JOURNAL OF ALGEBRA **176**, 128–138 (1995)

A Left Loop on the 15-Sphere

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Communicated by E. Kleinfeld

Received February 28, 1994

The sedenions, elements of a 16-dimensional real division semialgebra with multiplicative Euclidean norm, are presented. The induced multiplication on the 15-sphere forms a left loop that is a loop almost everywhere. Suitable left multiplications in the left loop provide an 8-dimensional non-vanishing vector field on the 15-sphere. © 1995 Academic Press, Inc.

1. INTRODUCTION

The sequence of normed real division algebras—real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and Cayley numbers $\mathbb{K}$—exhibits a successive degradation of properties. The complex numbers are no longer ordered, the quaternions no longer commutative, and the Cayley numbers no longer associative. There is a parallel degradation of the properties of the induced multiplications on the corresponding unit spheres. Thus S^0 is a cyclic group, S^1 is a non-cyclic abelian group, S^3 is a non-abelian group, and S^7 is a Moufang loop. This degradation, along with results such as Hurwitz' [Hu] on composition algebras and Adams' [Ad] on odd maps, has led to a consensus that the nested sequence of "hypercomplex numbers" $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{K}$ necessarily terminates at the Cayley numbers [Eb, Sect. 10.3.4]. The only work outside the consensus appears to have been that of Pfister [Pf] on the composition of quadratic forms, although this work was done in more of a number-theoretic context without regard for the algebraic properties of the composition.

The purpose of the current paper is to present an algebraic structure (4.2) on a 16-dimensional Euclidean space $\mathbb{S} = \mathbb{K} \oplus \mathbb{K}f$ (of "sedenions"; cf. "quaternions," "octonions"), such that the Euclidean norm is multiplicative (Theorem 4.1) and the Cayley numbers appear as a subalgebra. Passing from the Cayley numbers to the sedenions, the right distributive law is lost, although the left distributive law still holds. (In Pfister's terms

0021-8693/95 $12.00

[Pf, Definition (b)], the Euclidean norm is strongly multiplicative.) The induced multiplication on S^{15} no longer yields a loop, but only a left loop. However, from the standpoint of universal algebra, left loops are still as tractable as loops. Being endowed with a Mal'cev operation (2.10), they have a well-established representation theory, cohomology, and centrality [S1, S2]. In fact the left loop on S^{15} is almost a loop, since right division is defined on all but a subset of the hypertorus $S^{15} \times S^{15}$ having measure-zero closure (Theorem 6.5). Moreover, from the standpoint of topology, the left loop structure on S^{15} is optimal, since suitable left multiplications furnish an 8-dimensional non-vanishing vector field on S^{15}.

The plan of the paper is as follows. Section 2 supplies the requisite background on loops and left loops, while Section 3 gives background on algebras. The sedenions are presented in Section 4, and shown to have a multiplicative norm (Theorem 4.1). Section 5 discusses the left loop structure on S^{15}, including both negative and positive properties. The left loop is not power-associative, and thus does not satisfy any of the more familiar generalisations of the associative law. The left loop is discontinuous overall, but has sufficient continuity to give the 8-dimensional non-vanishing vector field. The left loop is not a loop, but does have subloops that are not confined to the Cayley numbers. Finally, Section 6 investigates right division in the left loop. For ordered pairs (z, w) of elements of S^{15} with non-zero Cayley number components, a scalar cross-ratio (Definition 6.2) is introduced. Theorem 6.5 then shows that the right quotient z/w exists whenever this scalar cross-ratio is undefined or distinct from 1.

2. LOOPS AND LEFT LOOPS

A *quasigroup* is a set Q with a binary operation of *multiplication*, denoted by $\cdot$ or juxtaposition, such that in the equation

$$x \cdot y = z, \tag{2.1}$$

knowledge of any two of x, y, z specifies the third uniquely. Thus the *left multiplication*

$$L(x) : Q \to Q; q \mapsto xq \tag{2.2}$$

and the *right multiplication*

$$R(x) : Q \to Q; q \mapsto qx \tag{2.3}$$

biject for all x in Q. Quasigroups may also be described as sets Q with operations of multiplication, left division $\backslash$, and right division $/$ such that the identities

$$\begin{aligned} x \backslash (x \cdot y) &= y \\ x \cdot (x \backslash y) &= y \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} (y \cdot x)/x &= y \\ (y/x) \cdot x &= y \end{aligned} \tag{2.5}$$

are satisfied. For example, the identities (2.4) are equivalent to the bijectivity of the left multiplications. In order to reduce the number of brackets, juxtaposition is taken to bind more strongly than $\cdot$, $/$, or $\backslash$. Thus the first identity (2.4) may be written as $x \backslash xy = y$. A *left quasigroup* is a set Q with multiplication and left division satisfying (2.4). Extreme examples of left quasigroups are obtained by taking both the multiplication and the left division to be the projection $(x, y) \mapsto y$. Because of such examples, left quasigroups tend to be algebraically intractable.

A *loop* is a quasigroup Q with an *identity* element 1 satisfying

$$1 \cdot x = x = x \cdot 1. \tag{2.6}$$

A *left loop* is a left quasigroup with an identity element satisfying (2.6). Note

$$x \backslash x = x \backslash (x \cdot 1) = 1 \tag{2.7}$$

in any left loop. Both quasigroups and left loops have rich algebraic theories arising from the existence of a *Mal'cev operation*, a derived ternary operation P satisfying

$$(x, x, y)P = y = (y, x, x)P \tag{2.8}$$

[S1]. In quasigroups, the Mal'cev operation may be taken to be

$$(x, y, z)P = (x/(y \backslash y)) \cdot (y \backslash z) \tag{2.9}$$

(cf. [Ma]). In left loops, the Mal'cev operation may be taken to be

$$(x, y, z)P = x \cdot (y \backslash z) \tag{2.10}$$

(cf. [Be, Lemma 2.4(a)]).

A left loop is said to be *power-associative* if the left subloops generated by singletons are (necessarily cyclic) groups. A left loop is said to be *diassociative* if the left subloops generated by doubletons are groups. A consequence of power-associativity is the existence of an involution

$x \mapsto x^{-1}$ such that

$$xx^{-1} = 1 = x^{-1}x. \tag{2.11}$$

Consequences of diassociativity are the *right inverse property*

$$y = yx \cdot x^{-1} \tag{2.12}$$

and the *left inverse property*

$$x^{-1} \cdot xy = y. \tag{2.13}$$

In any left loop Q, an *inner map*

$$T(x) : Q \to Q; q \mapsto x \backslash qx \tag{2.14}$$

is defined, fixing the identity element. If Q is a group, then the $T(x)$ are just the inner automorphisms.

3. NORMED SEMIALGEBRAS

An *algebra* is a vector space R with a binary product such that the left and right multiplications are linear. A *division algebra* is an algebra R for which the set R^* of non-zero elements forms a loop. For current purposes, it is convenient to broaden these standard definitions.

DEFINITION 3.1. A *(left) semialgebra* is a vector space S with a binary product such that the left multiplications are linear. A *(left) division semialgebra* is a (left) semialgebra S for which the set S^* of non-zero elements forms a left loop.

(Compare the concept of an "abelian left near-ring" [Pi].)

Suppose that the underlying vector space of a left semialgebra S is real, and has a norm $|x|$. The norm is said to be *multiplicative* if

$$|x \cdot y| = |x| \cdot |y|. \tag{3.1}$$

Note that if S is finite dimensional with multiplicative norm, then S^* is a left quasigroup. Indeed (3.1) shows that the kernels of the linear left multiplications are zero, and one may then take $x \backslash y = yL(x)^{-1}$.

The work of the following sections depends heavily on the properties of the algebra $\mathbb{K}$ of Cayley numbers or octonions with multiplicative Euclidean norm. These properties are summarised in [Co, Eb]. The key properties are the existence of an involutory linear conjugation

$$K : \mathbb{K} \to \mathbb{K}; x \mapsto \bar{x} \tag{3.2}$$

such that

$$x\bar{x} = |x|^2 \tag{3.3}$$

and the fact that the loop $\mathbb{K}^*$ of non-zero Cayley numbers is diassociative. A typical consequence of these properties is the equation

$$KR(x)K = L(\bar{x}) = L(x)^T. \tag{3.4}$$

Note that for x in $\mathbb{K}^*$, the maps $|x|^{-1}L(x)$ and $|x|^{-1}R(x)$ are orthogonal.

4. THE SEDENIONS

Let $\mathbb{S} = \mathbb{K} \oplus \mathbb{K}f$ denote the direct sum of two copies of the Cayley numbers, with respective identities 1 and f. Elements of $\mathbb{S}$ are described as *sedenions*. (Note that the (semi-) algebra structure being considered here, with multiplicative norm, is quite distinct from the flexible algebra structure on $\mathbb{K} \oplus \mathbb{K}f$ obtained by applying the Cayley–Dickson process to the octonions, although the term "sedenions" has occasionally been used for that algebra.) The Cayley numbers themselves are embedded in the sedenions by

$$\mathbb{K} \to \mathbb{K} \oplus \mathbb{K}f; x \mapsto x + 0f. \tag{4.1}$$

For elements $z = x + yf$ and $w = u + vf$ of $\mathbb{K} \oplus \mathbb{K}f$, a product $z \cdot w$ or zw is defined as

$$\begin{cases} zu + vzf & \text{if } z \in \mathbb{K}, \text{ and} \\ (xy \cdot uy^{-1} - y\bar{v}) + (y\bar{u} + vy^{-1} \cdot xy)f & \text{otherwise.} \end{cases} \tag{4.2}$$

By the first of these expressions, $\mathbb{K}$ forms a subalgebra under the product. Since the expressions (4.2) are linear in u and v, the product satisfies the left distributive law $z(w_1 + w_2) = zw_1 + zw_2$. Clearly $1 \cdot w = w$ for any w in $\mathbb{S}$, and $z \cdot 1 = z$ for z in $\mathbb{K}$. For z not in $\mathbb{K}$, one has $z \cdot 1 = xy \cdot y^{-1} + yf = x + yf = z$ by the right inverse property in the Cayley numbers. Furthermore, since the expressions (4.2) are homogeneous in z and w, the real numbers are central in the semialgebra $\mathbb{S}$ (i.e., commute and associate with all elements).

A norm is defined on $\mathbb{K} \oplus \mathbb{K}f$ by

$$|z|^2 = x\bar{x} + y\bar{y}. \tag{4.3}$$

Conjugation in $\mathbb{K}$ may be extended to a conjugation in $\mathbb{K} \oplus \mathbb{K}f$ by

$$\bar{z} = \bar{x}T(y) - yf \qquad \text{for } z \notin \mathbb{K} \tag{4.4}$$

(using the notation of (2.14)). Clearly $z\bar{z} = |z|^2$ for z in $\mathbb{K}$, and for other z one has $z\bar{z} = xy \cdot \bar{x}T(y)y^{-1} + y\bar{y} + (y \cdot xT(\bar{y})^{-1} - yy^{-1} \cdot xy)f = |z|^2$ by the diassociativity of the Cayley numbers, so that the formula

$$z\bar{z} = |z|^2 \tag{4.5}$$

holds in $\mathbb{S}$. Note, however, that the conjugation (4.4) is non-linear. For example, $\overline{i + jf} = \bar{i}T(j) - jf = i - jf \neq -i - jf = \bar{i} + \overline{jf}$. On the positive side, one has the following

THEOREM 4.1. *Under* (4.2) *and* (4.3), *the* 16-*dimensional real vector space* $\mathbb{S}$ *of sedenions forms a left division semialgebra with multiplicative norm.*

Proof. It remains to be checked that $|zw|^2 = |z|^2|w|^2$, or in other terms that, for $|z| \neq 0$, the scalar multiple $|z|^{-1}L(z)$ of the $\mathbb{R}$-linear map

$$L(z)\colon w \mapsto zw \tag{4.6}$$

is orthogonal with respect to the quadratic form (4.3). For z in $\mathbb{K}$, one has $|zw|^2 = |zu|^2 + |vz|^2 = |z|^2(|u|^2 + |v|^2) = |z|^2|w|^2$. Otherwise, $L(z)$ has components $R(y)^{-1}L(xy) \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}, \mathbb{K})$, $-KL(y) \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}f, \mathbb{K})$, $KL(y) \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}, \mathbb{K}f)$, and $R(y)^{-1}R(xy) \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}f, \mathbb{K}f)$. The corresponding components of the transpose $L(z)^T$ are then $L(\bar{y}\bar{x})R(\bar{y})^{-1} \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}, \mathbb{K})$, $L(\bar{y})K \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}f, \mathbb{K})$, $-L(\bar{y})K \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}, \mathbb{K}f)$, and $R(\bar{y}\bar{x})R(\bar{y})^{-1} \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}f, \mathbb{K}f)$. The diagonal components of $L(z)^TL(z)$ are $L(\bar{y}\bar{x})R(\bar{y})^{-1}R(y)^{-1}L(xy) + L(\bar{y})KKL(y) = |x|^2 + |y|^2 \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}, \mathbb{K})$ and $R(\bar{y}\bar{x})R(\bar{y})^{-1}R(y)^{-1}R(xy) + L(\bar{y})KKL(y) = |x|^2 + |y|^2 \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{K}f, \mathbb{K}f)$. The off-diagonal component in $\operatorname{Hom}_{\mathbb{R}}(\mathbb{K}f, \mathbb{K})$ is $L(\bar{y})KR(y)^{-1}L(xy) - R(\bar{y}\bar{x})R(\bar{y})^{-1}KL(y)$. The image of a Cayley number v under this component is $xy \cdot (\bar{v}y \cdot y^{-1}) - y[y^{-1}(xy \cdot \bar{v})] = 0$. Since $L(z)^TL(z)$ is symmetric, the other off-diagonal component also vanishes. Thus $|z|^{-1}L(z)$ is orthogonal, as required. ∎

COROLLARY 4.2. *The product* (4.2) *restricts to an odd map*

$$S^{15} \times S^{15} \to S^{15}; (z, w) \mapsto zw \tag{4.7}$$

on the 15-*sphere* $S^{15} = \{z \in \mathbb{K} \oplus \mathbb{K}f \mid z\bar{z} = 1\}$.

5. THE LEFT LOOP

For non-zero z in $\mathbb{S}$, Theorem 4.1 shows that the map $|z|^{-1}L(z)$ is orthogonal. The 15-sphere $S^{15} = \{z \in \mathbb{S} \mid z\bar{z} = 1\}$ thus has a left loop structure under the product (4.2) and the left division $z \setminus w$ given by

$$\begin{cases} \bar{z}u + v\bar{z}f & \text{if } z \in \mathbb{K}, \text{ and} \\ (\bar{y}\bar{x} \cdot u)\bar{y}^{-1} + \bar{v}y + \left(-\bar{u}y + (v \cdot y^{-1}\bar{x})y\right)f & \text{otherwise}. \end{cases} \tag{5.1}$$

Negative properties of this left loop

$$(S^{15}, \cdot, \setminus, 1) \tag{5.2}$$

are summarised in

PROPOSITION 5.1. *The left loop* (5.2) *is*:

(i) *not power-associative*,

(ii) *not continuous*,

(iii) *not a loop*.

Proof. (i) For $z = x + yf \notin \mathbb{K}$, one has $\bar{z}z = (\bar{x}T(y) - yf)(x + yf) = (y^{-1}\bar{x}(xT(y^{-2}))y + \bar{y}y) + y^{-1}[-\bar{x}T(y^{-2}) + \bar{x}]y^2f$. Thus $\bar{z}z$ is real only if x commutes with y^2. In particular, $z = 2^{-1/2}(i + e^{j\pi/4}f)$ in S^{15} does not commute with $\bar{z}$, although $z\bar{z} = 1$. Thus $2^{-1/2}(i + e^{j\pi/4}f)$ does not generate a subgroup of the left loop (5.2).

(ii) For real ε, define $z_\varepsilon = (1 + \varepsilon^2)^{-1/2}(1 + \varepsilon if) \in S^{15}$. Then

$$\lim_{\varepsilon \to 0}(z_\varepsilon j) = \lim_{\varepsilon \to 0}\left[(1 + \varepsilon^2)^{-1/2}(-j - \varepsilon ijf)\right] = -j \neq j = \left(\lim_{\varepsilon \to 0} z_\varepsilon\right)j.$$

(iii) Using the Cayley number notation of [Co], suppose that $r = s + tf$ is one solution of (6.1) below with $u = 2^{-1/2}j$, $v = 2^{-1/2}h$, $x = 2^{-1/2}i$, $y = 2^{-1/2}(i + kh)$. Note that $t \neq 0$. Indeed, since (6.11) below holds, t is a root of the second equation of (6.12) below, i.e., of (6.9). As noted in the proof of Proposition 6.4, a non-zero root t_0 of (6.9) yields a solution $r_0 = s_0 + t_0 f$ of (6.1) on taking s_0 as given by (6.10). Now the linear map $A = L(y)L(v)^{-1}L(u)R(x)^{-1}$ has $\frac{1}{2}(1 + k - i + je)$ as a unit eigenvector of eigenvalue one. Since t solves (6.9), $\xi = y^{-1} \cdot vt^{-1}$ solves $\xi(A - 1) = (xy)^{-1}$, i.e., (6.7). Thus $\xi_0 = \xi + (\alpha/2)(1 + k - i + je)$ also solves (6.7) for any value of α. It follows that $t_0(\alpha) := \xi_0^{-1}y^{-1} \cdot v$ solves (6.9) for any value of α. In particular, any non-zero value of α such that t_0 is non-zero gives a second solution $r_0 = s_0 + t_0 f$ of (6.1) on taking s_0 as given by (6.10). ∎

Proposition 5.1(iii) shows that S^{15} does not form a (two-sided) subloop of (the multiplicative reduct of) the normed semialgebra $\mathbb{S}$, in the way that S^7 and the set $\mathbb{K}^*$ of non-zero Cayley numbers form subloops of the Cayley numbers. Nevertheless, there are subloops of $\mathbb{S}$ that are not confined to $\mathbb{K}$. For example, the following proposition shows that $\mathbb{K}^* \cup \mathbb{K}^*f$ is a two-sided subloop of $\mathbb{S}$, and that there is a loop structure on $S^7 \times S^0$, realised as $S^7 \cup S^7f$, forming a two-sided subloop of the left loop S^{15}.

PROPOSITION 5.2. *Define the split loop extension* $\mathbb{K}^* \sqsupset S^0$ *by*

$$\begin{aligned}(x,1)(x,-1) &= (yx,-1)\\ (x,-1)(y,1) &= (x\bar{y},-1)\\ (x,-1)(y,-1) &= (-x\bar{y},1).\end{aligned} \tag{5.3}$$

Then the map

$$\mathbb{K}^* \sqsupset S^0 \to \mathbb{K} \oplus \mathbb{K}f; (x,\varepsilon) \mapsto xf^{(1-\varepsilon)/2} \tag{5.4}$$

is a multiplicative monomorphism.

Despite the discontinuity recorded by Proposition 5.1(ii), the product (5.2) on S^{15} is sufficiently smooth to demonstrate the inequality

$$\operatorname{Span}(S^{15}) \geq 8 \tag{5.5}$$

(cf. [Eb, Chap. 10]). Indeed, (4.2) restricts to a bilinear product

$$\begin{aligned}&(\mathbb{K} \oplus \mathbb{R}f) \times (\mathbb{K} \oplus \mathbb{K}f) \to \mathbb{K} \oplus \mathbb{K}f;\\ &(x+yf, u+vf) \mapsto (xu - y\bar{v}) + (y\bar{u} + vx)f.\end{aligned} \tag{5.6}$$

Using the Cayley number notation of [Co], the set $\{L(e_1), \ldots, L(e_7), L(f)\}$ then gives an 8-dimensional non-vanishing vector field on S^{15}.

6. RIGHT DIVISION IN THE LEFT LOOP

By Proposition 5.1(iii), the left loop (5.2) on the 15-sphere is not a loop, since right division cannot be defined everywhere. The goal of the current section is to investigate the extent to which the right division is defined. In particular, it emerges that right division is in fact defined almost everywhere. In this sense, the left loop on S^{15} is almost a loop. For elements $z = x + yf$ and $w = u + vf$ of S^{15}, the issue is the existence of a unique solution $r = s + tf$ to the equation

$$r \cdot w = z. \tag{6.1}$$

If such a solution exists, it is the right quotient $r = z/w$.

PROPOSITION 6.1. *If* $xyuv = 0$, *then* (6.1) *has a unique solution.*

Proof. If two of x, y, u, v are zero, then (6.1) has solutions given by Proposition 5.2. These solutions are easily verified to be unique. If no two of x, y, u, v are zero, then straightforward calculations in the Moufang loop of Cayley numbers show that (6.1) has unique solutions given as follows in the respective cases $x, y, u, v = 0$:

$$\begin{aligned} yf/(u + vf) &= (yu \cdot \bar{v})y/yv + yuf; \\ x/(u + vf) &= x(xv \cdot \bar{u})/xv - xvf; \\ (x + yf)/vf &= xy/xv - xvf; \\ (x + yf)/u &= xy/yu + yuf. \quad \blacksquare \end{aligned} \tag{6.2}$$

DEFINITION 6.2. If $xyuv \neq 0$, define the *vector cross-ratios* of the pair (z, w) to be

$$\sigma = -(x/\bar{v})(\bar{u}/y) \tag{6.3}$$

and

$$\tau = (x/u)(y \backslash v). \tag{6.4}$$

Define the *scalar cross-ratio* to be

$$\mu = |xv/yu|, \tag{6.5}$$

i.e., the norm of (6.4).

PROPOSITION 6.3. *Suppose that* $xyuv \neq 0$.

(i) *The cross-ratio* $\sigma = 1$ *if and only if* (6.1) *has a solution* $(y/\bar{u})f = -(x/\bar{v})f$ *in* $\mathbb{K}f$ (*i.e.*, $s = 0$).

(ii) *The cross-ratio* $\tau = 1$ *if and only if* (6.1) *has a solution* $x/y = v \backslash y$ *in* $\mathbb{K}$ (*i.e.*, $t = 0$).

PROPOSITION 6.4. *Suppose that* $xyuv \neq 0$. *If there are no non-zero Cayley numbers* η *with*

$$v \backslash y\eta = u \backslash \eta x, \tag{6.6}$$

then (6.1) *has a unique solution satisfying* $t \neq 0$.

Proof. A Cayley number η is a root of (6.6) if and only if it lies in the kernel of the endomorphism $L(y)L(v)^{-1}L(u)R(x)^{-1} - 1$ of the real vector space $\mathbb{K}$. If (6.6) has no nonzero roots, then this endomorphism is an automorphism. There is thus a unique Cayley number ξ such that

$$\xi\big(L(y)L(v)^{-1}L(u)R(x)^{-1} - 1\big) = (xy)^{-1}. \tag{6.7}$$

For $t = \xi^{-1}y^{-1} \cdot v$, i.e., $\xi^{-1} = tR(v)^{-1}R(y) = tv^{-1} \cdot y$, Eq. (6.7) is equivalent to $ut^{-1} - \xi x = y^{-1}$, i.e., to $(tv^{-1} \cdot y) \cdot ut^{-1} - x = tv^{-1}$ or

$$tv^{-1} \cdot tu^{-1} = tv^{-1} \cdot y - x \cdot tu^{-1}. \tag{6.8}$$

Since $1 = |u|^2 + |v|^2$ and $|u| \cdot |v| \neq 0$, Eq. (6.8) in turn is equivalent to

$$x \cdot tu^{-1} + t\bar{v} \cdot tu^{-1} = tv^{-1} \cdot y - tv^{-1} \cdot t\bar{u}. \tag{6.9}$$

For the unique non-zero root $t_0 = \xi^{-1}y^{-1} \cdot v$ of (6.9), setting

$$s_0 = \big(x \cdot t_0u^{-1} + t_0\bar{v} \cdot t_0u^{-1}\big)t_0^{-1} \tag{6.10}$$

gives a solution $r_0 = s_0 + t_0 f$ to (6.1). Conversely, suppose that (6.1) has a solution $r = s + tf$ with $t \neq 0$. Then

$$\begin{aligned} st \cdot ut^{-1} &= x + t\bar{v} \\ vt^{-1} \cdot st &= y - t\bar{u}, \end{aligned} \tag{6.11}$$

so that

$$st = x \cdot tu^{-1} + t\bar{v} \cdot tu^{-1} = tv^{-1} \cdot y - tv^{-1} \cdot t\bar{u}, \tag{6.12}$$

i.e., t is the unique non-zero root t_0 of (6.9), and s in turn is the s_0 of (6.10). Thus (6.1) has a unique solution $s_0 + t_0 f$ not in $\mathbb{K}$. ∎

Consider the hypertorus $S^{15} \times S^{15}$. It has two subspaces

$$Z = \{(z,w) \in S^{15} \times S^{15} | xyuv = 0\} \tag{6.13}$$

and

$$T = \{(x,w) \in S^{15} \times S^{15} | |yu| = |xv|\}. \tag{6.14}$$

Each of these is a closed subspace of measure zero.

THEOREM 6.5. *The left loop (5.2) on S^{15} has a right division $(z,w) \mapsto z/w$ for all pairs (z,w) whose scalar cross-ratio is either undefined or distinct from unity. Thus the left loop on the 15-sphere is a loop almost everywhere.*

Proof. By Proposition 6.1, the right division is defined on Z. Consider a pair (z, w) in the complement of $Z \cup T$. Since $\mu = |\tau| \neq 1$, Proposition 6.3(ii) shows that (6.1) has no solution with $t = 0$. On the other hand, (6.6) cannot have a non-zero solution η, for otherwise taking norms would yield $|v|^{-1}|y||\eta| = |u|^{-1}|\eta||x|$, i.e., $\mu = |xv|/|yu| = 1$. Then Proposition 6.4 gives a solution to (6.1), unique subject to $t \neq 0$, that is in fact absolutely unique. Thus the right division is also defined on $S^{15} \times S^{15} - (T \cup Z)$. ∎

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