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Multilinear algebras and Lie's Theorem for formal n-loops

By

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1. Introduction. In modern language, Lie's Third Fundamental Theorem [10, p. 396] states that each local real analytic (in particular Lie) group determines a Lie algebra in its tangent space at the identity element. Furthermore, to each finite dimensional real Lie algebra, there exists a corresponding local real analytic group, determining the given Lie algebra in its tangent space at the identity. Bochner [2] isolated the algebraic content of this theorem, interpreting it as giving correspondences between real Lie algebras and real formal groups. These correspondences may be refined to category equivalences.

Starting with Mal'cev's work in the 1950's [11], Lie's Theorem was extended to correspondences between local real analytic or formal Moufang loops and algebras that are now known as Mal'cev algebras. Moufang loops are diassociative, so the Baker-Campbell-Hausdorff formula still serves to give a category equivalence, just as in the group case (cf. Kuz'min [9]). Again from the 1950's onwards, Chevalley, Dieudonné and others investigated Lie's Theorem for Lie algebras over fields of prime characteristic [4], [5]. Here the Baker-Campbell-Hausdorff formula, which entails division by all primes, cannot be used, and indeed the category equivalence breaks down.

Motivated by the differential-geometric study of 3-webs, Akivis [1] sought an analogue of Lie algebras and Mal'cev algebras in the tangent space to the identity of an arbitrary real analytic loop. He referred to the algebras that he found as "W-algebras". They are now known as *Akivis algebras.* An Akivis algebra over an arbitrary ring R is an R-module A equipped with a bilinear operation $[x, y]$ known as the *(binary) commutator* and a trilinear operation (x, y, z) known as the *associator*, such that the commutator is *anticommutative*:

$$(1.1)\qquad [x, y] + [y, x] = 0,$$

and the *Akivis identity*

$$(1.2)\qquad [[x, y], z] + [[y, z], x] + [[z, x], y] = (x, y, z) + (y, z, x) + (z, x, y) - (y, x, z) - (z, y, x) - (x, z, y)$$

is satisfied.

The corresponding analogue of Lie's Theorem was given by Hofmann and Strambach ([3, Chapter IX], [8], Theorems 2.1 and 2.2 below). The category equivalence of the characteristic zero group or Moufang loop case again broke down, because of the failure of the Baker-Campbell-Hausdorff formula. This time it was the implicit (di-)associativity on which the formula depends that was not available. A discussion of the breakdown is given in [8]. (One point to add to this discussion is that a more detailed comparison with the prime-characteristic group case may prove fruitful here – perhaps incorporating co-algebra structures.)

V. V. Goldberg [3, Problem X.3.9] [12, p. 16] has raised the problem of finding an algebraic construction in the tangent bundle of the coordinate n-ary loop of an $(n+1)$-web $W(n+1, n, r)$ similar to the construction of the Akivis algebra for 3-webs or (binary) loops. The purpose of this paper is to propose such a construction in general, and in detail for the case $n = 3$ of ternary loops. The proposal has two salient features. The first, given in the third section, is the specification of two ternary analogues of the binary commutator. These are called the "commutator" (3.2) and the "translator" (3.3). The commutator is left-alternative (3.4), the translator satisfies the Jacobi identity (3.5), and together the commutator and translator satisfy a new identity called the "comtrans" identity (3.6). A ternary algebra in which these identities are satisfied is called a "comtrans algebra". Thus a comtrans algebra may be viewed as an analogue of Lie, Mal'cev, and Akivis algebras.

The second salient feature, which has no analogue in the binary cases, is that multilinear operations do not arise directly from n-ary loops with $n > 2$. They are extracted from ternary loops in the fourth section by a technique known as "masking". Setting respective single arguments of a ternary loop equal to the identity gives three binary loops called the "masks" of the ternary loop. Trilinear operations forming a comtrans algebra then appear once the effects of the masks are removed.

In the fifth section Lie's Third Fundamental Theorem for formal ternary loops is given. The first part, Theorem 5.1, implies that a local analytic ternary loop, finite dimensional over a complete normed field, determines a comtrans algebra and the three Akivis algebras of its masks in the tangent space at the identity. The second part, Theorem 5.2, constructs a local analytic ternary loop, given a comtrans algebra and three Akivis algebras, such that the constructed loop in turn determines these algebras according to Theorem 5.1. As in the case of general binary loops, there is no category equivalence. The language of formal n-ary loops is used. This language, and the results of the binary case that are needed, are reviewed in the second section. The sixth and final section indicates briefly how to deal with n-ary loops having $n > 3$. The masking technique is used to give $\binom{n}{2}$ Akivis algebras of binary loops and $\binom{n}{3}$ comtrans algebras of masked ternary loops.

2. Formal loops. This paper is based on the fundamental algebraic concepts of an r-dimensional formal n-ary loop over a ring R (with 1_R). Consider r-tuples $U_1 = (X_1, \ldots, X_r), \ldots, U_i = (X_{ir-r+1}, \ldots, X_{ir}), \ldots, U_n = (X_{nr-r+1}, \ldots, X_{nr})$ of indeterminates $X_1, \ldots, X_{nr}$. Then an *r-dimensional formal n-ary loop F over R* is an r-tuple

(2.1) $$F = (F^1, \ldots, F^r)$$

of formal power series

(2.2) $$F^i = F^i(U_1, \ldots, U_n)$$

in $R[[X_1, \ldots, X_{nr}]]$ such that

$$X_{ir-r+i} = F^i(0, \ldots, 0, U_i, 0, \ldots, 0) \tag{2.3}$$

and $F^i - X_i - X_{r+i} - \cdots - X_{nr-r+i}$ contains no terms of degree zero or one. For the purposes of proving the analogue of Lie's Third Fundamental Theorem, it is helpful to have an explicit notation for the power series (2.2). A multi-index notation similar to that of [7, A.1] is used. Let K be the index set $\{1, 2, \ldots, nr\}$. Then a *multi-index* $\mathbf{n}$ is a function $\mathbf{n}: K \to \mathbb{N}$ or element of the power $\mathbb{N}^K$ (note that the set $\mathbb{N}$ of natural numbers here, as the set of cardinalities of finite sets, includes 0). Set $|\mathbf{n}| = \sum_{i \in K} \mathbf{n}(i)$. The *multi-power* $X^{\mathbf{n}}$ denotes the monomial of total degree $|\mathbf{n}|$

$$X^{\mathbf{n}} = \prod_{i \in K} X_i^{\mathbf{n}(i)} \tag{2.4}$$

where $X_i^0 = 1_R$ by convention. The power series (2.2) may then be written explicitly as

$$F^i = X_i + \cdots + X_{nr-r+i} + \sum \{\lambda_{\mathbf{n}}^i X^{\mathbf{n}} \mid \mathbf{n} \in K, |\mathbf{n}| > 1\}, \tag{2.5}$$

where $\lambda_{\mathbf{n}}^i = 0$ if $|\mathbf{n}(K)| = 1$ by (2.3). The *m-th degree chunk* F_m of the formal loop (2.1) (cf. [7, 5.7.1]) is the formal loop $F_m = (F_m^1, \ldots, F_m^r)$ with

$$F_m^i = X_i + \cdots + X_{nr-r+i} + \sum \{\lambda_{\mathbf{n}}^i X^{\mathbf{n}} \mid \mathbf{n} \in K, 1 < |\mathbf{n}| \leqq m\}. \tag{2.6}$$

Note that F_m^i is an element of the polynomial ring $R[X_1, \ldots, X_{nr}]$. Cubic chunks F_3, quadratic chunks F_2, and linear chunks F_1 are referred to below.

Formal n-ary loops over $\mathbb{R}$ arise from Taylor expansions of local real analytic n-ary loops about their identity elements. Such local loops in turn may appear as coordinate n-ary loops of $(n+1)$-webs $W(n+1, n, r)$, as discussed by V. V. Goldberg [3, X.2B]. Thus an $(n+1)$-web $W(n+1, n, r)$ determines an r-dimensional formal n-ary loop F over $\mathbb{R}$. The Akivis and comtrans algebras associated with F for $n > 2$ according to Sections 5 and 6 below then give algebra structures in the tangent bundle of the coordinate n-ary loop, offering solutions to V. V. Goldberg's problem [3, Problem X.3.9] [12, p. 16].

The relationship between Akivis algebras and r-dimensional formal (binary) loops F over a commutative ring R may be summarized as follows for the purposes of this paper. The quadratic chunk F_2 of F determines a commutator operation

$$[a_1, a_2] = F_2(a_1, a_2) - F_2(a_2, a_1) \tag{2.7}$$

on R^r. The cubic chunk F_3 of F determines an associator operation

$$(a_1, a_2, a_3) = F_3(F_3(a_1, a_2), a_3) - F_3(a_1, F_3(a_2, a_3)) \tag{2.8}$$

on R^r.

Theorem 2.1 [8, 2.7]. *If F is an r-dimensional formal loop over a commutative ring R, then the commutator* (2.7) *and associator* (2.8) *form an Akivis algebra*

$$(R^r, [\ ,\ ], (\ ,\ ,\ )) \tag{2.9}$$

on R^r.

Theorem 2.2 [8, 2.8]. *Let R be a field of characteristic prime to 6. Let r be a positive integer. Suppose given an Akivis algebra*

$$(2.10)\qquad (R^r, [\,,\,], (\,,\,,\,)).$$

Then there is an r-dimensional formal loop F over R such that its Akivis algebra (2.9) *as determined by Theorem* 2.1 *is the given Akivis algebra* (2.10).

3. Comtrans structures. Let A be an abelian group and E a set. (In many cases, E coincides with A.) Suppose that there is a ternary operation

$$(3.1)\qquad E \times E \times E \to A; \quad (x, y, z) \mapsto x\,y\,z$$

called *multiplication* or the (*genuine*) *product* from E to A. This operation leads to the definition of two related ternary operations from E to A, known respectively as the *commutator* and *translator* of the given multiplication. The commutator $[x, y, z]$ is defined by

$$(3.2)\qquad [x, y, z] = x\,y\,z - y\,x\,z,$$

while the translator $\langle x, y, z\rangle$ is defined by

$$(3.3)\qquad \langle x, y, z\rangle = x\,y\,z - y\,z\,x.$$

The commutator is *left alternative* in the sense that it satisfies the identity

$$(3.4)\qquad [x, y, z] + [y, x, z] = 0.$$

The translator satisfies the *Jacobi identity*

$$(3.5)\qquad \langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle = 0.$$

Finally, the commutator and translator together satisfy the *comtrans identity*

$$(3.6)\qquad [x, y, z] + [z, y, x] = \langle x, y, z\rangle + \langle z, y, x\rangle.$$

Remarks 3.1. (i) The product (3.1) is *symmetric* (in the sense of being invariant under permutations of the direct factors in $E \times E \times E$) if the commutator and translator are identically zero.

(ii) The classical Jacobi identity for the binary commutator $[x, y] = x\,y - y\,x$ of an associative bilinear product $A \times A \to A; (x, y) \mapsto x\,y$ is a special case of the current Jacobi identity (3.5). It is obtained by observing that the repeated binary commutator $[[x, y], z]$ is the translator for the ternary multiplication

$$(3.7)\qquad A \times A \times A \to A; \quad (x, y, z) \mapsto -\,z\,x\,y - y\,x\,z.$$

Note that the commutator for this multiplication agrees with the translator, so that the comtrans identity is trivial here.

The properties (3.4)–(3.6) motivate the following definition.

Definition 3.2. Let R be a ring with 1. Then a *comtrans structure* over R is a unital R-module A with two ternary operations from a set E to A, known as the *commutator*

$[x, y, z]$ and the *translator* $\langle x, y, z\rangle$ respectively, such that the commutator is left alternative (3.4), the translator satisfies the Jacobi identity (3.5), and the two together satisfy the comtrans identity (3.6). The comtrans structure A is called a *comtrans algebra* if the commutator and translator are trilinear operations $A \times A \times A \to A$.

In a comtrans structure over a ring R in which 6 is a unit, a new operation known as the *bogus product* $/x, y, z/$ may be defined by

$$6/x, y, z/ = [x, y, z] + [y, z, x] + [z, x, y] + 2\langle x, y, z\rangle - 2\langle z, x, y\rangle \tag{3.8}$$

using the invertibility of 6 in the underlying ring. The commutator $/x, y, z/ - /y, x, z/$ of the bogus product is called the *bogus commutator*; its translator $/x, y, z/ - /y, z, x/$ is called the *bogus translator*.

Proposition 3.3. *Let R be a ring in which 6 is a unit. Then in a comtrans structure over R:*

(i) *the bogus commutator agrees with the commutator;*
(ii) *the bogus translator agrees with the translator.*

Proof. (i) The bogus commutator is one-sixth of

$$\begin{aligned}&[x, y, z] + [y, z, x] + [z, x, y] + 2\langle x, y, z\rangle - 2\langle z, x, y\rangle\\ &\quad - [y, x, z] - [x, z, y] - [z, y, x] - 2\langle y, x, z\rangle + 2\langle z, y, x\rangle.\end{aligned}$$

By the comtrans identity, the translator terms may be replaced by corresponding commutator terms. By left alternativity, each commutator is a multiple of $[x, y, z]$, $[y, z, x]$, or $[z, x, y]$. One obtains

$$\begin{aligned}&[x, y, z] + [y, z, x] + [z, x, y] + 2[x, y, z] - 2[z, x, y]\\ &\quad + [x, y, z] + [z, x, y] + [y, z, x] + 2[x, y, z] - 2[y, z, x],\end{aligned}$$

which cancels to $6[x, y, z]$.

(ii) The bogus translator is one-sixth of

$$\begin{aligned}&[x, y, z] + [y, z, x] + [z, x, y] + 2\langle x, y, z\rangle - 2\langle z, x, y\rangle\\ &\quad - [y, z, x] - [z, x, y] - [x, y, z] - 2\langle y, z, x\rangle + 2\langle x, y, z\rangle.\end{aligned}$$

The commutator terms cancel immediately, while the translator terms reduce to $6\langle x, y, z\rangle$ by the Jacobi identity.

4. Masking. Let F be an r-dimensional formal ternary loop over a commutative ring R. The cubic chunk F_3 defines an operation

$$R^r \times R^r \times R^r \to R^r; \quad (a_1, a_2, a_3) \mapsto F_3(a_1, a_2, a_3). \tag{4.1}$$

However, the commutator and translator of this operation are not multilinear. Indeed, the commutator and translator of the operation F_3 are not even guaranteed to vanish if any one of their arguments vanishes. To overcome this problem, the technique of *masking* is used.

The *masks* of the formal ternary loop F are the formal (binary) loops

$$F\{1\}(U_2, U_3) = F(0, U_2, U_3),$$
(4.2) $$F\{2\}(U_1, U_3) = F(U_1, 0, U_3),$$
$$F\{3\}(U_1, U_2) = F(U_1, U_2, 0),$$

known respectively as the 1-*mask*, the 2-*mask*, and the 3-*mask* (cf. [6, (17)]). The *masked version* $M = M(F)$ of the formal ternary loop F is then defined by

(4.3) $$M = F\{1\} + F\{2\} + F\{3\} - F.$$

Proposition 4.1. *Let M be the masked version of the formal ternary loop F.*

(i) *M is a formal ternary loop.*
(ii) *The cubic chunk M_3 of M is the masked version $M(F_3)$ of the cubic chunk F_3 of F.*
(iii) *The commutator and translator of the ternary operation M_3 on R^r are trilinear.*

Proof. (i) If $F^i = X_i + X_{r+i} + X_{2r+i} + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}^K, |\mathbf{n}| > 1\}$ as in (2.4), with $K = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$ and $\mathrm{I} = \{1, \ldots, r\}$, $\mathrm{II} = \{r+1, \ldots, 2r\}$, $\mathrm{III} = \{2r+1, \ldots, 3r\}$, then

$$F\{1\}^i = X_{r+i} + X_{2r+i} + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{I}) = \{0\}, |\mathbf{n}| > 1\},$$
(4.4) $$F\{2\}^i = X_i + X_{2r+i} + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{II}) = \{0\}, |\mathbf{n}| > 1\},$$
$$F\{3\}^i = X_i + X_{r+i} + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{III}) = \{0\}, |\mathbf{n}| > 1\}.$$

Thus by (4.3)

$$M^i = X_i + X_{r+i} + X_{2r+i} - \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}^K, \mathbf{n}(\mathrm{I}) \neq \{0\}, \mathbf{n}(\mathrm{II}) \neq \{0\}, \mathbf{n}(\mathrm{III}) \neq \{0\}\},$$

where the latter sum includes no terms of degree less than 3. It follows that the linear chunk of M^i is just $X_i + X_{r+i} + X_{2r+i}$, so that M is a formal ternary loop.

(ii) By (4.4) the cubic chunk M_3 of M has

(4.5) $$M^i_3 = X_i + X_{r+i} + X_{2r+i} - \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \sum\mathbf{n}(\mathrm{I}) = \sum\mathbf{n}(\mathrm{II}) = \sum\mathbf{n}(\mathrm{III}) = 1\}.$$

The cubic chunk F_3 of F has

(4.6) $$\begin{aligned} F^i_3 = {} & X_i + X_{r+i} + X_{2r+i} \\ & + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{II}) \neq \mathbf{n}(\mathrm{I}) = \{0\} \neq \mathbf{n}(\mathrm{III}), |\mathbf{n}| \leqq 3\} \\ & + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{I}) \neq \mathbf{n}(\mathrm{II}) = \{0\} \neq \mathbf{n}(\mathrm{III}), |\mathbf{n}| \leqq 3\} \\ & + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{I}) \neq \mathbf{n}(\mathrm{III}) = \{0\} \neq \mathbf{n}(\mathrm{II}), |\mathbf{n}| \leqq 3\} \\ & + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \sum\mathbf{n}(\mathrm{I}) = \sum\mathbf{n}(\mathrm{II}) = \sum\mathbf{n}(\mathrm{III}) = 1\}. \end{aligned}$$

Since

(4.7)
(i) $$F_3\{1\}^i = X_{r+i} + X_{2r+i} + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{II}) \neq \mathbf{n}(\mathrm{I}) = \{0\} \neq \mathbf{n}(\mathrm{III}), |\mathbf{n}| \leqq 3\},$$
(ii) $$F_3\{2\}^i = X_i + X_{2r+i} + \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{I}) \neq \mathbf{n}(\mathrm{II}) = \{0\} \neq \mathbf{n}(\mathrm{III}), |\mathbf{n}| \leqq 3\},$$

(iii) $F_3\{3\}^i = X_i + X_{r+i}$
$$+ \sum\{\lambda^i_{\mathbf{n}} X^{\mathbf{n}} \mid \mathbf{n}(\mathrm{II}) \dotplus \mathbf{n}(\mathrm{I}) = \{0\} \dotplus \mathbf{n}(\mathrm{III}), |\mathbf{n}| \leqq 3\},$$

it follows that

$$M_3 = F_3\{1\} + F_3\{2\} + F_3\{3\} - F_3 = M(F_3), \tag{4.8}$$

as required.

(iii) By (4.5) and the summation convention,

$$M_3^i(a_1^j, a_2^k, a_3^l) = a_1^i + a_2^i + a_3^i - \lambda^i_{jkl} a_1^j a_2^k a_3^l, \tag{4.9}$$

writing $\lambda^i_{jkl} = \lambda^i_{\mathbf{n}}$ for $\mathbf{n}(j) = \mathbf{n}(r+k) = \mathbf{n}(2r+l) = 1$, the mapping $\mathbf{n}: K \to \mathbb{N}$ having $\sum \mathbf{n}(\mathrm{I}) = \sum \mathbf{n}(\mathrm{II}) = \sum \mathbf{n}(\mathrm{III}) = 1$. Thus the i-th component of the commutator is

$$[a_1^j, a_2^k, a_3^l]^i = (\lambda^i_{kjl} - \lambda^i_{jkl})\, a_1^j a_2^k a_3^l, \tag{4.10}$$

while the i-th component of the translator is

$$\langle a_1^j, a_2^k, a_3^l\rangle^i = (\lambda^i_{klj} - \lambda^i_{jkl})\, a_1^j a_2^k a_3^l. \tag{4.11}$$

Both of these are trilinear. (Note that the commutator and translator of $a_1^i + a_2^i + a_3^i$ vanish in accordance with Remark 3.1 (i).)

Corollary 4.2. *The commutator and translator of $M_3 = M(F_3)$ determine a comtrans algebra on R^r.*

Definition 4.3. The comtrans algebra of Corollary 4.2 is called the *comtrans algebra of the ternary loop F.*

5. The Fundamental Theorem. Let F be an r-dimensional formal ternary loop over a commutative ring R. Each mask of F is a formal binary loop, and so determines an Akivis algebra on R^r by Theorem 2.1. Denote the Akivis algebra of the α-mask by $(R^r, [\,,]_\alpha, (\,,\,,)_\alpha)$ for $\alpha = 1, 2, 3$. The formal analogue of the first part of Lie's Third Fundamental Theorem for formal ternary loops may then be stated as follows, summarizing the above results.

Theorem 5.1. *An r-dimensional formal ternary loop F over a commutative ring R determines an algebra structure*

$$(R^r, [\,,]_\alpha, (\,,\,,)_\alpha, [\,,\,,], \langle\,,\,,\rangle) \tag{5.1}$$

on R^r, $1 \leqq \alpha \leqq 3$, comprising its comtrans algebra $(R^r, [\,,\,,], \langle\,,\,,\rangle)$ and the three Akivis algebras $(R^r, [\,,]_\alpha, (\,,\,,)_\alpha)$ of its α-masks.

The converse of Theorem 5.1 completes the analogue of Lie's Third Fundamental Theorem for ternary loops.

Theorem 5.2. *Let R be a field of characteristic prime to 6. Let r be a positive integer. Suppose given Akivis algebras*

$$(R^r, [\,,]_\alpha, (\,,\,,)_\alpha) \tag{5.2}$$

for $\alpha = 1, 2, 3$ *and a comtrans algebra*

(5.3) $$(R^r, [\ ,\ ,\ ], \langle\ ,\ ,\ \rangle).$$

Then there is an r-dimensional formal ternary loop F over R such that (5.3) *is its comtrans algebra and such that the Akivis algebras* (5.2) *are the respective Akivis algebras of its α-masks for* $\alpha = 1, 2, 3$.

Proof. Use notation as in the proof of Proposition 4.1. By Theorem 2.2, the Akivis algebras (5.2) determine r-dimensional formal loops $G_1^i(U_2, U_3) = (3.7)$ (i), $G_2^i(U_1, U_3) = (3.7)$ (ii), and $G_3^i(U_1, U_2) = (3.7)$ (iii), whose Akivis algebras are again the Akivis algebras (5.2). Now consider the commutator and translator (5.3). They determine a trilinear bogus product

(5.4) $$R^r \times R^r \times R^r \to R^r; \quad (a_1, a_2, a_3) \mapsto /a_1, a_2, a_3/$$

according to (3.8), say

(5.5) $$/a_1^j, a_2^k, a_3^l/ = -\lambda_{jkl}^i a_1^j a_2^k a_3^l$$

with the summation convention. By Proposition 3.3, the components of the commutator and translator (5.3) are then given by (4.10) and (4.11) respectively. Set $\lambda_{jkl}^i = \lambda_{\mathbf{n}}^i$ for $\mathbf{n}(j) = \mathbf{n}(r+k) = \mathbf{n}(2r+l) = 1$, the mapping $\mathbf{n}: K \to \mathbb{N}$ having $\sum \mathbf{n}(\mathrm{I}) = \sum \mathbf{n}(\mathrm{II}) = \sum \mathbf{n}(\mathrm{III}) = 1$. Define the r-dimensional formal ternary loop F by

(5.6) $$F^i = X_i + X_{r+i} + X_{2r+i} + \sum \{\lambda_{\mathbf{n}}^i X^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}^k, 1 < |\mathbf{n}| \leqq 3\}.$$

Since F is equal to its cubic chunk F_3, it may also be written in the form (4.6). Thus the α-masks are G_1^i, G_2^i, G_3^i respectively, having (5.2) as their Akivis algebras. The masked version of F is given by (4.8) and (4.5), having (4.10) and (4.11) as its commutator and translator. The comtrans algebra of F is thus the given (5.3).

6. The general case. Let $F = F(U_1, \ldots, U_n)$ be a formal n-ary loop. For each $(n-2)$-element subset α of $\{1, 2, \ldots, n\}$ with complement $\{i, j\}$, the *α-mask* of F is the formal (binary) loop

(6.1) $$F\alpha(U_i, U_j) = F(0, \ldots, U_i, \ldots, U_j, \ldots, 0)$$

obtained by setting $U_k = 0$ for k in α. For each $(n-3)$-element subset β of $\{1, 2, \ldots, n\}$ with complement $\{i, j, k\}$, the *β-mask* of F is the formal ternary loop

(6.2) $$F\beta(U_i, U_j, U_k) = F(0, \ldots, U_i, \ldots, U_j, \ldots, U_k, \ldots, 0)$$

obtained by setting $U_l = 0$ for l in β. Note that for a given β, the three α-masks of F with $\beta \subset \alpha$ are the masks of the formal ternary loop $F\beta$ in the sense of (4.2). Then the formal n-ary loop F determines $\binom{n}{2}$ Akivis algebras and $\binom{n}{3}$ comtrans algebras. The Akivis algebras come from the α-masks, and the comtrans algebras come from the masked versions of the β-masks. Conversely, given $\binom{n}{2}$ Akivis algebras and $\binom{n}{3}$ comtrans algebras, indexed appropriately by $(n-2)$-element subsets α and $(n-3)$-element subsets β of $\{1, \ldots, n\}$, one may build (the cubic chunk of) a formal n-ary loop yielding these given Akivis and comtrans algebras according to the above construction.

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