Lambda-rings of automorphisms

Jonathan D. H. Smith

Abstract. The set of automorphisms of a single algebra forms a group under composition. This paper studies the full set of all automorphisms of all algebra structures in a given abstract class on a fixed finite set. A lambda-ring, known as the automorphism type ring, is associated with this full set of automorphisms, and its structure is described. For certain classes (such as the variety of lattices), the full set of automorphisms is represented up to conjugation in the symmetric group by the automorphisms of a single, so-called representative algebra. In this case, the automorphism type ring is a subring of the ring of class functions on the automorphism group of the representative algebra.

1. Introduction

One of the standard exercises in universal algebra considers the set of automorphisms of a given algebra. This set of automorphisms forms a group under composition. The intention of the current paper is to study the set $\mathrm{Aut}(A, \mathcal{C})$ of all automorphisms of all those algebras on a fixed set A that lie in a certain abstract class $\mathcal{C}$ of algebras. This global approach to automorphisms (and autotopies) has recently received increasing attention in the theory of quasigroups (cf. e.g. [2, 4]).

Taking $\mathcal{C}$ as an abstract class means that the set $\mathrm{Aut}(A, \mathcal{C})$ is a union of conjugacy classes in the group $A!$ of all permutations (bijective self-maps) of the set A (§2). For a finite set A, each such conjugacy class is specified by its cycle type, the partition of the size $|A|$ of A given by the lengths of the orbits on A under the action of a member of the conjugacy class. Thus one of the goals of the program is to specify, for a given abstract class $\mathcal{C}$ and for each natural number n, a list of the partitions of n that arise as cycle types of automorphisms of $\mathcal{C}$-algebras of order n. Since this set of cycle types will be empty if and only if n does not lie in the finite spectrum of $\mathcal{C}$, the specification of the automorphism cycle types subsumes the specification of the finite spectrum of the given class $\mathcal{C}$. Tables 1–4 present sample lists of automorphism cycle types for Boolean algebras, GF(2)-spaces, abelian groups, and quasigroups, for orders up to 8. It is apparent that decreasing structure (e.g., fewer identities or fewer derived operations) progressively admits new cycle types.

For certain abstract classes $\mathcal{C}$ and cardinalities n, there is a single algebra of order n whose automorphisms include all the possible automorphism cycle types of $\mathcal{C}$-algebras of order n. Such an algebra is known as a representative algebra (Definition 3.1). The varieties of semilattices and lattices have a representative

1991 *Mathematics Subject Classification*: 08A35, 19A22.
Key words and phrases: variety, automorphism, λ-ring, categorical in power, fusion.

algebra in each (finite) cardinality (Theorems 3.2, 3.4). An example of a small abstract class with no representative algebra is also given. In general, it may be difficult to determine whether a given abstract class has a representative algebra of a given finite order (Problem 3.6).

While the set of automorphisms of a single algebra has a group structure under composition, the question of the algebraic structure of the full sets $\mathrm{Aut}(A,\mathcal{C})$ is not so easily resolved. In Sections 6 and 7, the structure of a λ-ring, the so-called automorphism type ring, is associated with $\mathrm{Aut}(A,\mathcal{C})$ for each abstract class $\mathcal{C}$ and set A of finite order n. As recalled in Section 4, λ-rings are commutative, unital rings or $\mathbb{Z}$-algebras with additional unary operations, the lambda operations λ^k defined for each natural number k. In the ring of integers, $\lambda^k(l)$ is the binomial coefficient $\binom{l}{k}$. If the underlying ring is actually a $\mathbb{Q}$-algebra, then an alternative axiomatization is offered by the unary Adams operations ψ^k for positive integers k (Section 5). The automorphism type rings are actually $\mathbb{C}$-algebras, whose complex dimensions agree with the total number of automorphism cycle types. The Adams operations then give vestiges of the positive powers in automorphism groups. The two final sections are devoted to the statement and proof of the main Theorem 8.3, which describes the structure of the automorphism type rings.

Readers may consult [7] for concepts and conventions that are not otherwise referenced in the paper.

2. Automorphisms

An algebra A with operator domain Ω will often be written as a triple (A,Ω,ρ), where ρ denotes the action of Ω on A. Thus if ω is a k-ary operator, the operation of ω on A is written as

$$A^k \to A; (a_1,\dots,a_k) \mapsto a_1 \dots a_k\omega^\rho$$

in algebraic notation. The following lemma is of course extremely well known, but it will serve as a basis for subsequent considerations.

Lemma 2.1. *The set* $\mathrm{Aut}(A,\Omega,\rho)$ *of all automorphisms of an algebra* (A,Ω,ρ) *forms a group under composition. In particular, if* r *is a natural number and* α *is an automorphism of* (A,Ω,ρ)*, then so is* α^r*.*

Let $\mathcal{C}$ be a class of algebras. Let A be a set, with group $A!$ of permutations. Write $(A,\mathcal{C})$ for the set of all algebra structures on the set A that belong to the class $\mathcal{C}$. Further, write

$$\begin{aligned}\mathrm{Aut}(A,\mathcal{C}) &= \{\alpha \in A! \mid \exists\ (A,\Omega,\rho) \in (A,\mathcal{C})\,.\ \alpha \in \mathrm{Aut}(A,\Omega,\rho)\}\\ &= \bigcup_{(A,\Omega,\rho)\in(A,\mathcal{C})} \mathrm{Aut}(A,\Omega,\rho)\,.\end{aligned}$$

Recall that a class $\mathcal{C}$ of algebras is *abstract* if it is closed under the isomorphism closure operator I, so that $B \cong A \in \mathcal{C}$ implies $B \in \mathcal{C}$. The (combinatorial) *finite spectrum* of an abstract class $\mathcal{C}$ is the set of natural numbers n for which there is a $\mathcal{C}$-algebra of order n. One of the main concerns of this paper is the study of the sets

$\mathrm{Aut}(A, \mathcal{C})$ for an abstract class $\mathcal{C}$. Note that for a finite set A, the set $\mathrm{Aut}(A, \mathcal{C})$ is nonempty if and only if $|A|$ lies in the finite spectrum of $\mathcal{C}$. Thus $\mathrm{Aut}(A, \mathcal{C})$ certainly embodies all the information in the finite spectrum.

If π is an element of the permutation group $A!$, a new algebra $(A, \Omega, \rho\pi)$ is defined by

$$a_1 \dots a_k \omega^{\rho\pi} := (a_1\pi^{-1}) \dots (a_k\pi^{-1})\omega^{\rho}\pi \tag{2.1}$$

so that $\pi : (A, \Omega, \rho) \to (A, \Omega, \rho\pi)$ is an isomorphism.

Lemma 2.2. *Let $\mathcal{C}$ be an abstract class of algebras.*

(a) *$(A, \mathcal{C})$ is a right $A!$-set under the action $\pi : (A, \Omega, \rho) \mapsto (A, \Omega, \rho\pi)$.*
(b) *$\forall\ \alpha \in \mathrm{Aut}(A, \mathcal{C})\,,\ \forall\ \pi \in A!\,,\ \pi^{-1}\alpha\pi \in \mathrm{Aut}(A, \mathcal{C})$.*

Proof. (a) Consider permutations π, π', and a k-ary operator ω. Then for $a_1, \dots, a_k$ in A, (2.1) gives

$$\begin{aligned} a_1 \dots a_k \omega^{(\rho\pi)\pi'} &= (a_1\pi'^{-1}) \dots (a_k\pi'^{-1})\omega^{\rho\pi}\pi' = (a_1\pi'^{-1}\pi^{-1}) \dots (a_k\pi'^{-1}\pi^{-1})\omega^{\rho}\pi\pi' \\ &= a_1(\pi\pi')^{-1} \dots a_k(\pi\pi')^{-1}\omega^{\rho}(\pi\pi') = a_1 \dots a_k \omega^{\rho(\pi\pi')} \end{aligned}$$

as required.

(b) Consider the diagram

$$\begin{array}{ccc} (A, \Omega, \rho) & \xrightarrow{\ \pi\ } & (A, \Omega, \rho\pi) \\ {\scriptstyle\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi^{-1}\alpha\pi} \\ (A, \Omega, \rho) & \xrightarrow[\ \pi\ ]{} & (A, \Omega, \rho\pi) \end{array}$$

of isomorphisms of $\mathcal{C}$-algebras. □

Lemma 2.2 shows that for an abstract class $\mathcal{C}$, the elements of $\mathrm{Aut}(A, \mathcal{C})$ form complete conjugacy classes in the permutation group $A!$. If A has finite order n, then each such conjugacy class is specified as the class C_τ, for a partition τ of n, consisting of all the permutations of A that have cycle type τ. Tables 1–4 list the cycle types of automorphisms of small algebras for the respective classes of Boolean algebras, vector spaces over GF(2), abelian groups, and quasigroups (compare [4, 5]). Two further examples, semilattices and lattices, are analyzed in the following section.

3. Representative algebras

Definition 3.1. For an abstract class $\mathcal{C}$, an algebra (A, Ω, ρ) in $\mathcal{C}$ is said to be *representative* if each element of $\mathrm{Aut}(A, \mathcal{C})$ is conjugate in $A!$ to an automorphism of (A, Ω, ρ).

Recall the notation $\tau \vdash n$ to express that τ is a partition of a positive integer n. In this paper, partitions are written in product form (compare [7, p.50]).

Theorem 3.2. *For $1 < n \in \mathbb{Z}$, let A be a set of order n. Let $\mathcal{H}$ be the variety of semilattices. Then*

$$\mathrm{Aut}(A, \mathcal{H}) = \bigcup_{\mu \vdash (n-1)} C_{\mu \cdot 1} .$$

Proof. Each automorphism α of a meet semilattice structure $(A, \cdot)$ on A has to fix the lower bound $\prod_{a \in A} a$ of the poset $(A, \leq.)$. Thus α has cycle type $\mu \cdot 1$ with $\mu \vdash (n-1)$. For the converse, choose an element $\perp$ of A. Consider the ordinal sum $\{\perp\} \oplus (A \smallsetminus \{\perp\})$ of $\{\perp\}$ with the unordered set $A \smallsetminus \{\perp\}$ as a meet semilattice. Then each permutation of A that fixes $\perp$ is an automorphism of this semilattice structure on A. □

Corollary 3.3. *The semilattice $\{\perp\} \oplus (A \smallsetminus \{\perp\})$ is representative for $\mathcal{H}$.*

Theorem 3.4. *For $2 < n \in \mathbb{Z}$, let A be a set of order n. Let $\mathcal{L}$ be the variety of lattices. Then*

$$\mathrm{Aut}(A, \mathcal{L}) = \bigcup_{\mu \vdash (n-2)} C_{\mu \cdot 1^2} .$$

Proof. To establish the containment $\mathrm{Aut}(A, \mathcal{L}) \subseteq \bigcup_{\mu \vdash (n-2)} C_{\mu \cdot 1^2}$, note that each automorphism of a lattice structure $(A, +, \cdot)$ on A has to fix the lower bound $\prod_{a \in A} a$ and upper bound $\sum_{a \in A} a$ of the poset $(A, \leq)$. For the converse, choose distinct elements $\perp$ and $\top$ of A. Consider the ordinal sum $\{\perp\} \oplus (A \smallsetminus \{\perp, \top\}) \oplus \{\top\}$ of $\{\perp\}$, the unordered set $A \smallsetminus \{\perp, \top\}$, and $\{\top\}$ as a lattice. Then each permutation of A that fixes $\perp$ and $\top$ is an automorphism of this lattice structure on A. □

Corollary 3.5. *The lattice $\{\perp\} \oplus (A \smallsetminus \{\perp, \top\}) \oplus \{\top\}$ is representative for $\mathcal{L}$.*

The following problem appears to be difficult in general.

Problem 3.6. Given an abstract class $\mathcal{C}$, and a member n of the finite spectrum of $\mathcal{C}$, determine if there is a representative algebra of order n for $\mathcal{C}$.

To contrast the preceding positive results for the varieties of semilattices and lattices, we present one negative result.

Proposition 3.7. *For the symmetric group S_4 and additive group $(\mathbb{Z}/_{24}, +)$ of integers modulo 24, define $\mathcal{C} = \mathrm{I}\{S_4, (\mathbb{Z}/_{24}, +)\}$. Then there is no representative algebra for the abstract class $\mathcal{C}$.*

Proof. The automorphisms of the additive group $\mathbb{Z}/_{24}$, namely multiplications by residues coprime to 24, have cycle types $2^r 1^{24-2r}$ for $r = 0, 6, 8, 9, 10, 11$. On the other hand, the automorphism group of S_4 acts faithfully on the 4-element set of Sylow 3-subgroups of S_4. Since $Z(S_4)$ is trivial, $\mathrm{Aut}(S_4)$ consists entirely of the 24 inner automorphisms. These have cycle types 1^{24}, $2^{10}1^4$, $2^9 1^6$, $3^7 1^3$, and $4^4 2^2 1^4$. The two latter types are not represented in $\mathbb{Z}/_{24}$, while $2^6 1^{12}$, $2^8 1^8$, and $2^{11} 1^2$ are not represented in S_4. □

Recall that an abstract class $\mathcal{C}$ is *categorical in power* $|A|$ if, up to isomorphism, there is at most one $\mathcal{C}$-algebra structure (A, Ω, ρ) on the set A. If such an algebra

(A, Ω, ρ) does exist in this case, it is certainly representative. The following example serves to point out the distinction between $\mathrm{Aut}(A, \Omega, \rho)$ and $\mathrm{Aut}(A, \mathcal{C})$.

Example 3.8. The class $\mathcal{A}$ of abelian groups is categorical in power 3. However, while the automorphism group of the abelian group of integers $\{1, 2, 3\}$ modulo 3 under addition is the group $\{(1), (1\ 2)\}$, the set $\mathrm{Aut}(\{1, 2, 3\}, \mathcal{A})$ is the union $\{(1), (1\ 2), (2\ 3), (3\ 1)\}$ of S_3-conjugacy classes. This set does not form a subgroup of the symmetric group S_3.

4. λ-rings

In order to associate an algebra structure with the sets $\mathrm{Aut}(A, \mathcal{C})$ for a finite set A and an abstract class $\mathcal{C}$, we will consider λ-rings. A *λ-ring* R [1, §1][3, §3.1][1] is a commutative, unital ring equipped with unary *λ-operations* λ^n for each natural number n, such that the identities

$$\lambda^0(x) = 1\,, \qquad \lambda^1(x) = x\,,$$

and

$$\lambda^n(x + y) = \sum_{k=0}^{n} \lambda^k(x)\lambda^{n-k}(y) \tag{4.1}$$

are satisfied. Defining the generating function

$$\lambda_t(x) = \sum_{n=0}^{\infty} \lambda^n(x) t^n$$

for each element x of R, with indeterminate t, the identity (4.1) may be rewritten in the form $\lambda_t(x + y) = \lambda_t(x)\lambda_t(y)$.

Example 4.1. The ring $\mathbb{Z}$ of integers becomes a λ-ring with $\lambda_t(1) = 1 + t$, the identity (4.1) reducing to the relationship

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$$

between binomial coefficients.

Definition 4.2. An subset I of a λ-ring R is said to be a *λ-ideal* if

(a) I is an ideal of the ring R;
(b) For each element r of I, one has $\lambda^k(r) \in I$ for $k > 0$.

Using (4.1), one may readily verify the following.

Proposition 4.3. *If I is a λ-ideal of a λ-ring R, then for each natural number n,*

$$\lambda^n(r + I) = \lambda^n(r) + I$$

gives a well-defined λ-operation on R/I, making R/I a λ-ring.

[1] Some authors say *pre-λ-ring*, reserving the term *λ-ring* for the special λ-rings of Definition 4.4 — compare [6, pp.7, 13].

Now let $\xi_1, \dots, \xi_q$, $\eta_1, \dots, \eta_r$ be indeterminates. Use

$$\sum_{i=0}^{\infty} s_i t^i = \prod_{k=1}^{q} (1 + \xi_k t) \quad \text{and} \quad \sum_{i=0}^{\infty} \sigma_i t^i = \prod_{k=1}^{q} (1 + \eta_k t)$$

to define the *elementary symmetric functions*

$$s_i(\xi_1, \dots, \xi_q) \,, \qquad \sigma_i(\eta_1, \dots, \eta_r) \,.$$

Then define $P_n(s_1, \dots, s_n; \sigma_1, \dots, \sigma_n)$ to be the coefficient of t^n in

$$\prod_{i=1}^{q} \prod_{j=1}^{r} (1 + \xi_i \eta_j t) \ .$$

Define $P_{n,d}(s_1, \dots, s_{nd})$ to be the coefficient of t^n in

$$\prod_{1 \le i_1 < \dots < i_m \le d} (1 + \xi_{i_1} \dots \xi_{i_m} t) \ .$$

Definition 4.4. A λ-ring is said to be *special* if it satisfies the identities

$$\lambda^n(xy) = P_n\left(\lambda^1(x), \dots, \lambda^n(x); \lambda^1(y) \dots \lambda^n(y)\right)$$

and

$$\lambda^m\left(\lambda^n(x)\right) = P_{m,n}\left(\lambda^1(x), \dots, \lambda^{mn}(x)\right) \tag{4.2}$$

for all natural numbers m and n.

Remark 4.5. Setting $n = 0$ in (4.2) yields $\lambda_t(1) = 1 + t$. Thus the λ-ring structure of Example 4.1 is the unique special λ-ring structure on the ring $\mathbb{Z}$ of integers.

5. Adams operations and class functions

Let R be a commutative $\mathbb{Q}$-algebra. For the underlying ring structure R, an alternative specification of λ-rings is given by the unary *Adams operations* ψ^k for positive integers k. Without going into the full detail available in the standard references [1, 3, 6], it will suffice here to point out that the $\psi^k : R \to R$ are unital ring homomorphisms with $\psi^k \circ \psi^l = \psi^{k+l}$, and that

$$\lambda^k = \frac{1}{k!} \begin{vmatrix} \psi^1 & 1 & 0 & 0 & \dots & 0 \\ \psi^2 & \psi^1 & 2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ \psi^{k-1} & \psi^{k-2} & \dots & \ddots & \psi^1 & k-1 \\ \psi^k & \psi^{k-1} & \dots & & \psi^2 & \psi^1 \end{vmatrix} , \tag{5.1}$$

for $0 < k, l \in \mathbb{Z}$ [6, pp.49–50].

Let G be a finite group. A *complex class function* on G is a function

$$\chi : G \to \mathbb{C}; x \mapsto \langle x, \chi \rangle$$

such that

$$\forall\ g \in G\,,\ \langle g^{-1}xg, \chi\rangle = \langle x, \chi\rangle\,.$$

Consider the $\mathbb{C}$-algebra $\mathrm{Cl}(G)$ of all complex class functions on G, with pointwise ring structure, complex scalar multiplication, and pointwise complex conjugation. Define Adams operations on $\mathrm{Cl}(G)$ by

$$\langle x, \psi^k(\chi)\rangle = \langle x^k, \chi\rangle \tag{5.2}$$

for $x \in G$, $\chi \in \mathrm{Cl}(G)$ and $0 < k \in \mathbb{Z}$, noting

$$\langle g^{-1}xg, \psi^k(\chi)\rangle = \langle (g^{-1}xg)^k, \chi\rangle = \langle g^{-1}x^k g, \chi\rangle = \langle x^k, \chi\rangle = \langle x, \psi^k(\chi)\rangle$$

for g in G. It is also worth recording the *periodicity* identities

$$\psi^{k+|G|}(\chi) = \psi^k(\chi) \tag{5.3}$$

for $0 < k \in \mathbb{Z}$, following from

$$\langle x, \psi^{k+|G|}(\chi)\rangle = \langle x^{k+|G|}, \chi\rangle = \langle x^k, \chi\rangle = \langle x, \psi^k(\chi)\rangle$$

for x in G. Now λ-operations are given on $\mathrm{Cl}(G)$ by (5.1).

Theorem 5.1. [6, p.54] *The complex class function ring* $\mathrm{Cl}(G)$ *of a finite group* G *forms a special* λ*-ring.*

6. Automorphism annihilators

Let $\mathcal{C}$ be an abstract class of algebras.

Definition 6.1. A complex class function κ on the symmetric group S_n is said to be a $\mathcal{C}$-*automorphism annihilator* if $\langle \alpha, \kappa\rangle = 0$ for each automorphism α of a $\mathcal{C}$-algebra of order n.

For each natural number n, define the *automorphism kernel* $\mathrm{AK}_n(\mathcal{C})$ to be the complex vector space that consists of all the $\mathcal{C}$-automorphism annihilators on S_n.

Example 6.2. Let $\mathcal{Q}$ be the variety of quasigroups (Table 4).
(a) The space $\mathrm{AK}_3(\mathcal{Q})$ is trivial.
(b) The space $\mathrm{AK}_4(\mathcal{Q})$ is spanned by the characteristic function of the conjugacy class of permutations of cycle type 4^1.

Remark 6.3. There are differing conventions for the characteristic function $\chi_X : A \to \{0, 1\}$ of a subset X of a set A. Generally speaking, X is taken as the preimage of $\{1\}$ in mathematics, and as the preimage of $\{0\}$ in logic. This paper follows the mathematical convention.

The following proposition shows how automorphism annihilators may be used to identify $\mathcal{C}$-algebra automorphisms.

Proposition 6.4. *For a natural number* n*, let* π *be an element of* S_n*. Then* π *is an automorphism of an* n*-element* $\mathcal{C}$*-algebra if and only if*

$$\forall\ \kappa \in \mathrm{AK}_n(\mathcal{C})\,,\ \langle \pi, \kappa\rangle = 0\,. \tag{6.1}$$

Proof. The "only if" direction is immediate from Definition 6.1. For the converse, suppose that a permutation π satisfies (6.1). Let N be the subset of S_n consisting of permutations which are not $\mathcal{C}$-algebra automorphisms. The characteristic function χ_N of N is a class function on S_n by Lemma 2.2(b), and is a $\mathcal{C}$-automorphism annihilator. By (6.1), χ_N takes the value zero on π. Thus π is identified as a $\mathcal{C}$-algebra automorphism. □

Remark 6.5. To verify (6.1) for a permutation π, it of course suffices to check the condition $\langle \pi, \kappa \rangle = 0$ for κ from a basis of the automorphism kernel $\mathrm{AK}_n(\mathcal{C})$.

Theorem 6.6. *For each natural number n, the automorphism kernel $\mathrm{AK}_n(\mathcal{C})$ forms a λ-ideal in the special λ-ring $\mathrm{Cl}(S_n)$.*

Proof. Let α be a $\mathcal{C}$-algebra automorphism. Suppose that κ is a $\mathcal{C}$-automorphism annihilator. Then for each class function χ of S_n,

$$\langle \alpha, \kappa\chi \rangle = \langle \alpha, \kappa \rangle \langle \alpha, \chi \rangle = 0\,.$$

Thus the space $\mathrm{AK}_n(\mathcal{C})$ is an ideal of the ring $\mathrm{Cl}(S_n)$. Further, one has

$$\langle \alpha, \psi^k(\kappa) \rangle = \langle \alpha^k, \kappa \rangle = 0$$

by (5.2) and Lemma 2.1. Thus the space $\mathrm{AK}_n(\mathcal{C})$ is closed under the various Adams operations ψ^k for positive integers k. The corresponding λ-operations are given by the determinant (5.1). Since each term of the determinant's Laplace expansion down the first column involves at least one Adams operation with positive index, the set $\mathrm{AK}_n(\mathcal{C})$ is closed under the λ-operations λ^k for positive integers k. □

Corollary 6.7. *Let n be a natural number. Then the quotient $\mathrm{Cl}(S_n)/\mathrm{AK}_n(\mathcal{C})$ is a special λ-ring whose dimension is equal to the number of conjugacy classes of $\mathcal{C}$-algebra automorphisms in S_n.*

Proof. Apply Proposition 4.3. □

Definition 6.8. For a natural number n and abstract class $\mathcal{C}$, the special λ-ring

$$\mathrm{ATR}_n(\mathcal{C}) = \mathrm{Cl}(S_n)/\mathrm{AK}_n(\mathcal{C})$$

is known as the *automorphism type ring* for $\mathcal{C}$ of *degree* n.

Let $P(n)$ be the total number of integer partitions of n. Corollary 6.7 gives the following *Duality Principle*, which may assist in piecing together a full list of $\mathcal{C}$-algebra automorphism types.

Proposition 6.9. *Let n be a positive integer. Let T be a set of cycle types of automorphisms of n-element $\mathcal{C}$-algebras. Let L be a linearly independent subset of the automorphism kernel $\mathrm{AK}_n(\mathcal{C})$. Then*

$$|T| + |L| \leq P(n)\,. \tag{6.2}$$

Equality holds in (6.2) if and only if L spans $\mathrm{AK}_n(\mathcal{C})$ and T contains all the cycle types of automorphisms of $\mathcal{C}$-algebras of order n.

7. Automorphism type rings

Let $\mathcal{C}$ be an abstract class of algebras, and let n be a positive integer. The goal of the final two sections is to describe the structure of the automorphism type ring $\mathrm{ATR}_n(\mathcal{C})$.

Proposition 7.1. *Let (A,Ω,ρ) be a $\mathcal{C}$-algebra of order n, with automorphism group of order r. Let*

$$\{D_{ij} \mid 1 \le i \le l\,,\ 1 \le j \le m_i\}$$

be the full set of conjugacy classes of the automorphism group $\mathrm{Aut}(A,\Omega,\rho)$, indexed so that for $1 \le i \le l$, there are distinct conjugacy classes C_i of $A!$ such that the classes $D_{i1},\dots,D_{im_i}$ of $\mathrm{Aut}(A,\Omega,\rho)$ fuse into C_i in $A!$. Let $S(A,\Omega,\rho)$ be the subset of $\mathrm{Cl}\,(\mathrm{Aut}(A,\Omega,\rho))$ consisting of those complex class functions χ on $\mathrm{Aut}(A,\Omega,\rho)$ that satisfy

$$\forall\, 1 \le i \le l\,,\ \forall\, 1 \le j_1, j_2 \le m_i\,,\ \forall\, \alpha_1 \in D_{ij_1}\,, \forall\, \alpha_2 \in D_{ij_2}\,,\ \langle \alpha_1, \chi\rangle = \langle \alpha_2, \chi\rangle\,. \quad (7.1)$$

Then:

(a) *$S(A,\Omega,\rho)$ is an l-dimensional $\mathbb{C}$-subalgebra of $\mathrm{Cl}\,(\mathrm{Aut}(A,\Omega,\rho))$.*
(b) *$S(A,\Omega,\rho)$ is a λ-subring of $\mathrm{Cl}\,(\mathrm{Aut}(A,\Omega,\rho))$.*
(c) *$S(A,\Omega,\rho)$ satisfies the periodicity identities $\psi^{k+r} = \psi^k$ for $0 < k \in \mathbb{Z}$.*

Proof. Abbreviate $S(A,\Omega,\rho)$ to S.
(a) Suppose that χ and χ' are elements of S, so the quantified equations

$$\langle \alpha_1, \chi\rangle = \langle \alpha_2, \chi\rangle \quad \text{and} \quad \langle \alpha_1, \chi'\rangle = \langle \alpha_2, \chi'\rangle$$

of (7.1) are satisfied. Then

$$\big\langle \alpha_1, \chi\chi'\big\rangle = \big\langle \alpha_1, \chi\big\rangle\big\langle \alpha_1, \chi'\big\rangle = \big\langle \alpha_2, \chi\big\rangle\big\langle \alpha_2, \chi'\big\rangle = \big\langle \alpha_2, \chi\chi'\big\rangle$$

and for $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned}\big\langle \alpha_1, z_1\chi + z_2\chi'\big\rangle &= z_1\big\langle \alpha_1, \chi\big\rangle + z_2\big\langle \alpha_1, \chi'\big\rangle \\ &= z_1\big\langle \alpha_2, \chi\big\rangle + z_2\big\langle \alpha_2, \chi'\big\rangle = \big\langle \alpha_2, z_1\chi + z_2\chi'\big\rangle\,.\end{aligned}$$

Finally, for $1 \le i \le l$, define the subset

$$D_i = \bigcup_{j=1}^{m_i} D_{ij}$$

of $\mathrm{Aut}(A,\Omega,\rho)$. Then the characteristic functions $\chi_{D_1},\dots,\chi_{D_l}$ form a basis of S, so $\dim_{\mathbb{C}} S = l$.

(b) Let k be a positive integer, and let χ be an element of S. For $1 \le i \le l$ and $1 \le j_1, j_2 \le m_i$, consider elements α_1 of D_{ij_1} and α_2 of D_{ij_2}, so that α_1 and α_2 are conjugate in $A!$. Since α_1^k and α_2^k are also conjugate in $A!$, there is an index $1 \le i' \le l$ and indices $1 \le j_1', j_2' \le m_{i'}$ such that $\alpha_1^k \in D_{i'j_1'}$ and $\alpha_2^k \in D_{i'j_2'}$. Then

$$\big\langle \alpha_1, \psi^k(\chi)\big\rangle = \big\langle \alpha_1^k, \chi\big\rangle = \big\langle \alpha_2^k, \chi\big\rangle = \big\langle \alpha_2, \psi^k(\chi)\big\rangle\,,$$

by (5.2), the central equality holding since $\chi \in S$. It follows that $\psi^k(\chi)$ also lies in S, as required.

(c) follows from (b) and (5.3). □

Definition 7.2. The $\mathbb{C}$-algebra and special λ-ring $S(A,\Omega,\rho)$ specified by condition (7.1) in Proposition 7.1 is called the $A!$-*fusion subring* of $\mathrm{Cl}\,(\mathrm{Aut}(A,\Omega,\rho))$.

Theorem 7.3. *For a $\mathcal{C}$-algebra (A,Ω,ρ), let*

$$\iota_{(A,\Omega,\rho)} : \mathrm{Aut}(A,\Omega,\rho) \hookrightarrow A!$$

denote the embedding of the automorphism group in the permutation group. Then the restriction map

$$\mathrm{res}_{(A,\Omega,\rho)} : \mathrm{Cl}(A!) \to S(A,\Omega,\rho); \chi \mapsto \iota_{(A,\Omega,\rho)}\chi \tag{7.2}$$

is a well-defined surjective $\mathbb{C}$-linear λ-ring homomorphism that factors through a homomorphism

$$\pi_{(A,\Omega,\rho)} : \mathrm{ATR}_n(\mathcal{C}) \to S(A,\Omega,\rho) \tag{7.3}$$

from the automorphism type ring.

Proof. Continue the notation of Proposition 7.1. Since a class function χ of $A!$ satisfies the condition (7.1), the restriction map (7.2) is well-defined. It is clearly a $\mathbb{C}$-linear homomorphism of λ-rings. To show that it surjects, consider an element χ of S. Then a class function $\widetilde{\chi}$ of $A!$ is well-defined by

$$\langle \beta, \widetilde{\chi}\rangle = \begin{cases} \langle \gamma^{-1}\beta\gamma, \chi\rangle & \text{if } \exists\, 1 \le i \le l\,.\ \exists\, \gamma \in A!\,.\ \gamma^{-1}\beta\gamma \in D_{i1}\,; \\ 0 & \text{otherwise} \end{cases}$$

for β in $A!$. Consider an automorphism α of (A,Ω,ρ), say $\alpha \in D_{hj}$ for $1 \le h \le l$ and $1 \le j \le m_h$. Consider a permutation γ such that $\gamma^{-1}\alpha\gamma \in D_{h1}$. Then

$$\alpha\iota_{(A,\Omega,\rho)}\widetilde{\chi} = \langle \gamma^{-1}\alpha\gamma, \chi\rangle = \langle \alpha, \chi\rangle$$

since χ lies in S. Thus $\mathrm{res}_{(A,\Omega,\rho)} : \widetilde{\chi} \mapsto \chi$, as required for the surjectivity. Now let κ be an automorphism annihilator. Then for an automorphism α of (A,Ω,ρ), one has $\alpha\iota_{(A,\Omega,\rho)}\kappa = \langle \alpha, \kappa\rangle = 0$, so that κ lies in the kernel of $\mathrm{res}_{(A,\Omega,\rho)}$. It follows that $\mathrm{res}_{(A,\Omega,\rho)}$ factors through the projection $\pi_{(A,\Omega,\rho)}$ of (7.3) as claimed. □

If there is a representative $\mathcal{C}$-algebra of order n, Theorem 7.3 gives an immediate description of the automorphism type ring.

Corollary 7.4. *Let $\mathcal{C}$ be an abstract class of algebras. Let n be a positive integer, and let (A,Ω,ρ) be a representative $\mathcal{C}$-algebra of order n. Let r be the order of $\mathrm{Aut}(A,\Omega,\rho)$, and let l be the cardinality of its set of $A!$-fused conjugacy classes.*

(a) *The projection $\pi_{(A,\Omega,\rho)} : \mathrm{ATR}_n(\mathcal{C}) \to S(A,\Omega,\rho)$ is an isomorphism.*
(b) *The $\mathbb{C}$-algebra $\mathrm{ATR}_n(\mathcal{C})$ has dimension l.*
(c) *The λ-ring $\mathrm{ATR}_n(\mathcal{C})$ satisfies the identities $\psi^{k+r} = \psi^k$ for $0 < k \in \mathbb{Z}$.*

Proof. Since (A,Ω,ρ) is representative, the number of $\mathcal{C}$-algebra automorphism types is equal to l . Thus by Proposition 7.1(a), the respective dimensions of the domain and codomain of the surjective $\mathbb{C}$-linear map $\pi_{(A,\Omega,\rho)}$ agree. It follows that $\pi_{(A,\Omega,\rho)}$ is an isomorphism. The periodicity identities in $\mathrm{ATR}_n(\mathcal{C})$ then hold by Proposition 7.1(c). □

Example 7.5. As a simple illustration of the working of Corollary 7.4, consider the class $\mathcal{P}$ of nontrivial right C_p-sets, for the cyclic group C_p of prime order p. The regular right C_p-set (C_p, C_p) is a representative $\mathcal{P}$-algebra of order p. Its automorphism group is again C_p, acting as the left regular representation on the set C_p. While the conjugacy classes of the abelian group C_p are singletons, the $p-1$ nonidentity classes fuse in S_p. Thus Corollary 7.4 identifies the automorphism type ring $\mathrm{ATR}_p(\mathcal{P})$ as a 2-dimensional $\mathbb{C}$-algebra, and as a special λ-ring satisfying the periodicities $\psi^{k+p} = \psi^k$. Of course, this concurs with the direct observation that the $\mathcal{P}$-automorphism types of order p are just 1^p and p^1.

8. Representative sets

The structural description of a general automorphism type ring requires the concept of a representative algebra to be extended.

Definition 8.1. Let $\mathcal{C}$ be an abstract class of algebras, and let A be a finite set. Then a set R of $\mathcal{C}$-algebras on A is said to be *representative* if each element of $\mathrm{Aut}(A, \mathcal{C})$ is conjugate in $A!$ to an automorphism of some member (A, Ω, ρ) of R.

The following result is immediate.

Proposition 8.2. *Let $\mathcal{C}$ be an abstract class of algebras, and let A be a finite set. Then the set $\mathfrak{R}$ of all representative sets of $\mathcal{C}$-algebras on A forms a join semilattice under set-theoretical union.*

Note that $\mathfrak{R}$ is the singleton $\{\varnothing\}$ if $|A|$ is not in the spectrum of $\mathcal{C}$. If $|A|$ is in the finite spectrum, and a minimal element of $\mathfrak{R}$ is a singleton $\{(A, \Omega, \rho)\}$, then the algebra (A, Ω, ρ) is representative. In that case, Corollary 7.4 identified the automorphism type ring as an isomorphic copy of $S(A, \Omega, \rho)$. The following theorem gives the general structural description of automorphism type rings. Corollary 7.4 is recovered as the case $d = 1$ of this theorem.

Theorem 8.3. *Let $\mathcal{C}$ be an abstract class of algebras. Let A be a finite set of positive cardinality n. Let $R = \{(A, \Omega_1, \rho_1), \dots, (A, \Omega_d, \rho_d)\}$ be a minimal representative set of $\mathcal{C}$-algebras on A. For $1 \le i \le d$, let r_i be the order of $\mathrm{Aut}(A, \Omega_i, \rho_i)$, and let l_i be the cardinality of its set of $A!$-fused conjugacy classes. Let $r = \mathrm{lcm}\{r_1, \dots, r_d\}$.*

- (a) *The automorphism type ring $\mathrm{ATR}_n(\mathcal{C})$ is a subdirect product of the $A!$-fusion algebras $S(A, \Omega_i, \rho_i)$ for $1 \le i \le d$.*
- (b) *The $\mathbb{C}$-algebra $\mathrm{ATR}_n(\mathcal{C})$ has dimension satisfying*

$$d - 1 + \max\{l_i \mid 1 \le i \le d\} \le \dim_{\mathbb{C}} \mathrm{ATR}_n(\mathcal{C}) \le 1 - d + \sum_{i=1}^{d} l_i \,. \tag{8.1}$$

- (c) *The λ-ring $\mathrm{ATR}_n(\mathcal{C})$ satisfies the identities $\psi^{k+r} = \psi^k$ for $0 < k \in \mathbb{Z}$.*

Proof. (a) Consider the product

$$p : \mathrm{Cl}(A!) \to \prod_{i=1}^{d} S(A, \Omega_i, \rho_i)$$

of the surjective restriction homomorphisms $\mathrm{res}_{(A,\Omega_i,\rho_i)}$ of (7.2). Suppose that a class function κ of $A!$ lies in the kernel of p. Then $\langle \alpha, \kappa \rangle = 0$ for each element α of each of the representative automorphism groups $\mathrm{Aut}(A, \Omega_i, \rho_i)$. It follows that κ is an automorphism annihilator, so $\mathrm{Ker}\, p \leq \mathrm{AK}_n(\mathcal{C})$. Conversely, Theorem 7.3 shows that

$$\mathrm{Ker}\, p = \bigcap_{i=1}^{d} \mathrm{Ker}\, \mathrm{res}_{(A,\Omega_i,\rho_i)} \geq \mathrm{AK}_n(\mathcal{C}) \, .$$

Thus $\mathrm{Ker}\, p = \mathrm{AK}_n(\mathcal{C})$, and application of the First Isomorphism Theorem to p yields the required subdirect embedding of $\mathrm{ATR}_n(\mathcal{C})$.

(b) Since $\dim_{\mathbb{C}} S(A, \Omega_i, \rho_i) = l_i$ for $1 \leq i \leq d$ by Proposition 7.1(a), the codomain of p has dimension $\sum_{i=1}^{d} l_i$. In each factor of the codomain, a 1-dimensional subspace is spanned by the characteristic function of the subset $\{\mathrm{id}_A\}$. The image of p only covers the 1-dimensional diagonal subspace of the product of these 1-dimensional subspaces. Thus the upper bound in (8.1) holds. For the lower bound on the dimensionality of $\mathrm{ATR}_n(\mathcal{C})$, or equivalently (by Corollary 6.7) on the number of automorphism types of $\mathcal{C}$-algebras on A, consider the $l_j = \max\{l_i \mid 1 \leq i \leq d\}$ types represented by $\mathrm{Aut}(A, \Omega_j, \rho_j)$. By the minimality of the representative set R, each of its $d-1$ remaining members must contribute at least one new automorphism type that is not represented by any of the other members.

(c) Since $r_i \mid r$ for $1 \leq i \leq d$, Proposition 7.1(c) shows that $S(A, \Omega_i, \rho_i)$ satisfies $\psi^{k+r} = \psi^k$ for $0 < k \in \mathbb{Z}$. The subdirect product $\mathrm{ATR}_n(\mathcal{C})$ then satisfies each of these identities as well. □

Acknowledgement

The author is grateful to an anonymous referee for a meticulous reading of the manuscript.

References

[1] Atiyah, M.F., Tall, D.O.: Group representations, λ-rings, and the J-homomorphism. Topology **8**, 253–297 (1969)

[2] Bryant, D., Buchanan, M., Wanless, I.M.: The spectrum for quasigroups with cyclic automorphisms and additional symmetries. Discrete Math. **309**, 821–833 (2009)

[3] Dieck, T. tom: Transformation Groups and Representation Theory. Springer, Berlin (1979)

[4] Falcón, R.M.: Cycle structures of autotopisms of the Latin squares of order up to 11. To appear in Ars Combinatorica. `arXiv:0709.2973v2 [math.CO]` (2009)

[5] Kerby, B., Smith, J.D.H.: Quasigroup automorphisms and symmetric group characters. Comment. Math. Univ. Carol. **51**, 279–286 (2010)

[6] Knutson, D.: λ-rings and the Representation Theory of the Symmetric Group. Springer, Berlin (1973)

[7] Smith, J.D.H., Romanowska, A.B.: Post-Modern Algebra. Wiley, New York, NY (1999)

n	Cycle types of n-element Boolean algebra automorphisms
1	1^1
2	1^2
4	$1^4,\ 2^1 1^2$
8	$1^8,\ 2^2 1^4,\ 3^2 1^2$

Table 1. Boolean algebra automorphisms

n	Cycle types of n-element GF(2)-space automorphisms
1	1^1
2	1^2
4	$1^4,\ 2^1 1^2,\ 3^1 1^1$
8	$1^8,\ 2^2 1^4,\ 2^3 1^2,\ 3^2 1^2,\ 4^1 1^4,\ 7^1 1^1$

Table 2. GF(2)-space automorphisms

n	Cycle types of n-element abelian group automorphisms
1	1^1
2	1^2
3	$1^3,\ 2^1 1^1$
4	$1^4,\ 2^1 1^2,\ 3^1 1^1$
5	$1^5,\ 2^2 1^1,\ 4^1 1^1$
6	$1^6,\ 2^2 1^2$
7	$1^7,\ 2^3 1^1,\ 3^2 1^1,\ 6^1 1^1$
8	$1^8,\ 2^2 1^4,\ 2^3 1^2,\ 3^2 1^2,\ 4^1 1^4,\ 7^1 1^1$

Table 3. Abelian group automorphisms

Department of Mathematics, Iowa State University, Ames, Iowa 50011, U.S.A.
E-mail address: jdhsmith@iastate.edu
URL: http://www.orion.math.iastate.edu/jdhsmith/

n	Cycle types of n-element quasigroup automorphisms
1	1^1
2	1^2
3	$1^3,\ 2^1 1^1,\ 3^1$
4	$1^4,\ 2^1 1^2,\ 2^2,\ 3^1 1^1$
5	$1^5,\ 2^2 1^1,\ 3^1 1^2,\ 4^1 1^1,\ 5^1$
6	$1^6,\ 2^2 1^2,\ 3^1 1^3,\ 3^2,\ 4^1 1^2,\ 5^1 1^1$
7	$1^7,\ 2^2 1^3,\ 2^3 1^1,\ 3^2 1^1,\ 4^1 1^3,\ 4^1 2^1 1^1,\ 5^1 1^2,\ 6^1 1^1,\ 7^1$
8	$1^8,\ 2^2 1^4,\ 2^3 1^2,\ 2^4,\ 3^2 1^2,\ 4^1 1^4,\ 4^1 2^1 1^2,\ 4^1 2^2,\ 4^2,\ 5^1 1^3,\ 6^1 1^2,\ 7^1 1^1$

Table 4. Quasigroup automorphisms