MATCHED PAIRS, PERMUTATION REPRESENTATIONS, AND THE BOL PROPERTY

K. W. JOHNSON[1] AND J. D. H. SMITH[2]

Abstract. The paper examines Bol loops that are constructed from matched pairs of groups, including a new simple, non-Moufang Bol loop of order 120. Certain permutation representations of the loops under consideration provide nontrivial examples of doubly stochastic action matrices. A generalization of the matched-pair loop construction yields proper Bol actions of groups.

1. Introduction

In recent work, G. Nagy gave a construction for simple Bol loops based on faithful group factorizations [5]. His analysis used group-theoretical methods, exhibiting loop transversals. In connection with certain Hopf algebra constructions, Takeuchi discussed the equivalence of group factorizations and so-called matched pairs of subgroups (with mutual actions) [12]. The intention of the present paper is to provide a conceptual understanding of Nagy's Bol loop construction, employing direct quasigroup-theoretical methods and the matched-pair approach to group factorizations. Since the context goes beyond the construction of simple Bol loops (encompassing, for example, Bol loops like those of [3], [4]), the factorizations are not required to be faithful.

Section 2 sets up the notation for (exact) group factorizations, and summarizes the equivalence with matched pairs of groups. The left Bol loop construction (3.1) is then presented in Theorem 3.1 as a slightly twisted version of the reconstruction (2.11) of a factorized group from the equivalent matched-pair data. Left and right multiplications in the Bol loop are obtained explicitly from the matched pair. Among the examples offered in Section 4, we present a simple, non-Moufang left Bol loop of order 120 (Example 4.4). Section 5 exhibits certain Lagrangean subgroups of the matched-pair loops. The final sections are devoted to permutation actions. Motivated by the homogeneous spaces of the subgroups from Section 5, Theorem 6.1 shows that action

2000 *Mathematics Subject Classification.* 20N05, 20D40, 20B99.

Key words and phrases. Bol loop, group factorization, group action, matched pair, permutation representation, dual code, doubly stochastic.

matrices of the homogeneous space of a Lagrangean subquasigroup are doubly stochastic. Two examples discuss the homogeneous spaces of the subgroup C_4 in the simple left Bol loop of order 24. In the right homogeneous space, nontrivial doubly stochastic action matrices are shown to arise. On the other hand, left actions of the left Bol loop on the left homogenous space are shown not to be left Bol actions such as those considered by Sbitneva [7]. Nevertheless, Proposition 7.4 uses the main idea of Theorem 3.1 to obtain left Bol actions of groups.

Readers are referred to [10] and [11] for conventions, concepts and notations that are not otherwise explained explicitly in the paper. In order to minimize the number of brackets, we use algebraic notation with maps placed to the right of their arguments, either in line or as a superscript.

2. Matched pairs

Consider a group G with two subgroups C and C^*, such that

$$G = C \cdot C^* \quad \text{and} \quad C \cap C^* = \{1\}\,. \tag{2.1}$$

The group G is said to be (*exactly*) *factorized* by the subgroups C and C^* (or to form a *Zappa-Szép product* [2, §VI.4]). The subgroup C is called the *primal* subgroup, while the subgroup C^* is called the *dual* subgroup. Given a factorization (2.1) of G, the factorization $G = C^* \cdot C$ is called the *dual* factorization, while (2.1) itself is then called the *primal* factorization. The factorization (2.1) yields a set isomorphism

$$(\delta, \delta^*) : G \to C \times C^*; g \mapsto (g\delta, g\delta^*) \tag{2.2}$$

which is inverse to the multiplication

$$C \times C^* \to G; (c, c^*) \mapsto cc^*\,. \tag{2.3}$$

Example 2.1 (**Dual codes**)**.** Let G be an abelian group considered as a channel — for instance the set $G = (\mathbb{Z}/2\mathbb{Z})^l$ of binary words of length l, under componentwise addition. Then (2.1) corresponds to the direct sum decomposition $G = C \oplus C^*$. Here the primal subgroup C is a linear code, and the dual subgroup C^* is a dual code (with respect to suitable statistics in the channel). The map $\delta : G \to C$ of (2.2) decodes a received word g to the codeword $g\delta$, while δ^* provides the dual decoding.

It is convenient to use the coding theory terminology of Example 2.1 in the general nonabelian case (2.1).

Example 2.2 (**Symmetric groups**)**.** For each natural number n, consider the symmetric group S_{n+1} as the group $\{0, 1, \ldots, n\}!$ of all permutations of the set of integers modulo $n+1$. Identify each residue r modulo $n+1$ with the addition

$$\mathbb{Z}/(n+1)\mathbb{Z} \to \mathbb{Z}/(n+1)\mathbb{Z}; x \mapsto x + r$$

modulo $n+1$. These additions form the cyclic subgroup C_{n+1} of S_{n+1}. The stabilizer of 0 in S_{n+1} forms a symmetric subgroup S_n of S_{n+1}. The factorization $S_{n+1} = S_n \cdot C_{n+1}$ gives an example of (2.1), with $g\delta^* = 0g$ for g in S_{n+1}.

The following fundamental proposition is well known, but worth recording in detail to establish the notational techniques that we use.

Proposition 2.3. *Consider a factorization* (2.1) *of a group* G.

(a) *A left action of* G *on the primal subgroup* C *is defined by*

$$xL'(g) = (gx)\delta \tag{2.4}$$

for x *in* C *and* g *in* G.

(b) *A right action of* G *on the dual subgroup* C^* *is defined by*

$$x^*R^*(g) = (x^*g)\delta^* \tag{2.5}$$

for x^* *in* C^* *and* g *in* G.

Proof. For g_1, g_2 in G and x in C, one has

$$\begin{aligned} xL'(g_2g_1) &= (g_2 \cdot g_1x)\delta = \big(g_2(g_1x)^\delta(g_1x)^{\delta^*}\big)\delta \\ &= \big(g_2x^{L'(g_1)} \cdot (g_1x)^{\delta^*}\big)\delta \\ &= \Big(\big(g_2x^{L'(g_1)}\big)^\delta\big(g_2x^{L'(g_1)}\big)^{\delta^*}(g_1x)^{\delta^*}\Big)\delta \\ &= \big(g_2x^{L'(g_1)}\big)\delta = xL'(g_1)L'(g_2)\,, \end{aligned}$$

proving (a). The proof of (b) is dual. $\square$

Corollary 2.4. *Let* $L : G \to G!$ *and* $R : G \to G!$ *denote the respective left and right multiplications in the group* G. *Then*

$$yL(x) = yL'(x) \quad \textit{and} \quad x^*R(y^*) = x^*R^*(y^*)$$

for x, y *in* C *and* x^*, y^* *in* C^*.

The left and right actions of Proposition 2.3(a), (b) are respectively isomorphic to the group homogeneous spaces G/C^* and $C\backslash G$. The respective isomorphisms of left and right G-sets are

$$C \to G/C^*; x \mapsto xC^*$$

and

$$C^* \to C\backslash G; x^* \mapsto Cx^*\,.$$

Thus the respective actions (2.4) and (2.5) of a group element g may be written as $L_{G/C^*}(g)$ and $R_{C\backslash G}(g)$.

Definition 2.5. A group factorization (2.1) is said to be *faithful* if the actions L_{G/C^*} and $R_{C\backslash G}$ of G are faithful, thus if the subgroups C and C^* of G have trivial cores.

The actions of G in Proposition 2.3 restrict respectively to a left action of the subgroup C^* on C and a right action of C on C^*.

Proposition 2.6. *Consider a factorization* (2.1) *of a group* G*.*

(a) *The multiplication in* G *is recovered from the mutual actions* (2.4) *and* (2.5) *of* C *and* C^**, together with the multiplications on the individual subgroups* C *and* C^**, by*

$$xx^* \cdot yy^* = xy^{L'(x^*)} \cdot x^{*R^*(y)}y^* \tag{2.6}$$

for x*,* y *in* C *and* x^**,* y^* *in* C^**.*

(b) *The mutual actions* (2.4) *and* (2.5) *of* C *and* C^* *satisfy*

$$(xy)L'(x^*) = xL'(x^*) \cdot yL'(x^*R^*(x)) \tag{2.7}$$

and

$$(x^*y^*)R^*(x) = x^*R^*(xL'(y^*)) \cdot y^*R^*(x) \tag{2.8}$$

for x*,* y *in* C *and* x^**,* y^* *in* C^**.*

Proof. **(a):** Note $xx^* \cdot yy^* = x(x^*y)y^* = x(x^*y)^\delta \cdot (x^*y)^{\delta^*}y^*$. Then (2.6) follows by the definitions (2.4) and (2.5).
(b): The associative law in the group G gives

$$\begin{aligned}(xy)L'(x^*) &= \big(x^*(xy)\big)\delta = \big((x^*x)y\big)\delta = \big((x^*x)^\delta \cdot (x^*x)^{\delta^*}y\big)\delta \\ &= (x^*x)^\delta \cdot \big((x^*x)^{\delta^*}y\big)\delta = xL'(x^*) \cdot yL'(xR^*(x^*))\,,\end{aligned}$$

namely (2.7). The proof of (2.8) is dual. □

Definition 2.7. An ordered pair (C, C^*) of groups is said to form a *matched pair*

$$(C, C^*; L', R^*)$$

if there is a left action $L' : C^* \to C!$ of C^* on C and a right action $R^* : C \to C^*!$ of C on C^*, such that the conditions (2.7) and (2.8) are satisfied.

Remark 2.8. The concept of a matched pair of groups appears in [12, Defn. 2.1]. Takeuchi explicitly imposed the additional conditions

$$1_C L'(x^*) = 1_C \quad \text{and} \quad 1_{C^*}R^*(x) = 1_{C^*} \tag{2.9}$$

for x in C and x^* in C^*. However, these conditions result from setting $x = 1_C$ in (2.7) and $y^* = 1_{C^*}$ in (2.8) respectively. In connection with a group factorization such as (2.1), Huppert [2, §VI.4] speaks of "complicated functional equations" — presumably with (2.7) and (2.8) in mind. Note that the conditions (2.9) do not hold for general elements x or x^* of G.

Proposition 2.6 has the following converse [12, Prop. 2.2]:

Proposition 2.9. *Suppose that* $(C, C^*; L', R^*)$ *is a matched pair of groups. Define a product*

$$(x, x^*) \cdot (y, y^*) = \big(x \cdot yL'(x^*), x^*R^*(y) \cdot y^*\big) \tag{2.10}$$

on $C \times C^*$*. Then* $(C \times C^*, \cdot)$ *is a group which is factorized by subgroups* $C \times \{1_{C^*}\}$ *and* $\{1_C\} \times C^*$*, respectively isomorphic to* C *and* C^**.*

Together, Propositions 2.6 and 2.9 provide an equivalence between (exact) group factorizations and matched pairs of groups.

Remark 2.10. Given a group factorization (2.1), it is often convenient to identify the sets G and $C \times C^*$ by means of the mutually inverse set isomorphisms (2.2) and (2.3). By Corollary 2.4, the product gh of two group elements $g = (x, x^*)$ and $h = (y, y^*)$ may then be written in the form

$$gh = \big((gh)\delta, (gh)\delta^*\big) = (yL'(g), x^*R^*(h)) \tag{2.11}$$

using the formula (2.10), since $L'(x^*)L(x) = L'(x^*)L'(x) = L'(xx^*)$ and $R^*(y)R(y^*) = R^*(y)R^*(y^*) = R^*(yy^*)$.

3. Bol loops

Recently, G. Nagy used a group-theoretical method to show how faithful (exact) group factorizations may be used to construct simple Bol loops [5] — compare Remark 3.2 below. We now give a more general, direct and conceptual description of this construction, not just with simple Bol loops in mind. For a group $(G, \cdot)$, the opposite will be denoted by $(G, \check{\circ})$, so that $g \check{\circ} h = h \cdot g$ with g, h in G. (This is the notation of [11, (1.4)], not that of [10, (1.5)].) To comprehend the Bol loop construction, compare (2.11) with (3.1) below.

Theorem 3.1. *Consider a group factorization* (2.1)*. Identify the set* G *with* $C \times C^*$ *by means of the mutually inverse set isomorphisms* (2.2) *and* (2.3)*. Then a left Bol loop structure* $G_\circ$ *or* $(G, \circ, 1_G)$ *is defined by*

$$g \circ h = \big((gh)\delta, (g\check{\circ}h)\delta^*\big) = (yL'(g), y^*R^*(g)) \tag{3.1}$$

for $g = (x, x^*)$ *and* $h = (y, y^*)$ *in* $G = C \times C^*$*.*

Proof. Proposition 2.3 shows that the left multiplication

$$L_\circ(g) = (L'(g), R^*(g)) \tag{3.2}$$

in $(G, \circ, 1_G)$ or $(C \times C^*, \circ, 1_G)$ has $L_\circ(g^{-1})$ as its inverse, and $L_\circ(1_G) = 1_{G!}$. Also $g \circ 1_G = (g\delta, g\delta^*) = g$, so $(G, \circ, 1_G)$ is a left loop. Now (3.1), (2.11) and (3.2) yield

$$\begin{aligned} ghg &= \big((ghg)\delta, (ghg)\delta^*\big) = \Big(\big[g \cdot (hg)^\delta\big]\delta, \big[(gh)^{\delta^*} \cdot g\big]\delta^*\Big) \\ &= \big(xL'(h)L'(g), x^*R^*(h)R^*(g)\big) = g \circ (h \circ g)\,. \end{aligned}$$

Thus (3.2) gives

$$\begin{aligned} L_\circ(g)L_\circ(h)L_\circ(g) &= \big(L'(g)L'(h)L'(g), R^*(g)R^*(h)R^*(g)\big) \\ &= \big(L'(ghg), R^*(ghg)\big) = L_\circ(ghg) \\ &= L_\circ\big(g \circ (h \circ g)\big)\,, \end{aligned} \tag{3.3}$$

the left Bol property in the left loop $(G, \circ, 1_G)$.

Finally, given h and $k = (z, z^*)$ in G, consider the problem of finding g in G with $g \circ h = k$. By (3.1), this amounts to finding elements t in C and t^* in C^* such that

$$g = zt^*h^{-1} = h^{-1}tz^*\,,$$

since one then has $(gh)\delta = (zt^*)\delta = z$ and $(hg)\delta^* = (tz^*)\delta^* = z^*$. Now

$$g \in zC^*h^{-1} \cap h^{-1}Cz^* \quad \Leftrightarrow \quad hgh \in hzC^* \cap Cz^*h\,.$$

There is a unique element

$$(hz)\delta \cdot (z^*h)\delta^* = \big(zL'(h), z^*R^*(h)\big) = h \circ k$$

of $hzC^* \cap Cz^*h$, so there is a unique solution

$$g = h^{-1}(h \circ k)h^{-1} = h^{-1} \circ \big[(h \circ k) \circ h^{-1}\big]$$

to the equation $g \circ h = k$. In other words, $G_\circ$ is a (two-sided) loop. □

Remark 3.2. The method used by G. Nagy was to show that the twisted diagonal $\{(g, g^{-1}) \mid g \in G\}$ is a Bol loop transversal to $C \times C^*$ in $G \times G$. The inversion in the second component of the twisted diagonal corresponds to our use of the opposite multiplication in the second component of the middle term of (3.1). Consider the action of $G \times G$ on the cosets of $C \times C^*$. Projection onto the first factor gives the action L' of G on C, corresponding to the first component of the right hand side of (3.2). Projection onto the second factor gives the action R^* of G on C^*, corresponding to the second component of the right hand side of (3.2).

Corollary 3.3. *In the loop $G_\circ$, the map*

$$R_\circ(h) = L(h)R(h)L_\circ(h)^{-1} = L(h)R(h)\big(L'(h)^{-1}, R^*(h)^{-1}\big)$$

is the right multiplication by an element h.

Definition 3.4. The left Bol loop $G_\circ$ constructed by (3.1) from the group factorization $G = C{\cdot}C^*$ or equivalent matched pair $(C, C^*; L', R^*)$ is described as the corresponding *matched-pair loop*.

In certain cases, it is possible to relate the inner multiplication groups of a matched-pair group and the corresponding matched-pair loop.

Proposition 3.5. *Let* (2.1) *be a factorization of a group G generated by involutions.*

(a) *The inner automorphism group of the group G is a subgroup of the inner multiplication group of the loop $G_\circ$..*
(b) *The conjugacy classes of the group G fuse to the conjugacy classes of the loop $G_\circ$.*

Proof. By (3.2) and Corollary 3.3,

$$R_\circ(t)L_\circ(t) = L(t)R(t) \tag{3.4}$$

for elements t of G. If t is an involution, then the right-hand side of (3.4) is the inner automorphism $T(t)$ by t in the group G. Thus the inner automorphism group $\operatorname{Inn} G$ of the group G is a subgroup of the inner multiplication group $\operatorname{Inn} G_\circ$ of the loop $G_\circ$. In particular, the orbits of $\operatorname{Inn} G_\circ$, the loop conjugacy classes, are fusions of the orbits of $\operatorname{Inn} G$. □

4. Examples

Example 4.1 (Group products). Let A and B be groups. Take G to be the direct product $A \times B$. Then a factorization (2.1) is provided by $C = A \times \{1\}$ and $C^* = \{1\} \times B$. The matched-pair loop $(G, \circ)$ becomes the group $(A, \cdot) \times (B, \check{\circ})$, the product of A with the opposite of B. Note that C and C^* are normal subgroups of G, so the factorization is not faithful in the sense of Definition 2.5.

Example 4.2 (Split extensions). If the factorized group G of (2.1) is obtained as the split extension of a normal subgroup C^* by a subgroup C, then the construction of Theorem 3.1 specializes to a version of the constructions of [3], [4]. Indeed, in this case, the respective actions (2.4) and (2.5) take the form $xL'(yy^*) = xL(y)$ and $xR^*(yy^*) = xT(y)R(y^*)$ with $xT(y) = y^{-1}xy$, for x, y in C and x^*, y^* in C^*. Thus (3.1) reduces to

$$(x, x^*) \circ (y, y^*) = \big(xy, y^*R^*(x) \cdot x^*\big)\,, \tag{4.1}$$

while [3, (3)] may be rewritten as

$$(x, x^*) \circ (y, y^*) = \big(xy, x^* \cdot y^* R^*(x)\big)\,. \tag{4.2}$$

In particular, the loop products (4.1) and (4.2) coincide if the normal subgroup C^* is abelian.

Example 4.3 (**The simple Bol loop of order 24**)**.** The factorization of $G = S_4$ from Example 2.2 gives a simple Bol loop of order 24 [5, Ex. II]. Nagy's argument for the simplicity of this loop used the known classification of all Bol loops of order at most 12 (which are solvable). However, direct computations aided by Proposition 3.5(b) show that the loop is a rank 2 quasigroup, and therefore simple (compare [10, §6.8]).

Example 4.4 (**A simple, non-Moufang Bol loop of order 120**)**.** Consider the factorization of $G = S_5 = \{1, 2, 3, 4, 5\}!$ with subgroups $C = \langle(1\ 2)(3\ 4\ 5)\rangle$ and $C^* = \langle(1\ 2\ 3\ 4\ 5), (1\ 2\ 4\ 3)\rangle$ [2, VI.4.2]. The loop $G_\circ$ is simple (e.g. by [5, Th. 3.6]). Because $G_\circ$ has elements of order 6 from C, it is not isomorphic to the simple Moufang loop of order 120. Indeed, the nonidentity elements of that loop have orders 2 or 3. Note that an application of Theorem 3.1 to the factorization of S_5 from Example 2.2 gives a Bol loop which is not simple.

Example 4.5 (**Simple groups generated by involutions**)**.** One obtains simple Bol loops from the (exact) group factorizations

$$M_{11} = C_{11} \cdot M_{10} = (C_{11} \times C_5) \cdot (M_9 \cdot 2)$$

and

$$M_{23} = C_{23} \cdot M_{22} = (C_{23} \times C_{11}) \cdot P\Sigma L(3, 4) = (C_{23} \times C_{11}) \cdot (2^4 \times A_7)$$

— compare [1, Table 4], [5, Th. 3.6]. Other examples include $PSL(2, 11)$ factorized by a Sylow 11-subgroup and a subgroup isomorphic to A_5.

5. Lagrangean subloops

Consider the matched-pair loop $G_\circ$ given by a group factorization $G = C \cdot C^*$ or equivalent matched pair $(C, C^*; L', R^*)$. In Example 4.1, the direct product matched-pair loop had subgroups $C \times \{1_{C^*}\}$ and $\{1_C\} \times C^*$, respectively isomorphic to the groups $(C, \cdot)$ and $(C^*, \breve{\circ})$. The following proposition records the fact that this behavior is quite typical for general matched-pair loops.

Proposition 5.1. *Consider the matched-pair loop $G_\circ$ or $(C \times C^*, \circ)$ given by a general matched pair $(C, C^*; L', R^*)$. The subsets $C \times \{1_{C^*}\}$ and $\{1_C\} \times C^*$ form associative subloops, respectively isomorphic to the groups $(C, \cdot)$ and $(C^*, \breve{\circ})$.*

Proof. Using the notation of Theorem 3.1, one has

$$x \circ y = \big((xy)\delta, (yx)\delta^*\big) = (xy, 1_{C^*}) = xy$$

and

$$x^* \circ y^* = \big((x^*y^*)\delta, (y^*x^*)\delta^*\big) = (1_C, y^*x^*) = y^*x^*$$

for x, y in C and x^*, y^* in C^*. □

Definition 5.2. In the context of Proposition 5.1, the associative subloops $C \times \{1_{C^*}\}$ and $\{1_C\} \times C^*$ of $G_\circ$ are respectively known as the *primal* and *dual* subloops of $G_\circ$.

Both the primal and dual subloops of a finite matched-pair Bol loop are certainly "Lagrange-like" in the sense of [6, Defn. I.2.13]. (Indeed, [3] records that the Bol loops of Example 4.2 satisfy the strong Lagrange property of [6, Defn. I.2.15].) Now recall that a subquasigroup P of a quasigroup Q is said to be (*right*) *Lagrangean* in Q if the relative left multiplication group $\mathrm{LMlt}_Q P$ acts semitransitively on Q (so all the orbits have the same size) [8], [10, §4.5].

Proposition 5.3. *Consider a group factorization* (2.1) *in which the primal subgroup C is commutative. Then for each element $h = (y, y^*)$ of G, the right coset $h\mathrm{LMlt}_{G_\circ}P$ of h by the primal subloop P takes the form*

$$P \circ h = \{p \circ h \mid p \in P\}\,.$$

In particular, the primal subloop P is (right) Lagrangean in the left Bol loop $G_\circ$.

Proof. Equation (3.1) gives $p \circ h = \big(yL'(p), y^*R^*(p)\big)$ for p in P. Then

$$\begin{aligned} q \circ (p \circ h) &= \big(yL'(p), y^*R^*(p)\big)L_\circ(q) = \big(yL'(p)L'(q), y^*R^*(p)R^*(q)\big) \\ &= \big(yL'(qp), y^*R^*(pq)\big) = \big(yL'(qp), y^*R^*(qp)\big) = (qp) \circ h \end{aligned}$$

for a second element q of P. The group product qp lies in P. □

Dually, one obtains the following.

Corollary 5.4. *Consider a group factorization* (2.1) *in which the dual subgroup C^* is commutative.*

(a) *For each element h of G, the right coset $h\mathrm{LMlt}_{G_\circ}P^*$ of h by the dual subloop P^* takes the form*

$$P^* \circ h = \{p^* \circ h \mid p^* \in P^*\}\,.$$

(b) *The subloop C^* is (right) Lagrangean in the left Bol loop $G_\circ$.*

6. Doubly stochastic actions

By Proposition 5.3 and Corollary 5.4, many group factorizations yield a right Lagrangean subloop P of the matched-pair loop $G_\circ$. A general, but hitherto unremarked property of right Lagrangean subquasigroups, is that the action matrices of their homogeneous spaces are always doubly stochastic. (As permutation matrices, the action matrices of groups are doubly stochastic, but general action matrices of (left) quasigroup homogeneous spaces are only row-stochastic [9, Th. 4.2], [10, Th. 4.1].)

Theorem 6.1. *Let P be a right Lagrangean subquasigroup of a finite quasigroup Q. Then for each element q of Q, the action matrix $R_{P\backslash Q}(q)$ of q on the homogeneous space $P\backslash Q$ is doubly stochastic.*

Proof. If P is empty, then the homogeneous space $P\backslash Q$ is regular, and the action matrices are permutation matrices. Now suppose that P is nonempty. Let A_P be the $|Q|\times|P\backslash Q|$ incidence matrix for the elements of Q in the points of the homogeneous space $P\backslash Q$ (the orbits of $\mathrm{LMlt}_Q P$ on Q.) Thus a matrix entry $[A_P]_{xX}$ is 1 if the quasigroup element x lies in the $\mathrm{LMlt}_Q P$-orbit X, and otherwise the entry is 0. Since P is right Lagrangean, each orbit of $\mathrm{LMlt}_Q P$ on Q has cardinality $|P|$. Thus the generalized inverse A_P^+ of A_P has an entry $[A_P^+]_{Xx}$ of $|P|^{-1}$ if x lies in X, and 0 otherwise.

The action matrix of a quasigroup element q on the homogeneous space $P\backslash Q$ is given as

$$R_{P\backslash Q}(q) = A_P^+ R_Q(q) A_P$$

in terms of the (permutation matrix of the) right multiplication $R_Q(q)$ by q on Q. Thus the sum of the elements in the column of $R_{P\backslash Q}(q)$ labeled by the point Y of $P\backslash Q$ is

$$\begin{aligned}\sum_{X\in P\backslash Q}[R_{P\backslash Q}(q)]_{XY} &= \sum_{X\in P\backslash Q}\sum_{x\in Q}\sum_{y\in Q} A^+_{Xx}R(q)_{xy}A_{yY}\\ &= \sum_{y\in Y}|P|^{-1} = 1\,,\end{aligned}$$

as required for the column stochasticity. □

Example 6.2. Consider the dual subloop $P^* \cong C_4$ of the matched-pair loop $G_\circ$ of Example 4.3. The action matrix of the 3-cycle (1 2 3) on the homogeneous space $P^*\backslash G_\circ$ is

$$\frac{1}{4}\begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 & 0 \end{bmatrix},$$

a nonpermutational, doubly stochastic matrix.

7. Bol actions

The following definition adapts a concept from [7].

Definition 7.1. Let $(Q, \circ)$ be a left Bol loop. Consider a function

$$L_X : Q \to X! \tag{7.1}$$

from Q to the group $X!$ of permutations of X, an action of Q on a set X. Then the action (7.1) is said to be a *left Bol action* if

$$L_X(q)L_X(p)L_X(q) = L_X\big(q \circ (p \circ q)\big) \tag{7.2}$$

for all elements p and q of Q.

Remark 7.2. Sbitneva [7, Defn. 1] required the injectivity of (7.1). She spoke of a "left Bol loop action," attributing the concept to Sabinin.

The left regular action of a left Bol loop is a left Bol action. Trivially, left group actions are left Bol actions. The following "opposite" version of Example 6.2 indicates that left homogeneous spaces of matched-pair loops are unlikely to yield nontrivial left Bol actions.

Example 7.3. Consider the dual subloop $P^* \cong C_4$ of the matched-pair loop $G_\circ$ of Example 4.3. The left homogeneous space $G_\circ/P^*$ consists of points which are the orbits of the relative right multiplication group $\mathrm{RMlt}_{G_\circ}P^*$ of P^* in $G_\circ$. Let x be an element of $G_\circ$ that does not lie in P^*. Then the action matrix of x on $G_\circ/P^*$ is

$$\frac{1}{5}\begin{bmatrix} 0 & 5 \\ 1 & 4 \end{bmatrix}.$$

In particular, the action is not a left Bol action. Moreover, in contrast with Proposition 5.4(b), the dual subloop is not left Lagrangean in the left Bol loop $G_\circ$.

On the positive side, the following generalizations of the main idea of Theorem 3.1 give a rich source of left Bol actions of groups.

Proposition 7.4. *Let G be a group. Let X be a left G-set by a left action (group antihomomorphism) $L' : G \to X!$, and let Y be a right G-set by a right action (group homomorphism) $R^* : G \to Y!$.*

(*a*) *Define an action $L_{X\times Y}$ of G on the direct product $X \times Y$ of X and Y by*
$$L_{X\times Y}(g) = \big(L'(g), R^*(g)\big)$$
for g in G. Then $L_{X\times Y}$ is a left Bol action.

(*b*) *Define an action L_{X+Y} of G on the disjoint union $X + Y$ of X and Y by the restrictions*
$$L_{X+Y}(g)|_X = L'(g) \quad \text{and} \quad L_{X+Y}(g)|_Y = R^*(g)$$
for g in G. Then L_{X+Y} is a left Bol action.

Proof. **(a):** Use the argument of the first two lines of (3.3).
(b): One has
$$\begin{aligned}&\big(L_{X+Y}(g)L_{X+Y}(h)L_{X+Y}(g)\big)|_X\\ &\qquad = L'(g)L'(h)L'(g) = L'(ghg) = L_{X+Y}(ghg)|_X\end{aligned}$$
and
$$\begin{aligned}&\big(L_{X+Y}(g)L_{X+Y}(h)L_{X+Y}(g)\big)|_Y\\ &\qquad = R^*(g)R^*(h)R^*(g) = R^*(ghg) = L_{X+Y}(ghg)|_Y\end{aligned}$$
for g and h in G, so that $L_{X+Y}(g)L_{X+Y}(h)L_{X+Y}(g) = L_{X+Y}(ghg)$ as required by (7.2). □

Acknowledgements

We are grateful to M. Kinyon and an anonymous referee for helpful observations on the content of this paper.

References

[1] M. Giudici, *Factorisations of sporadic simple groups*, J. Alg. **304** (2006), 311–323. (2006).
[2] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.
[3] K.W. Johnson and B.L. Sharma, *On a family of Bol loops*. Boll. Un. Mat. It. Suppl. Alg. e Geom. **2** (1980), 119–126.
[4] K.W. Johnson and B.L. Sharma, *Bol loops from group extensions*. Bull. Korean Math. Soc. **18** (1981/82), 65–68.
[5] G.P. Nagy, *A class of simple proper Bol loops*, Manuscripta Math. **127** (2008), 81–88. Available online at `doi:10.1007/s00229-008-01881`.
[6] H.O. Pflugfelder, *Quasigroups and Loops: Introduction*, Heldermann, Berlin, 1990.
[7] L. Sbitneva, *Bol loop actions*, Comment. Math. Unic. Carolinae **41** (2000), 405–408.

[8] J.D.H. Smith, *Permutation representations of loops*, J. Alg. **264** (2003), 342–357.
[9] J.D.H. Smith, *Permutation representations of left quasigroups*, Alg. Univ. **55** (2006), 387–406.
[10] J.D.H. Smith, *An Introduction to Quasigroups and Their Representations*, Chapman and Hall/CRC, Boca Raton, FL, 2007.
[11] J.D.H. Smith and A. B. Romanowska, *Post-Modern Algebra*, Wiley, New York, NY, 1999.
[12] M. Takeuchi, *Matched pairs of groups and bismash products of Hopf algebras*, Comm. Alg. **9** (1981), 841–882.

[1] Department of Mathematics, Pennsylvania State University Abington, 1600 Woodland Avenue, Abington, PA19001, U.S.A.

[2] Department of Mathematics, Iowa State University, Ames, Iowa 50011, U.S.A.
E-mail address: [1]kwj1@psu.edu, [2]jdhsmith@iastate.edu
URL: http://www.orion.math.iastate.edu/jdhsmith/