MODAL THEORY, PARTIAL ORDERS, AND DIGITAL GEOMETRY

Jonathan D. H. Smith
Department of Mathematics
Iowa State University
Ames, Iowa 50011, U.S.A.

1. INTRODUCTION A long-standing problem in domain theory has been the search for algebra structures that ride naturally on the ordered sets involved. Indeed, the constructions in the theory of complete partial orders and continuous lattices, as applied to the recursive definition of a data structure, are usually independent of any algebra carried by the data structure, and do not mesh nicely with the algebra. The aim of the current paper is to provide an introduction to the kind of algebra that does guarantee a good mesh with partial orderings, and to present topological ideas and category-theoretical relationships showing how the algebra is automatically reproduced under order-theoretical constructions such as power domains. As a potential application of the algebra, a new direct approach to the programming of geometry and scientific models is proposed.

Giving a brief survey of an extensive theory, this paper is necessarily somewhat condensed. Fuller details of many of the topics treated here, as well as an introduction to the universal algebraic notations used, may be found in [6]. Note, too, that the whole theory as described here has so far only been worked out on the basis of finitary universal algebraic methods. There is great potential for future development of the theory using the monadic approach to algebra. Another aspect of the theory that has hardly been investigated at all is that of duality. Here, too, a great deal remains to be done.

2. MODES The basic algebraic concept is that of a mode. A _mode_ is an algebra (A,Ω) satisfying the following conditions:

(2.1) The algebra is _idempotent_, i.e. each singleton subset $\{a\}$ of A is a subalgebra $(\{a\},\Omega)$ of (A,Ω); and

(2.2) the algebra is <u>entropic</u>, i.e. each operation ω in Ω (of arity $\omega\tau$), already a set mapping $\omega : A^{\omega\tau} \to A$; $(x_1,\dots,x_{\omega\tau}) \mapsto x_1 \dots x_{\omega\tau}\omega$, is also a homomorphism $\omega : (A^{\omega\tau},\Omega) \to (A,\Omega)$.

Some examples will serve to demonstrate the scope of this apparently restrictive definition, showing how various familiar mathematical concepts are brought into the purview of modal theory.

EXAMPLE 2.3. If Ω is empty, the conditions (2.1) and (2.2) are vacuously satisfied. Thus unstructured sets A are modes.

EXAMPLE 2.4. A semilattice $(L,\cdot)$ is a mode. Idempotence reduces to $x\cdot x = x$ for x in L, while the entropic law $(x\cdot y)\cdot(z\cdot t) = (x\cdot z)\cdot(y\cdot t)$ for the single infix binary operation $\cdot$ follows from the commutative and associative laws. The semilattice $(L,\cdot)$ has a partial order specified by

(2.5) $x \leq y$ if and only if $x\cdot y = x$.

With this order, $(L,\cdot)$ is called a <u>meet semilattice</u>. If a partial order $\leq$ is given on a set L, and this partial order is known to come from a meet semilattice structure on L (i.e. each pair of elements of L has a greatest lower bound), then (2.5) may be read backwards to specify the semilattice operation $\cdot$ on L. Sometimes the dual notion of <u>join semilattice</u> $(L,+)$ is used: $x \leq y$ if and only if $x + y = y$. Meet and join semilattices are the means by which order is dealt with in modal theory.

EXAMPLE 2.6. Let E be a vector space over a field R, or more generally a unital module over a commutative ring R with 1. For each r in R, define a binary operation $\underline{r}$ by

(2.7) $\underline{r} : E \times E \to E;\ (x,y) \mapsto x(1-r) + yr.$

Interpreting R as the set of these binary operations $\underline{r}$, the algebra (E,R) becomes a mode. Idempotence follows since $xx\underline{r} = x(1-r) + xr = x1 = x$, and a straightforward calculation establishes the entropic laws $xy\underline{r}zt\underline{r}\underline{s} = xz\underline{s}yt\underline{s}\underline{r}$. These algebras (E,R) serve to relate linear algebra to modal theory.

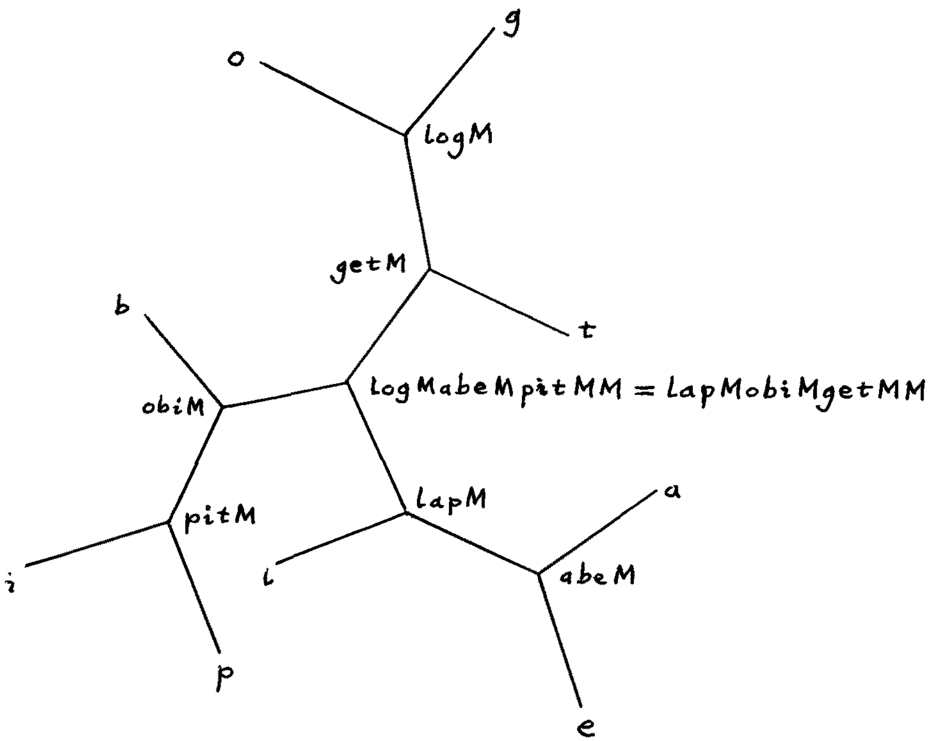
EXAMPLE 2.8. Let I° denote the open unit interval in the set $\mathbb{R}$ of real numbers. Given a real vector space E thought of as a mode $(E,\mathbb{R})$ according to Example 2.6, the reduct (E,I°), admitting just those operations $\underline{r}$ for which $0 < r < 1$, is also a mode. Subalgebras (S,I°) of this mode are then just convex subsets of E. In this way convexity becomes part of modal theory. An important application of these examples is to non-determinism. A convex subset S of E may represent (possibly) non-deterministic states of a machine. An extreme point x of E represents a deterministic state. Given two states x, y in S, and $0 < p < 1$, the state $xy\underline{p}$ represents a non-deterministic state obtained by choosing x with probability $1 - p$ and y with probability p.

EXAMPLE 2.9. Let T be the vertex set of a tree. Given three vertices x, y, z of T, let xyzM denote the unique vertex that lies on each of the paths from x to y, from y to z, and from z to x. Note that if any two of x, y, z coincide to equal t, then xyzM = t. In particular xxxM = x, so the algebra (T,M) with the ternary operation M is idempotent. The entropic law

(2.10) logMabeMpitMM = lapMobiMgetMM

for (T,M) may also be verified (by a tedious case analysis). It is illustrated on the tree of Figure 1.

FIGURE 1



Given a tree, one thus obtains a mode (T,M), known as a tree algebra [3]. Conversely, conditions on a ternary algebra (T,M) ensuring that it comes from a tree T in this way have been given [3, 1.2-3]. Thus trees may be regarded as modes.

3. POWER DOMAINS Given a mode (A,Ω), let AS denote the set of non-empty subalgebras of (A,Ω). For each ω in Ω (of arity $\omega\tau$), an operation known as complex product or complex ω-product may be defined on AS by

(3.1) $X_1 \ldots X_{\omega\tau}\omega = \{x_1 \ldots x_{\omega\tau}\omega \mid x_i \in X_i\}$.

That $X_1 \ldots X_{\omega\tau}\omega$ is a subalgebra of (A,Ω) is a typical consequence of the entropic law. One then obtains the fundamental self-reproducing property of modes under this power domain construction that shows how well these concepts fit together.

PROPOSITION 3.2 [6, 146]. If (A,Ω) is a mode, then under the complex product operations of (3.1), (AS,Ω) is again a mode.

Let AP denote the set of polytopes or finitely generated non-empty subalgebras of (A,Ω). (If (A,Ω) is (E,I°) as in Example 2.8, then EP is precisely the set of polytopes of E in the geometric sense.) If non-empty subalgebras $S_1, \ldots, S_{\omega\tau}$ of (A,Ω) are generated by sets $X_1, \ldots, X_{\omega\tau}$ respectively, then an inductive proof [6, 147] shows that the complex product $S_1 \ldots S_{\omega\tau}\omega$ is generated by $X_1 \ldots X_{\omega\tau}\omega$. As a consequence of this, the set AP under the complex product operations forms a submode (AP,Ω) of (AS,Ω). Further, there is a homomorphic canonical embedding

(3.3) $\iota : (A,\Omega) \to (AP,\Omega);\ a \mapsto \{a\}$

of (A,Ω) into (AP,Ω), or indeed of (A,Ω) into (AS,Ω).

The sets AP and AS also carry additional structure, namely ordering by set inclusion. This order determines join semilattice structures $(AP,+)$ and $(AS,+)$, as discussed in Example 2.4. There is a special relationship between the Ω-algebra structures and the join semilattice structures on AP and AS, expressed by the following definition and proposition.

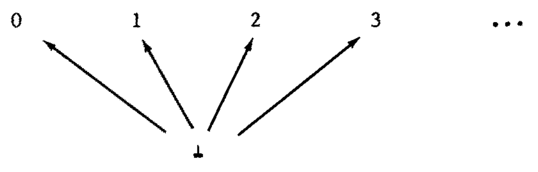
DEFINITION 3.4. An Ω-algebra structure (D,Ω) on a set D is said to distribute over a semilattice structure $(D,+)$ on D if, for each ω (of arity $\omega\tau$), and for each $1 \leq j \leq \omega\tau$ and $x_1, \ldots, x_j, \ldots, x_{\omega\tau}, x_j'$ in D,

$$x_1 \ldots (x_j+x_j') \ldots x_{\omega\tau}\omega = x_1 \ldots x_j \ldots x_{\omega\tau}\omega + x_1 \ldots x_j' \ldots x_{\omega\tau}\omega.$$

PROPOSITION 3.5. [6, 313]. For a mode (A,Ω), the complex product structure (AS,Ω) distributes over the semilattice $(AS,+)$.

This distributivity is surprising, as a priori there is no reason to expect any nice interaction between the two structures on AS and AP. It is further evidence of the good behaviour of mode algebra with respect to order. Some consequences of the distributivity are worked out in the next section. This section concludes with a simple example illustrating the power domain construction AP.

EXAMPLE 3.6. Let $(A,\cdot)$ denote the meet semilattice structure on $A = \{\perp\} \dot{\cup} \mathbf{N}$ (natural numbers with bottom element) in which $\mathbf{N}$ forms an antichain and $\perp$ is below each element of $\mathbf{N}$. The Hasse diagram of $(A,<)$, i.e. the directed graph of the relation "is covered by", is as follows:



The power domain AP consists of all finite non-empty subsemilattices of $(A,\cdot)$. Aside from the singletons $\{n\}$ for n in $\mathbf{N}$, this comprises the unions of $\{\perp\}$ and finite subsets of $\mathbf{N}$. Now it follows from the stronger version of Proposition 3.2 given in [6, 146] (and requiring universal algebraic concepts for its formulation) that $(AP,\cdot)$ forms a (meet) semilattice under the complex product. Thus the power domain AP carries two semilattice structures, the meet semilattice $(AP,\cdot)$ and the join semilattice $(AP,+)$. In each of these, there is a copy of the lattice of finite subsets of $\mathbf{N}$, a finite subset F of $\mathbf{N}$ being represented by $F\dot{\cup}\{\perp\}$. In $(AP,\cdot)$, the singletons $\{n\}$ for n in $\mathbf{N}$ appear as maximal elements covering the corresponding $\{n,\perp\}$. In $(AP,+)$, these singletons $\{n\}$ appear as minimal elements covered by the corresponding $\{n,\perp\}$. Such algebras are investigated in detail in [5] and [6].

4. MODALS Let (A,Ω) be a mode, and let $(D,+,\Omega)$ denote either $(AP,+,\Omega)$ or $(AS,+,\Omega)$. Then the algebra $(D,+,\Omega)$ satisfies the following three conditions:

(4.1) (D,Ω) is a mode;

(4.2) $(D,+)$ is a join semilattice; and

(4.3) (D,Ω) distributes over $(D,+)$.

An algebra $(D,+,\Omega)$ satisfying these conditions (4.1) - (4.3) is said to be a modal. The name emphasizes the connection with modes, as well as being reminiscent of "modules", which are also algebras having a +-structure and operations distributing over it. Modal theory may be described as the study of modes, modals, and the relationships between them. Examples of modals beyond the $(AP,+,\Omega)$ and $(AS,+,\Omega)$ coming from a mode (A,Ω) are given below.

EXAMPLE 4.4. If Ω is empty, then a modal $(D,+,\Omega)$ is just a join semilattice. In particular, if A is a set considered as a mode $(A,\emptyset)$, then AP and AS are power sets of A considered as join semilattices (and without the empty set as bottom element).

EXAMPLE 4.5. Distributive lattices $(L,+,\cdot)$, for example the lattice reducts of Boolean algebras, are modals.

EXAMPLE 4.6. Let $(L,\cdot)$ be a semilattice. Then the stammered semilattice $(L,\cdot,\cdot)$, with the operation $\cdot$ taken twice, is a modal. Note that this method of considering semilattices as modals differs from the method of Example 4.4.

EXAMPLE 4.7 There is a common generalization of the distributive lattices of Example 4.5 and the stammered semilattices of Example 4.6. This is the concept of dissemilattice -- an algebra $(D,+,\cdot)$ in which $(D,+)$ and $(D,\cdot)$ are semilattices, and in which $(D,\cdot)$ distributes over $(D,+)$, i.e. the law $x\cdot(y+z) = x\cdot y + x\cdot z$ is satisfied. The power domain $(AP,+,\cdot)$ of Example 3.6 has such a structure. If lattices are regarded as generalizations of distributive lattices obtained by dropping the requirement of distributivity, then dissemilattices may be considered as parallel generalizations in which distributivity is retained but the absorption law is relaxed.

EXAMPLE 4.8. Let $\vee$ denote the binary operation of maximum on the set $\mathbf{R}$ of real numbers. Then $(\mathbf{R},\vee,I^{\circ})$, under this operation and the set I° of convex combinations as in Example 2.8, forms a modal.

EXAMPLE 4.9. Any space of functions mapping into the reals inherits the modal structure on the reals given in Example 4.8. For instance, if X is a topological space, the set $C(X)$ of continuous functions $f : X \to \mathbf{R}$ forms a modal $(C(X),\vee,I^{\circ})$.

The distributivity of (D,Ω) over $(D,+)$ in a modal $(D,+,\Omega)$ has a number of direct consequences for the way the mode structure (D,Ω) interacts with the join semilattice ordering $\leq$ on D. These are listed in the following lemmas. Note that a function $f : (A,\Omega) \to (D,+,\Omega)$ is said to be convex if $a_1 \dots a_{\omega\tau}\omega f \leq a_1 f \dots a_{\omega\tau} f\omega$ for each operation ω in Ω and $\omega\tau$-tuple $(a_1,\dots,A_{\omega\tau})$ in $A^{\omega\tau}$. A function $f = (\mathbf{R},I^{\circ}) \to (\mathbf{R},\vee,I^{\circ})$ is convex in this sense if and only if it is convex in the classical sense as a function $f : \mathbf{R} \to \mathbf{R}$.

MONOTONICITY LEMMA 4.10 [6, 315]. Each operation ω on a modal $(D,+,\Omega)$ is monotone as a mapping $\omega : (D^{\omega\tau},\leq) \to (D,\leq)$, i.e. $a_1 \leq b_1, \dots, a_{\omega\tau} \leq b_{\omega\tau}$ imply $a_1 \dots a_{\omega\tau}\omega \leq b_1 \dots b_{\omega\tau}\omega$.

CONVEXITY LEMMA 4.11 [6, 317]. For each positive integer r, the mapping

$$\sum_r : (D^r,\Omega) \to (D,+,\Omega) \; ; \; (a_1,\dots,a_r) \to a_1 + \dots + a_r$$

into a modal $(D,+,\Omega)$ is convex.

SUM-SUPERIORITY LEMMA 4.12 [6, 318]. For each operation ω on a modal $(D,+,\Omega)$, and for each $\omega\tau$-tuple $(a_1,\ldots,a_{\omega\tau})$,

$$a_1 \ldots a_{\omega\tau}\omega < a_1 + \ldots + a_{\omega\tau}.$$

In Section 3, the construction of the modal $(AP,+,\Omega)$ from a mode (A,Ω) was given. From the category theory standpoint, such constructions are indicative of a left adjoint functor. The adjointness present here, expressed by the theorem below, lies at the very heart of modal theory. The theorem may be interpreted as showing how functions are extended from the domain A to the power domain AP (cf. [2],[4],[5,Th. 3.1]).

THEOREM 4.12 [6, 351]. The construction of the modal $(AP,+,\Omega)$ from a mode (A,Ω) is left adjoint to the forgetful functor assigning the mode (D,Ω) to a modal $(D,+,\Omega)$. In other words, each mode homomorphism $f : (A,\Omega) \to (D,\Omega)$ may be extended to a unique modal homomorphism $\overline{f} : (AP,+,\Omega) \to (D,+,\Omega)$ whose composite $\iota\overline{f}$ with the canonical embedding ι of (3.3) is f.

For a subalgebra S of (A,Ω) finitely generated by a set X, the element $S\overline{f}$ of D is defined to be $\sum_{x \in X} xf$. That this is a good definition may be shown using the Sum-Superiority Lemma 4.12. Other details of the proof are given in [6].

5. APPROXIMATION Theorem 4.12 gives the abstract significance of the modal $(AP,+,\Omega)$ as the free modal over the mode (A,Ω). Questions then arise as to the abstract significance of the modal $(AS,+,\Omega)$, the abstract relationship between $(AP,+,\Omega)$ and $(AS,+,\Omega)$, and the possibility of extending functions from A through the power domain AP to the power domain AS. These questions are addressed by the concepts and results of this section, which build up to the equivalance of categories given in Theorem 5.8 below.

A join semilattice $(D,+)$ is said to be <u>complete</u> if arbitrary (non-empty) subsets of D have suprema. An Ω-algebra structure (D,Ω) on a set D is said to be completely distributive over a complete joint semilattice structure $(D,+)$ on D if for each ω in Ω, $1 \leq j \leq \omega\tau$, and subset

$\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{\omega\tau}\} \cup X$ of D $(X \neq \emptyset)$,

$$\sup \{x_1 \ldots x_{j-1} x x_{j+1} \ldots x_{\omega\tau}\omega \mid x \in X\}$$

$$= x_1 \ldots x_{j-1}(\sup X)\, x_{j+1} \ldots x_{\omega\tau}\omega.$$

A non-empty subset X of the complete join semilattice $(D,+)$ is said to cover an element d if $d \leq \sup X$. An element d of $(D,+)$ is called compact if each subset X of D covering d has a finite subset also covering d. The set of compact elements of the complete join semilattice $(D,+)$ will be denoted by DQ.

In analogy with [1, Defn. I. 4.6], the following definition is made.

DEFINITION 5.1. A modal $(D,+,\Omega)$ is said to be arithmetical if it satisfies the following conditions:

(i) $(D,+)$ is a complete join semilattice;

(ii) (D,Ω) is completely distributive over $(D,+)$;

(iii) each element of D is the supremum of the compact elements less than or equal to it;

(iv) DQ is a submode of (D,Ω).

A modal homomorphism $f : (D,+,\Omega) \to (E,+,\Omega)$ between arithmetical modals is said to be an arithmorphism if $DQf \subseteq EQ$.

For a mode (A,Ω), the modal $(AS,+,\Omega)$ of all non-empty submodes is arithmetical [6, 335]. The compact elements of AS are precisely the finitely generated non-empty submodes of (A,Ω), and each submode is the supremum (i.e. set-theoretical union) of the finitely generated submodes it contains. Indeed, each submode S of (A,Ω) can be expressed as $S = \sup \{\{x\} \mid x \in S\}$.

For a subset B of a partial order $(D,\leq)$, the <u>subordinate set</u> $\downarrow B$ is $\{d \in D \mid \exists b \in B.\ b \leq d\}$. For a modal $(D,+,\Omega)$, let DR denote the set of non-empty subsemilattices B of $(D,+)$ that are their own subordinate sets. By the Sum-superiority Lemma 4.12, such subsemilattices of $(D,+)$ are automatically submodes of (D,Ω). Given B, B' in DR, define $B + B'$ to be $\downarrow \{b+b' \mid b \in B, b' \in B'\}$. Under this operation, $(DR,+)$ becomes a complete join semilattice. Given ω in Ω and elements $B_1, \ldots, B_{\omega\tau}$ of DR, define a new element $(B_1,\ldots,B_{\omega\tau})\omega$ of DR as the subordinate set of the complex ω-product $B_1 \ldots B_{\omega\tau}\omega$ of the submodes $B_1, \ldots, B_{\omega\tau}$ of (D,Ω). This gives a mode structure (DR,Ω) which is completely distributive over $(DR,+)$, so that in particular $(DR,+,\Omega)$ is a modal [6, 332]. In fact, $(DR,+,\Omega)$ is an arithmetical modal [6, 341], having as its compact elements the elements $\downarrow\{d\}$ for d in D. Given a modal homomorphism $f = (D,+,\Omega) \to (E,+,\Omega)$ between two modals, the mapping

(5.2) $fR : DR \to ER;\ B \mapsto \sup\downarrow(Bf)$

becomes an arithmorphism $fR : (DR,+,\Omega) \to (ER,+,\Omega)$ [6, 343].

The relevance of the construction R for the power domains AP and AS of a mode (A,Ω) is described by the following result.

PROPOSITION 5.3. [6, 333]. For a mode (A,Ω), the arithmetical modals $(AS,+,\Omega)$ and $(APR,+,\Omega)$ are isomorphic via the arithmorphisms

(5.4) $\delta : AS \to APR;\ S \mapsto \{F \leq S \mid F \in AP\}$

and

(5.5) $\gamma : APR \to AS;\ B \mapsto \sup \{\{a\} \mid \exists F \in B.\ a \in F\}$.

If (A,Ω) is a convex subset (A,I°) of a real vector space E, as in Example 2.8, then Proposition 5.3 may be interpreted as showing how arbitrary convex subsets S of A are approximated by polytopes F contained within S.

Let $\mathfrak{D}$ be a variety of modals $(D,+,\Omega)$ in the sense of universal algebra -- the class of all modals satisfying some (possibly empty) set of identities in Ω. Consider $\mathfrak{D}$ as a category having modal homomorphisms as its morphisms. Let $\mathcal{E}$ be the subcategory of $\mathfrak{D}$ whose objects are the arithmetical modals in $\mathfrak{D}$ and whose morphisms are the arithmorphisms between them. Then the construction R gives a functor $R : \mathfrak{D} \to \mathcal{E}$. In the other direction, there is a functor $Q : \mathcal{E} \to \mathfrak{D}$ assigning the modal $(DQ,+,\Omega)$ of compact elements to each arithmetical modal $(D,+,\Omega)$ in $\mathcal{E}$. For an arithmorphism $f : (D,+,\Omega) \to (E,+,\Omega)$, the modal homomorphism $fQ : (DQ,+,\Omega) \to (EQ,+,\Omega)$ is just the restriction of f to the subset DQ of D. For each object D of $\mathfrak{D}$ there is a natural isomorphism

(5.6) $\eta_D : D \to DRQ;\ d \mapsto \downarrow\{d\}$.

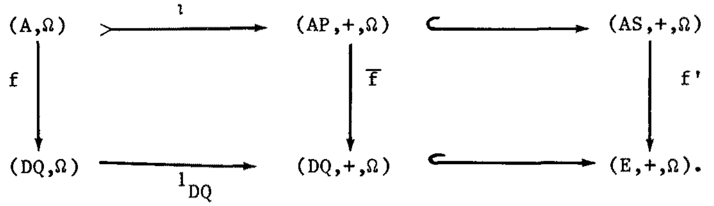
For each object E of $\mathcal{E}$, there is a natural isomorphism

(5.7) $\varepsilon_E : EQR \to E;\ B \mapsto \sup B$ [6, 344].

THEOREM 5.8 [6, 345]. The categories $\mathfrak{D}$ and $\mathcal{E}$ are equivalent via the adjoint equivalence $(R,Q;\eta,\varepsilon)$.

Theorem 5.8 may be used to extend functions from the mode A through the power domain AP to the power domain AS.

THEOREM 5.9 [6, 356]. Let (A,Ω) be a mode, and $(D,+,\Omega)$ an arithmetical modal. Then each mode homomorphism $f : (A,\Omega) \to (DQ,\Omega)$ may be extended to a unique arithmorphism $f' : (AS,+,\Omega) \to (E,+,\Omega)$ such that the following diagram commutes:



Here, $\overline{f}$ is given by Theorem 4.12, and ι is the canonical embedding (3.3). Then $f' : AS \to E$ is the composite

$$AS \xrightarrow{\;\delta\;} APR \xrightarrow{\;\overline{f}R\;} DQR \xrightarrow{\;\varepsilon_D\;} D$$

of the mappings δ of (5.4), $\overline{f}R$ of (5.2), and ε_D of (5.7).

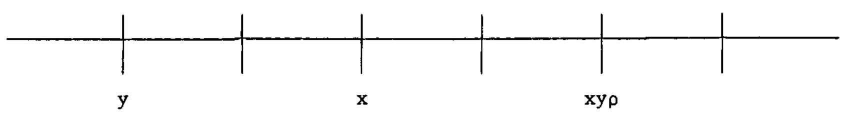
6. DIGITAL GEOMETRY Present-day scientific computing is a long and roundabout exercise. To begin with, theoretical scientists create mathematical models reducing natural laws to differential equations relating real or complex-valued functions. These differential equations are then solved using programs developed by numerical analysts. The real and complex numbers involved are represented in the programs by floating-point numbers. When the programs are implemented and run on a machine, these floating-point numbers are converted into binary digits processed by the machine's circuitry. Each stage of this lengthy process introduces new potential errors that have to be kept under control. These errors may be conceptual in the theoretical stages or arithmetical in the computational stages. In the theoretical models, the real and complex numbers represent idealizations of the results of physical measurements. In the numerical analysis, the floating-point numbers used represent approximations to these real and complex numbers. In the machine implementation, the natural binary logic is contorted to deal with the decimal representations that are more appropriate to finger counting and other calculation methods used by human beings. A typical breakdown of this process occurs when one "proves" the convergence of an algorithm using real analysis, only to have the algorithm diverge in an implementaion because of an accumulation of rounding errors.

One of the main uses of modal theory is as a guide in the search for simpler and safer ways of performing scientific computation. The idea is to formulate new mathematical models that are directly computable, avoiding the roundabout route via real numbers and decimal or floating-point representations. From the computational side, the starting point is at those numbers directly representable in a binary machine, namely dyadic rational numbers $m \cdot 2^{-n}$ (m, n integral) or "finite binary decimals". The question raised is:

(6.1) What mathematical structures are carried naturally by finite binary strings?

The answers given to this question should suggest to theoretical scientists the mathematical language in which their theories ought to be formulated if they are to facilitate the practical computations that are the ultimate applications of the theories. Here, two typical answers to question (6.1) are given. A brief discussion then shows how modal theory might be used to apply these mathematical structures to the formulation of readily computable models.

To begin with, consider an infinite sequence of points equally spaced out along a line:



Given two points x, y, on the line, let xyρ denote the reflexion of y in x - the point that is as far behind x as y is in front. Starting with two adjacent points labelled 0 and 1, all the other points, labelled by the set **Z** of integers in the usual order, may be obtained by repeated reflexions of the points obtained from 0 and 1 by reflexion. Using binary notation for the integers, two or 10 is obtained by reflexion of 0 in 1, i.e. as 10ρ. Three or 11 is obtained as 10ρ1ρ, and four or 100 is obtained as 10ρ0ρ. On the negative side, -1 is obtained as 01ρ, -10 as 01ρ0ρ, -11 as 01ρ1ρ, -100 as 01ρ0ρ0ρ, etc. A striking pattern begins to emerge, confirmed in Theorem 6.2 below. Define a kei mode to be a mode (A,ρ) with a single binary operation ρ satisfying the identity $xxy\rho\rho = y$. For a binary digit a, let a' denote the complementary digit, i. e. $0' = 1$ and $1' = 0$.

THEOREM 6.2 [6, 416]. The algebra (Z,ρ) of reflexions on the integers is the free kei mode on the set $\{0,1\}$. A negative integer with binary representation

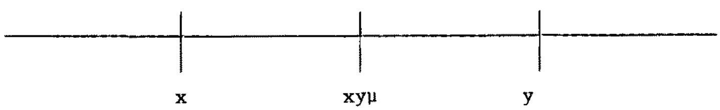
$-\sum_{i=0}^{r} a_i\, 2^{r-i}$ $(a_0 = 1)$ represents the word $01\rho a_1\rho \ldots a_r\rho$, while an integer

$n > 1$ for which $n - 1$ has the binary representation $\sum_{i=0}^{r} a_i\, 2^{r-i}$ $(a_0=1)$

represents the word $10\rho a_1'\rho \ldots a_r'\rho$.

The answer Theorem 6.2 gives to question (6.1) is that, if the binary strings represent integers, then the natural mathematical structure may be expressed geometrically as reflexion or algebraically as the free kei mode on 0 and 1.

Next, consider points x, y on a line. Let $xy\mu$ denote the midpoint of x and y.



Starting with two points labelled 0 and 1, taking all the possible midpoints successively gives the intersection D_1 of the real unit interval $[0,1]$ with the set $D = \{m\cdot 2^{-n} \mid m,n \in Z\}$ of dyadic rationals. The set D_1 is called the dyadic unit interval. Its elements are labelled by finite binary fractions. One-half or $\cdot 1$ is obtained as 10μ, three-quarters or $.11$ is obtained as $110\mu\mu$, three-eighths or $\cdot 011$ is obtained as $0110\mu\mu\mu$, etc. Define a commutative binary mode to be a mode (A,μ) with a single binary operation μ satisfying the commutative law $xy\mu = yx\mu$. Then in analogy with Theorem 6.2, one has the following.

THEOREM 6.3 [6, 424]. The algebra (D_1,μ) of midpoints on the dyadic unit interval is the free commutative binary mode on the set $\{0,1\}$. A proper binary fraction

$$\sum_{i=0}^{r} a_i\, 2^{i-r-1} \quad (a_0=1) \quad \text{represents the word} \quad a_r \ldots a_1\, 10\mu\mu \ldots \mu.$$

The answer Theorem 6.3 gives to question (6.1) is that, if the binary strings represent fractions, then the natural mathematical structure may be expressed geometrically as bisection or algebraically as the free commutative binary mode on 0 and 1.

Putting the structures of Theorems 6.2 and 6.3 together, one obtains the mode $(\mathbf{D},\mu,\rho)$, a commutative quasigroup on the set of dyadic rationals [6, 4.3], free on $\{0,1\}$. This structure often serves to replace the real numbers in the new formulations of scientific theories. For example, there are formulations of quantum mechanics based on convexity (mentioned and referenced in [6]). The reformulation designed for computing would then replace the convexity structure $(\mathbf{R},I^{\circ})$ on the reals by the algebra $(\mathbf{D},\mu)$. Note that the modal $(\mathbf{R},\vee,0\cdot5)$ (notations of (2.7) and Example 4.8) reappears as a submodal of the modal $(\mathbf{D}\mathrm{R},\vee,\mu)$ constructed as in Section 6. An interesting exercise in the application of the ideas discussed here would be to design a computer graphics package based on the modal structures on powers $\mathbf{D}^n$ of the dyadic rationals $\mathbf{D}$.

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