Omega)$ between arithmetic $\underline{\underline{V}}$-modals is a $\mathfrak{B}\underline{\underline{V}}$*-morphism* if

(a) $f : (D, \leq_+) \to (E, \leq_+)$ is a $\mathfrak{B}$-morphism and
(b) $f^Q : (EQ, \Omega) \to (DQ, \Omega)$ is a $\underline{\underline{V}}$-morphism. $\square$

Proposition 7.5. *Let $\underline{\underline{V}}$ be a regular linear variety of $\underline{\underline{\tau}}$-modes. Then there is a category $\mathfrak{B}\underline{\underline{V}}$ whose morphisms are the $\mathfrak{B}\underline{\underline{V}}$-morphisms between arithmetic $\underline{\underline{V}}$-modals.*

Proof. Let $f : (D, +, \Omega) \to (E, +, \Omega)$ and $g : (E, +, \Omega) \to (F, +, \Omega)$ be $\mathfrak{B}\underline{\underline{V}}$-morphisms. By Definition 7.4(a), the concave functions f and g are monotone, so their composite is also concave. By (7.7), one also has $(fg)^Q = g^Q f^Q$ as the composite of the $\underline{\underline{V}}$-morphisms $g^Q : (FQ, \Omega) \to (EQ, \Omega)$ and $f^Q : (EQ, \Omega) \to (DQ, \Omega)$. $\square$

Theorem 7.6. *Let $\underline{\underline{V}}$ be a regular linear variety of modes. Then there is a contravariant functor*

$$T : \underline{\underline{V}} \to \mathfrak{B}\underline{\underline{V}}; (f : A \to B) \mapsto (BT \to AT; X \mapsto f^{-1}X) \tag{7.8}$$

left adjoint to the forgetful functor

$$Q : \mathfrak{B}\underline{\underline{V}} \to \underline{\underline{V}}; (f : (D, +, \Omega) \to (E, +, \Omega)) \mapsto (f^Q : (EQ, \Omega) \to (DQ, \Omega)). \tag{7.9}$$

For a $\underline{\underline{V}}$-mode A, the unit

$$\eta_A : A \to ATQ; a \mapsto \{a\} \tag{7.10}$$

embeds A as the algebra of singletons. For an arithmetic $\underline{\underline{V}}$-modal $(D,+,\Omega)$, the counit is

$$\varepsilon_D : D \to DQT; d \mapsto \{c \in DQ | c \leq d\}. \tag{7.11}$$

Proof. In (7.8), consider a basic operation ω and subalgebras $X_1, \ldots, X_{\omega\tau}$ of (B,Ω). Then $(f^{-1}X_1)\ldots(f^{-1}X_{\omega\tau})\omega = \{a_1 \ldots a_{\omega\tau}\omega | a_i f \in X_i\} \subseteq f^{-1}(X_1 \ldots X_{\omega\tau}\omega)$, verifying the concavity of f^{-1}. The image under $(f^{-1})^Q$ of a finitely generated subalgebra X of A is $\inf(f^{-1})^{-1} \uparrow X = \inf\{S \in BT | X \subseteq f^{-1}S\} = \inf\{S \in BT | Xf \subseteq S\} = Xf$. By (7.5) and (6.2), it then follows that $(f^{-1})^Q$ is a $\underline{\underline{V}}$-morphism. The remaining verifications are straightforward. $\square$

The functor of (7.8) is known as the *contravariant subalgebra functor*.

Example 7.7. To see that $fT : (BT,\Omega) \to (AT,\Omega)$ in (7.8) need not be a $\underline{\underline{V}}$-morphism, take $\underline{\underline{V}}$ to be $\underline{\underline{\mathrm{Sl}}}$ and take $f : A \to B$ to be the constant homomorphism $f : \{x,y\}\mathrm{Sl} \to \{x,y\}\mathrm{Sl}$ with image $\{x.y\}$. Then $f^{-1}\{x\}.f^{-1}\{y\} = \varnothing.\varnothing = \varnothing \subset \{x,y\}\mathrm{Sl} = f^{-1}\{x.y\}$. $\square$

8. A structure theorem.

The structure theory of modes and modals is very rich. Theorem 8.3 below is intended to give one sample, showing how a general mode may be constructed from semilattices and affine spaces. Throughout this section, $\tau : \Omega \to \mathbb{N}$ will be a *plural* type, i.e. with $\Omega\tau$ a non-empty subset of $\{n \in \mathbb{N} | n > 1\}$. A join semilattice $(H,+)$ becomes a τ-algebra on setting

$$h_1 \ldots h_{\omega\tau}\omega = h_1 + \cdots + h_{\omega\tau}. \tag{8.1}$$

The algebra (H,Ω) is called an Ω-*semilattice.* Plurality of τ ensures that $(H,+)$ may be recovered from (H,Ω). Given a (covariant) functor $G : (H) \to \underline{\underline{\tau}}$, the *Płonka sum HG over the semilattice $(H,+)$ by the functor G* is the τ-algebra structure on the disjoint union $HF = \bigcup\limits^{\cdot}_{h \in H} hG$ given by disjoint summands of basic operations

$$\omega : h_1G \times \cdots \times h_nG \to kG; (x_1, \ldots, x_n) \mapsto x_1(h_1 \to k)G \ldots x_n(h_n \to k)G \tag{8.2}$$

for $n = \omega\tau$ and $k = h_1 + \cdots + h_n$. For a variety $\underline{\underline{V}}$ of τ-algebras, the *regularization* $\widetilde{\underline{\underline{V}}}$ is the variety of τ-algebras satisfying all the regular identities satisfied by $\underline{\underline{V}}$-algebras. The Płonka sum of a functor $G : (H) \to \underline{\underline{V}}$ lies in $\widetilde{\underline{\underline{V}}}$.

A variety $\underline{\underline{V}}$ is a *Mal'cev variety* if it has a ternary derived operator P, the *Mal'cev operator* [cf. (4.9)], such that the parallelogram laws (4.10) are satisfied in $\underline{\underline{V}}$. Mal'cev varieties of modes are just affine spaces [RS2, Theorem 254]:

Theorem 8.1. *Let $\underline{\underline{V}}$ be a Mal'cev variety of modes. Then the free $\underline{\underline{V}}$-algebra $\{0,1\}V$ carries a commutative ring structure such that $\underline{\underline{V}}$ is the variety $\underline{\underline{\{0,1\}V}}$ of affine spaces over $\{0,1\}V$.*

Proof. The ring product in $\{0,1\}V$ is defined by the first equation of (4.8). Subtraction in $\{0,1\}V$ is defined by $x - y = xy0P$. For further details, see [RS2, §2.5]. $\square$

Now let $\underline{\underline{V}}$ be a variety of modes of type τ. The tensor product variety $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{V}}$ is a Mal'cev variety, and thus by Theorem 8.1 is the variety of affine spaces over a commutative ring RV. Let F be a minimal cogenerator for the category RV-$\underline{\underline{\mathrm{Mod}}}$ of modules over RV [AF, §18]. The corresponding affine space F lies in $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{V}}$, and thus has a $\underline{\underline{V}}$-reduct F. Let F^∞ be the Płonka sum over the 2-element semilattice $0 \to 1$ by the functor to $\underline{\underline{V}}$ that sends the unique non-identity arrow to the $\underline{\underline{V}}$-morphism $F \to \{\infty\}$. Note that F^∞ lies in the regularization $\widetilde{\underline{\underline{V}}}$.

Definition 8.2. A $\underline{\underline{V}}$-algebra A is *separable* if

$$\forall\ a \neq b \in A,\ \exists\ f \in \widetilde{\underline{\underline{V}}}(A, F^\infty).\ af \neq bf. \quad \square \tag{8.3}$$

Theorem 8.3 [RS9]. *A $\underline{\underline{V}}$-algebra A is separable iff it is a subreduct of a Płonka sum of affine spaces over RV.* $\square$

Example 8.4. For the variety $\underline{\underline{B}}$ of barycentric algebras, the ring RB is the field $\mathbb{R}$. Moreover, the vector space $\mathbb{R}$ is a minimal cogenerator for the variety of real vector spaces. Flood [Fl] showed that barycentric algebras are separable. Theorem 8.3 then exhibits each barycentric algebra as a subalgebra of the I°-reduct of a Płonka sum of affine spaces. In fact, one may characterize barycentric algebras as subalgebras of Płonka sums of convex sets [RS6, Th. 4.5] $\square$

Example 8.5. A *commutative binary mode* $(G, .)$ is a mode with a single commutative binary operation (*multiplication*). Recall that a *quasigroup* $(Q, ., /, \backslash)$ is a set equipped with a binary *multiplication* . , *right division*/ and *left division* $\backslash$ satisfying

$$(8.4) \qquad \begin{cases} (x.y)/y = x = y \backslash (y.x) \\ (x/y).y = x = y.(y \backslash x). \end{cases}$$

By [RS7, Th. 6.4], each commutative binary mode is a subalgebra of a Płonka sum of multiplicative reducts of commutative quasigroup modes. Now quasigroups have the Mal'cev operation $xyzP = [x/(y \backslash y)].(y \backslash z)$. As an application of Theorem 8.1, one obtains [RS2, 433] the variety of commutative quasigroup modes as the variety of affine spaces over $\mathbb{Z}[2^{-1}]$. Thus each commutative binary mode is a subalgebra of the $\{2^{-1}\}$-reduct of a Płonka sum of affine spaces over $\mathbb{Z}[2^{-1}]$. But if $\underline{\underline{V}}$ is the variety of commutative binary modes, one may readily verify $RV = \mathbb{Z}[2^{-1}]$. Theorem 8.3 then shows that each commutative binary mode is separable. Commutative binary modes are ideal candidates for representing barycentric algebras (such as convex sets) in digital computers. For a further discussion of this issue, see [RS2, §4.6]. $\square$

9. Affine to projective geometry.

The classical passage from affine to projective geometry [Br, §81], consisting of the addition of a "hyperplane at infinity", is not an intrinsic construction. Part of the problem lies with the schizophrenia that is inherent in classical geometry, namely the split between a geometry and an algebra coordinatizing it. ("Pursuing analytic geometry based on abstract spaces whose points are systems of numbers, we should pay attention to a clear distinction between the geometric properties under investigation and the arithmetic properties of these systems" [Br, p. 6].) The importance of modal theory in geometry resides in its ability to provide algebraic structures ("systems of numbers" with "arithmetic properties" in Borsuk's terms) that may be directly identified with geometric structures, removing the dualism between a geometric space and its coordinatizing algebra. For example, affine spaces may be identified with the modes of §4(D), or convex sets with the modes of §4(E). As an illustration of the application of modal theory to geometry, this section demonstrates how it may be used to give an intrinsic, invariant passage from real affine space to real projective space.

Given a real vector space E, the corresponding projective space is the join semilattice $(L(E), +)$ of non-empty vector subspaces of E. On the other hand, the corresponding affine space E may be identified as a mode $(E, \mathbb{R})$ in the variety $\underline{\underline{\mathbb{R}}}$ of algebras of type $\mathbb{R} \times \{2\}$ satisfying (A2) and (A4) of (4.11). The affine space $(E, \mathbb{R})$ has a convex set reduct (E, I°). In particular, (E, I°) is a barycentric algebra. Using (4.18) or (8.1), a semilattice such as $(L(E), +)$ may be viewed as a barycentric algebra satisfying the identities

$$xy\underline{p} = xy\underline{q} \tag{9.1}$$

for p, q in I°. Indeed, the class $\underline{\underline{\mathrm{Sl}}}$ of I°-semilattices is precisely the variety of barycentric algebras satisfying the identities (9.1). According to [RS3], the invariant passage from real affine to projective geometry may then be described by

Theorem 9.1. *The projective geometry* $(L(E), I^\circ)$ *is the semilattice replica of the image* (ES, I°) *under the covariant submode functor (6.7) of the convex set reduct* (E, I°) *of the affine geometry* $(E, \mathbb{R})$. $\square$

Extensions of Theorem 9.1 to more general underlying fields and rings are discussed in [RS3], [PRS].

10. Differential groupoids.

A *differential groupoid* $(G, .)$ is a mode with a single binary operation of *multiplication*, satisfying the reductive law (2.3). Differential groupoids offer a modal-theoretic approach to certain topics in calculus and homological algebra. There are two basic congruence relations on a differential groupoid G: the $\underline{\underline{\mathrm{Lz}}}$-replica congruence [cf. §4(B)] relation β of *cobordism*, and the *cocyclism* relation γ, namely the kernel of

$$G \to \underline{\underline{\mathrm{Set}}}(G, G); y \mapsto (x \mapsto xy). \tag{10.1}$$

Note that $\beta \leq \gamma$, i.e. cobordic elements are cocylic. Each element x of G determines a *homology set*

$$x^\gamma \text{ nat } \beta = \{y^\beta | (x,y) \in \gamma\}. \tag{10.2}$$

Let $\underline{\underline{d}}$ denote the variety of differential groupoids. The tensor product variety $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{d}}$ is the variety $\underline{\underline{\underline{\mathbb{Z}[d]}}}$ of affine spaces over the ring $\mathbb{Z}[X]/\langle X^2 \rangle = \mathbb{Z}[d]$ of *integral dual numbers*, i.e. the ring of integers extended by the indeterminate d with $d^2 = 0$. In Mac Lane's terminology [Ma, II.1], a (left) $\mathbb{Z}[d]$-module K is a *differential group*, i.e. an abelian group $(K, +)$ equipped with an endomorphism d satisfying $d^2 = 0$.

Elements of Ker d are called *cycles*, and elements of Im d are called *boundaries*. The *homology group* $H(K)$ of K is defined as the quotient Ker d/Im d. (In topological applications, K is graded by dimension. The homology group inherits the grading, becoming a direct sum $H(K) = \bigoplus_{n\in\mathbb{Z}} H_n(K)$ of "(co)homology groups" $\dots, H_{-1}(K), H_0(K), H_1(K), \dots$, etc.) Now the differential groupoid multiplication in $\underline{\underline{\underline{\mathbb{Z}[d]}}}$ is given by

$$x.y = x - dx + dy. \tag{10.2}$$

With this definition, a differential group K in Mac Lane's sense becomes a differential groupoid $(K, .)$. Elements of $(K, .)$ are cobordic iff they differ by a boundary in $(K, +)$, and cocyclic iff they differ by a cycle in $(K, +)$. The homology set of each element of $(K, .)$ may be identified with the homology group $H(K)$.

The ring $\mathbb{R}[d] = \mathbb{R} \otimes \mathbb{Z}[d]$ of *real dual numbers*, equipped with the multiplication (10.2), becomes a differential groupoid providing a convenient framework for certain aspects of real differential calculus. At each point a of $\mathbb{R}$, an everywhere-differentiable function $f : \mathbb{R} \to \mathbb{R}$ has the tangent line approximation $f_a : \mathbb{R} \to \mathbb{R}$ with

$$f_a(a + x) = f(a) + xf'(a). \tag{10.3}$$

These tangent line approximations may be used to extend the function $f : \mathbb{R} \to \mathbb{R}$ to a function $f : \mathbb{R}[d] \to \mathbb{R}[d]$ by means of the formula

$$f(a + dx) = f(a) + f'(a)dx \tag{10.4}$$

for real x. Now the endomorphism d of the differential group $\mathbb{R}[d]$ yields an exact sequence

$$0 \to d\mathbb{R} \to \mathbb{R}[d] \xrightarrow{d} \mathbb{R}[d] \xrightarrow{\pi} \mathbb{R} \to 0. \tag{10.5}$$

The cokernel $\pi : \mathbb{R}[d] \to \mathbb{R}; a + dx \mapsto a$ is described as *taking the finite part* of a dual number $a + dx$. Cobordic elements are described as being *infinitesimally close*. In general, the extended functions (10.4) are not homomorphisms of the differential groupoid structure on $\mathbb{R}[d]$, although affine functions such as the extension of (10.3) to $\mathbb{R}[d]$ are. The following theorem expresses the relationship of a differentiable function to its tangent line approximations by saying that these approximations "repair the failure of the function to be a differential groupoid homomorphism".

Theorem 10.1. *With notation as above, one has*

$$f(x.y) = f(x).f_{x\pi}(y) \tag{10.6}$$

for dual numbers x and y. Moreover, each solution $g : \mathbb{R}[d] \to \mathbb{R}[d]$ to the functional equation

$$\forall\ y \in \mathbb{R}[d],\ f(a.y) = f(a).g(y) \tag{10.7}$$

is infinitesimally close to the tangent line approximation f_a. $\square$

Inspired by the example of the real dual numbers, one may introduce concepts of differentiability and continuity for an arbitrary differential groupoid $(G, .)$.

Definition 10.2. Consider a function $f : G \to G$, and an element x of G.
(a) The function f is *differentiable at* x if there is an endomorphism f_x of $(G, .)$, called a *derivative* of f at x, such that

$$\forall\ y \in G,\ (x.y)f = xf.yf_x. \tag{10.8}$$

(b) The function f is *continuous at* x if

$$x\ \beta\ y \Rightarrow xf\ \beta\ yf. \tag{10.9}$$

(c) The function f is *differentiable* or *continuous* (*everywhere*) if it is differentiable or continuous at each element x of G. $\square$

With Definition 10.2(a), one obtains a Chain Rule: if f is differentiable at x and g is differentiable at xf, then fg is differentiable at x and

$$(fg)_x = f_x g_{xf}. \tag{10.10}$$

Moreover, differentiability implies continuity in the following sense.

Theorem 10.3. *If $f : G \to G$ is differentiable at each member of the cobordism class of an element x of G, then f is continuous at x.* $\square$

For further details of the topics of this section, see [RS8].

11. Support functions of bounded convex sets.

In this section, the space $\mathbb{R}^d$ is considered as Euclidean space with the Euclidean inner product $(x|y)$. For a non-empty convex subset A of $\mathbb{R}^d$, the set AK of non-empty compact convex subsets of A forms a submodal $(AK, +, I^\circ)$ of the image $(AS, +, I^\circ)$ of the barycentric algebra (A, I°) under the covariant submode functor $S : \underline{\underline{B}} \to \underline{\underline{E}}$ of (6.7). Let $C(\mathbb{R}^d)$ denote the subset of $\underline{\underline{\mathrm{Set}}}\,(\mathbb{R}^d, \mathbb{R})$ consisting of continuous functions. Then $C(\mathbb{R}^d)$ is a submodal of the modal $(\underline{\underline{\mathrm{Set}}}(\mathbb{R}^d, \mathbb{R}), +, I^\circ)$ induced pointwise from the modal $(\mathbb{R}, \max, I^\circ)$ or $(\mathbb{R}, +, I^\circ)$ of Example 5.2. For a non-empty convex subset A of $\mathbb{R}^d$, the modal $(AK, +, I^\circ)$ embeds homomorphically into $(C(\mathbb{R}^d), +, I^\circ)$ via the *support function*

$$H : AK \times \mathbb{R}^d \to \mathbb{R}; (x, y) \mapsto \sup\{(x|y) | x \in X\}. \tag{11.1}$$

Indeed, an element X of AK is specified as

$$X = \{x \in A | \forall\ y \in \mathbb{R}^d,\ x \le H(X, y)\} \tag{11.2}$$

[Bo, p.24]. Furthermore, for fixed X in AK, the *support function of X*

$$H_X : (\mathbb{R}^d, I^\circ) \to (\mathbb{R}, +, I^\circ); y \mapsto H(X, y) \tag{11.3}$$

is convex [Bo, p.24], and so continuous [Bo, p. 19]. Note that a convex function $f : (\mathbb{R}^d, I^\circ) \to (\mathbb{R}, +, I^\circ)$ such as (11.3) need not be differentiable, although for any x, y in $\mathbb{R}^d$, the limit

$$f'(x; y) = \lim_{h \to 0+} \frac{f(x + yh) - f(x)}{h} \tag{11.4}$$

does exist [Bo, p.19]. Moreover, the function $(\mathbb{R}^d, I^\circ) \to (\mathbb{R}, +, I^\circ); y \mapsto f'(x; y)$ is again convex for each x in $\mathbb{R}^d$. The embedding of $\mathbb{R}^d K$ into $C(\mathbb{R}^d)$ is then given by

$$(\mathbb{R}^d K, +, I^\circ) \to (C(\mathbb{R}^d), +, I^\circ); X \mapsto H_X. \tag{11.5}$$

An element f of $C(\mathbb{R}^d)$ is *positively homogeneous* if

$$\forall\ p \in (0, \infty), \forall\ x \in \mathbb{R}^d,\ f(px) = pf(x). \tag{11.6}$$

Note that if $f : \mathbb{R}^d \to \mathbb{R}$ is convex, then $y \mapsto f'(x; y)$ is positively homogeneous [Bo, §13(3)]. The support functions of compact convex subsets of $\mathbb{R}^d$ are then characterized as the functions that are positively homogeneous and convex [Bo, §17].

One of the research programmes in applied modal theory is to extend the support function concept from compact to arbitrary convex subsets of Euclidean spaces. The topic of this section is the extension to bounded (i.e. not necessarily closed) convex sets [RS4]. Let $(\mathbb{R}^d B, +, I^\circ)$ denote the submodal of $(\mathbb{R}^d S, +, I^\circ)$ consisting of bounded convex sets. The codomain of support functions (11.3) of compact convex sets is the modal

$$D = (\mathbb{R}, +, I^\circ). \tag{11.7}$$

By induction on the dimension d, a codomain modal D_d is constructed such that the support function of a bounded convex set is a function $\mathbb{R}^d \to D_d$. The construction uses the *ordinal product* $D \circ E$ of two $\underline{\underline{\tau}}$-modals $(D, +, \Omega)$ and $(E, +, \Omega)$. The mode reduct of $D \circ E$ is just the product $(D, \Omega) \times (E, \Omega)$. A partial order $\leq$ is defined lexicographically on $D \times E$ by

$$(d, e) \leq (d', e') \text{ iff } d <_+ d' \text{ or } (d = d' \text{ and } e \leq_+ e'). \tag{11.8}$$

Now $(D \times E, \leq_+)$ is a join semilattice order iff $(D,+)$ is a chain or $(E,+)$ has a least element [Sl]. A *zero element* of a modal $(E,+,\Omega)$ is defined to be a least element 0 of $(E,+)$ such that $\{0\}$ is a sink of (E,Ω). A τ-algebra (D,Ω) is said to be *cancellative* if

$$(11.9) \qquad \begin{cases} \forall\ \omega \in \Omega,\ \forall\ 1 \leq i \leq \omega\tau,\ \forall\ x_1,\dots,x_{\omega\tau},y,z \in D, \\ x_1 \dots x_{i-1} y x_{i+1} \dots x_{\omega\tau}\omega = x_1 \dots x_{i-1} a x_{i+1} \dots x_{\omega\tau}\omega \Rightarrow y = z. \end{cases}$$

Theorem 11.1 [RS4, 2.5]. *For a plural type $\tau : \Omega \to \mathbb{N}$, suppose that*

(a) the mode (D,Ω) is cancellative, and

(b) either $(D,+)$ is a chain, or $(E,+,\Omega)$ has a zero element.

Then the ordinal product $D \circ E$ is a $\underline{\tau}$-modal $(D \circ E,+,\Omega)$. $\square$

Along with the series of codomain modals D_d for $d \geq 1$, there are three other series of inductively defined modals $E_d (d \geq 0), F_d (d \geq 1)$, and $G_d (d \geq 0)$. The induction basis is the singleton G_0. The induction step begins with a modal G_{d-1} of support functions identified with $(\mathbb{R}^{d-1}B,+,I^\circ)$. Note that $\mathbb{R}^\circ B$ is the singleton $\{\{0\}\}$. The modal E_{d-1} is obtained from G_{d-1} by adding a zero element, identified with the empty subset of $\mathbb{R}^{d-1}$. The pair of modes (D, E_{d-1}) then satisfies the conditions of Theorem 11.1. The next codomain modal D_d is defined to be the ordinal product $D_d = D \circ E_{d-1}$. The modal F_d is the set of functions $\mathbb{R}^d \to D_d$ with the pointwise modal structure. The induction step is completed by giving five conditions on functions in F_d, the "G-conditions" of Definition 11.2 below, that define its submodal G_d of support functions identified with $(\mathbb{R}^d B,, I^\circ)$. The five G-conditions are the analogue of the two conditions, namely positive homogeneity and convexity, that characterize the support functions of compact convex sets.

Since the underlying set of D_d is $\mathbb{R} \times E_{d-1}$, an element f of F_d may be written as

$$(11.10) \qquad f : \mathbb{R}^d \to D_d; x \mapsto (H_f(x), C_f(x)).$$

Here $C_f(x)$, by the identification of E_{d-1} with $\mathbb{R}^{d-1}B \cup \{\varnothing\}$, is a (possibly empty) bounded convex subset of $\mathbb{R}^{d-1}$ called the *crust shadow* (*in the x direction*). Also $H_f : x \mapsto H_f(x)$ is a real-valued function on $\mathbb{R}^d$, called the *real function part* of f. The first two G-conditions on f are just the positive homogeneity and convexity of H_f. If f is taken as the support function of a non-empty bounded convex subset X of $\mathbb{R}^d$, then H_f is just the traditional support function of the non-empty compact convex closure $\overline{X}$ of X in $\mathbb{R}^d$. To complete the description of X, its intersection with the supporting hyperplane of $\overline{X}$ in the direction of each non-zero vector x of $\mathbb{R}^d$ must be given. This intersection is called the *crust* of X (*in the x direction*). The crust is a (possibly empty) bounded convex subset of the supporting hyperplane. It is described by f as the preimage of the crust shadow $C_f(x)$ in $\mathbb{R}^{d-1}$ under a certain affine isomorphism π_x from the supporting hyperplane to $\mathbb{R}^{d-1}$. Thus, before the remaining conditions on f can be given, the maps π_x must be specified. Their specification depends on the satisfaction of the first two G-conditions by f, so that H_f really is the support function of a non-empty compact convex subset of $\mathbb{R}^d$.

For each 1-dimensional vector subspace V of $\mathbb{R}^d$, pick a linear isomorphism $\theta_V : \mathbb{R}^d/V \to \mathbb{R}^{d-1}$. For each non-zero point x of $\mathbb{R}^d$, define

$$\pi_x : \{z \in \mathbb{R}^d | (z|x) = H_f(x)\} \to \mathbb{R}^{d-1} \tag{11.11}$$

to be the composite of the restriction of the projection $\mathbb{R}^d \to \mathbb{R}^d/x\mathbb{R}; z \mapsto z + x\mathbb{R}$ with the isomorphism $\theta_{x\mathbb{R}}$. Note that such π_x is an affine isomorphism. By convention, π_0 is defined to be the zero map $\mathbb{R}^d \to \mathbb{R}^{d-1}$. For a function f in F_d with positively homogeneous convex real function part H_f, and for x in $\mathbb{R}^d$, define the *supercrust* (*at x*)

$$K_f(x) = \{z \in \mathbb{R}^d | \forall\ y \in \mathbb{R}^d, (z|y) \leq H'_f(x; y)\}. \tag{11.12}$$

Note that $(K_f(x), I^\circ)$ is a subalgebra of $(\mathbb{R}^d, I^\circ)$. Let $K_f(x)^\circ$ be its smallest non-empty sink, i.e. the interior of the compact convex set $K_f(x)$.

Definition 11.2. The subset G_d of F_d, for $d > 0$, is defined to be the set of functions (11.10) satisfying the *G-conditions*:

(GC1) H_f is positively homogeneous;

(GC2) H_f is convex;

(GC3) $\forall\ x \in \mathbb{R}^d,\ C_f(x) \subseteq \pi_x(K_f(x))$;

(GC4) $\forall\ x \in \mathbb{R}^d,\ H_f(-x) = -H_f(x) \Rightarrow C_f(x) \subseteq \pi_x(K_f(x)^\circ)$;

(GC5) $\forall\, 0 \neq x, y \in \mathbb{R}^d$, $\pi_x^{-1}(C_f(x)) \cap \{z \in \mathbb{R}^d | (z|y) = H_f(y)\}$
$= \pi_y^{-1}(C_f(y)) \cap \{z \in \mathbb{R}^d | (z|x) = H_f(x)\}$. $\square$

Definition 11.3. Let X be a non-empty, bounded convex subset of $\mathbb{R}^d$, with closure $\overline{X}$. Let H_X be the support function of $\overline{X}$. For x in $\mathbb{R}^d$, let $C_X(x)$ be the image under π_x of the crust $X \cap \{z \in \mathbb{R}^d | (z|x) = H_X(x)\}$ of X in the x direction. Then the (*extended*) *support function* of X is the function

$$f_X : \mathbb{R}^d \to D_d; x \mapsto (H_X(x), C_X(x)). \quad \square \tag{11.13}$$

Theorem 11.4 [RS4]. *The map*

$$\varphi : (\mathbb{R}^d B, +, I^\circ) \to F_d; X \mapsto f_X \tag{11.14}$$

is a modal embedding with image G_d. $\square$

12. Multiplication tables of central quasigroups.

The definition (8.4) of a quasigroup implies that the multiplication table of a finite quasigroup is a bordered Latin square. The left hand border identifies the left hand factor in a product, while the top border identifies the right hand factor. Since these borders are not intrinsic to the Latin square, the same Latin square may arise from non-isomorphic quasigroups. A *homotopy*

$$(f_1, f_2, f_3) : (A, .) \to (B, *) \tag{12.1}$$

from a quasigroup A to a quasigroup B is a triple of functions $f_i : A \to B$ satisfying

$$\forall\, a_1, a_2 \in A,\ a_1 f_1 * a_2 f_2 = (a_1 \cdot a_2) f_3. \tag{12.2}$$

Homotopies compose componentwise. The category of homotopies between quasigroups is denoted by $\underline{\underline{\underline{\mathrm{Qtp}}}}$. An invertible morphism in the category $\underline{\underline{\underline{\mathrm{Qtp}}}}$ is called an *isotopy.* Then isomorphic Latin squares correspond to isotopic quasigroups.

Since algebra deals predominantly with homomorphisms, it has been hard to give an algebraic treatment of quasigroup homotopies. In fact, quasigroup homotopies have rather been considered as part of the geometric theory of nets [Bu]. However, recent work [S4] has proposed one purely algebraic approach to homotopies. A quasigroup is *semisymmetric* if it satisfies the identity

(12.3) $$(xy)x = y.$$

Let $\underline{\underline{P}}$ be the category of (homomorphisms between) semisymmetric quasigroups. There is a forgetful functor $\Sigma : \underline{\underline{P}} \to \underline{\underline{\mathrm{Qtp}}}$ sending a $\underline{\underline{P}}$-morphism $f : A \to B$ to the homotopy $(f, f, f) : A \to B$ with equal components. In the other direction, there is a *semisymmetrization functor* $\Delta : \underline{\underline{\mathrm{Qtp}}} \to \underline{\underline{P}}$. The object part of Δ sends a quasigroup Q to the set Q^3 equipped with the product

(12.4) $$(x_1, x_2, x_3)(y_1, y_2, y_3) = (y_3/x_2, y_1 \setminus x_3, x_1y_2).$$

The morphism part of Δ sends a quasigroup homotopy (12.1) to the homomorphism

(12.5) $$(a_1, a_2, a_3) \mapsto (a_1f_1, a_2f_2, a_3f_3).$$

Then Δ is right adjoint to Σ [S4, Th. 5.2].

An important class of quasigroups is the class of *central* quasigroups. A quasigroup Q is central if the diagonal $\widehat{Q} = \{(x, x)|x \in Q\}$ is a normal subquasigroup of Q^2, i.e. if there is a congruence on Q^2 with $\widehat{Q}$ as a congruence class. Entropic quasigroups are central. Each central quasigroup is isotopic (in fact "centrally isotopic" [S2, Defn. III.4.1]) to a module A over the free group on $\{R, L\}$ equipped with the multiplication

(12.6) $$x.y = xR + yL.$$

If the actions of R and L on A do not commute, then the quasigroup $(A, .)$ will not be entropic. However, semisymmetrizations of central quasigroups are entropic [S4, Th. 6.3].

The multiplication table of a quasigroup Q may be considered as the subset

(12.7) $$\{(x, y, z) \in Q|x.y = z\}$$

of the semisymmetrization $Q\Delta$. The elements of (12.7) are always idempotent in $Q\Delta$, but do not necessarily form a subquasigroup of $Q\Delta$. However, they do if Q is central.

Theorem 12.1. *The multiplication table (12.7) of a central quasigroup Q forms a semisymmetric quasigroup mode under the multiplication (12.4).*

Proof. The product of elements (x, y, xy) and (z, t, zt) of (12.7) is $(zt/y, z \setminus xy, xt)$. (Multiplication binds more strongly than the divisions.) By [S4, Prop. 6.2(c)], one then has $(zt/y)(z \setminus xy) = (xy/y)(z \setminus zt) = xt$. Thus the product again lies in (12.7). □

Corollary 12.2. *The multiplication table of a central quasigroup is an affine space over the ring $\mathbb{Z}[\omega]$, where $\omega = \exp(2\pi i/3)$.*

Proof. Letting $\underline{\underline{V}}$ be the variety of $\underline{\underline{P}}$-modes, one may apply Theorem 8.1. Then $RV = \mathbb{Z}[\omega]$. The quasigroup multiplication becomes $\underline{-\omega}$, while the divisions become $\underline{1+\omega}$. □

For Corollary 12.2, compare [RS2, 436].

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