LINEAR DIQUASIGROUPS

BOKHEE IM[1], K. MATCZAK[2], AND J. D. H. SMITH[3]

ABSTRACT. Following the prototype of dimonoids, diquasigroups are directed versions of quasigroups, where the structure is split into left and right quasigroups on the same set.

This is a preprint version. Please cite the published version as: B. Im, K. Matczak and J.D.H. Smith, Linear diquasigroups, *J. of Algebra and its Applications*, **16**, 1750057 (2017) [20 pages].

1. INTRODUCTION

Dimonoids $(S, \lhd, \rhd)$, as introduced by J.-L. Loday [1], break up a single associative multiplication into two separate binary operations, namely $x \lhd y$ corresponding to the left-hand argument of the original multiplication $x \cdot y$, and $x \rhd y$ corresponding to the right-hand argument. Recently, a general universal-algebraic procedure was developed for producing such so-called directional algebras from standard universal algebras, with a given set of fundamental operations and a given axiomatization in terms of identities [5]. Under this procedure, semigroups yield dimonoids, and groups yield digroups, which are dimonoids with additional structure. Quasigroups yield a number of different kinds of directional algebra, depending on the specific axiomatization chosen for quasigroups. The simplest of these are known as 4-diquasigroups: algebras $(Q, \lhd, \rhd, \swarrow, \searrow)$ embodying a right quasigroup structure $(Q, \lhd, \swarrow)$ and a left quasigroup structure $(Q, \rhd, \searrow)$ [2]. For simplicity, we will refer to 4-diquasigroups simply as diquasigroups.

The current paper is devoted to the study of linear diquasigroups, where both the left and right quasigroup structures are determined by endomorphisms of an abelian group.

The main theorem is preceded by two special cases.

Although the paper is intended to be reasonably self-contained, readers are referred to [6] for algebraic concepts that are not explicitly treated here. In particular, the action of functions on their arguments

2010 *Mathematics Subject Classification.* 20M20, 20N05.

Key words and phrases. dimonoid, quasigroup, diquasigroup.

generally observes the algebraic convention, with the function following the argument on the line or as a superfix.

2. Background

2.1. **Diquasigroups.** A *quasigroup* $(Q, \cdot, /, \backslash)$ is an algebra with three binary operations, the *multiplication* $\cdot$ and the *right* and *left divisions* $/, \backslash$, such that the identities

$$y\backslash(y \cdot x) = x = (x \cdot y)/y \quad \text{and} \tag{2.1}$$

$$y \cdot (y\backslash x) = x = (x/y) \cdot y \tag{2.2}$$

are satisfied. The further identities

$$y/(x\backslash y) = x = (y/x)\backslash y \tag{2.3}$$

hold in a (two-sided) quasigroup [4, §1.3]. A quasigroup Q is said to be a *loop* if it has an element e such that $e \cdot x = x = x \cdot e$ for all elements x of Q.

A *right quasigroup* $(Q, \cdot, /)$ is an algebra with a binary multiplication and right division satisfying the right-hand identities in (2.1), (2.2). Dually, a *left quasigroup* $(Q, \cdot, \backslash)$ is an algebra equipped with a binary multiplication and left division satisfying the left-hand identities in (2.1), (2.2).

Diquasigroups, directional quasigroups, were introduced in [5]. The following direct paraphrase is equivalent to the original definition [5, Proposition 9.2(a)].

Definition 2.1. Suppose that Q is a set. An algebra

$$(Q, \lhd, \rhd, \swarrow, \searrow)$$

is a *diquasigroup* if the set Q carries a right quasigroup structure $(Q, \lhd, \swarrow)$ and a left quasigroup structure $(Q, \rhd, \searrow)$.

2.2. **Multiplicatively undirected diquasigroups.**

Definition 2.2. A diquasigroup $(Q, \lhd, \rhd, \swarrow, \searrow)$ is said to be *multiplicatively undirected* if it satisfies the identity $x \lhd y = x \rhd y$.

Proposition 2.3. *For a diquasigroup $(Q, \lhd, \rhd, \swarrow, \searrow)$, the following conditions are equivalent:*

(a) $(Q, \lhd, \rhd, \swarrow, \searrow)$ *is multiplicatively undirected;*
(b) $(Q, \lhd, \swarrow, \searrow)$ *is a quasigroup;*
(c) $(Q, \rhd, \swarrow, \searrow)$ *is a quasigroup.*

On a diquasigroup $(Q, \lhd, \rhd, \swarrow, \searrow)$, the congruence μ generated by the set

$$V = \{(x \lhd y, x \rhd y) \mid x, y \in Q\} \tag{2.4}$$

of pairs of elements of Q yields a projection to the *quasigroup replica* or *multiplicatively undirected replica* Q^μ of the 4-diquasigroup Q. The congruence μ itself is known as the *quasigroup replica congruence* or *multiplicatively undirected replica congruence*.

2.3. **Right and left quasigroup homomorphisms.**

Proposition 2.4. *Let $(Q, \lhd, \rhd, \swarrow, \searrow)$ be a diquasigroup. Then the natural projection*

$$\operatorname{nat}\mu\colon Q \to Q^\mu; x \mapsto x^\mu$$

is a right quasigroup homomorphism $(Q, \lhd, \swarrow) \to (Q^\mu, \cdot, /)$ and a left quasigroup homomorphism $(Q, \rhd, \searrow) \to (Q^\mu, \cdot, \backslash)$.

Corollary 2.5. *Let $(Q, \lhd, \rhd, \swarrow, \searrow)$ be a diquasigroup.*

(a) *Any identity satisfied by the right quasigroup $(Q, \lhd, \swarrow)$ is satisfied by the quasigroup replica Q^μ.*
(b) *Any identity satisfied by the left quasigroup $(Q, \rhd, \searrow)$ is satisfied by the quasigroup replica Q^μ.*

Examples of identities to which the respective parts of Corollary 2.5 may apply include the right and left Bol identities [6, §I.4.2].

Now let $\mathbf{Q}$ be the category of quasigroup homomorphisms, and let $\mathbf{RQ}$ be the category of right quasigroup homomorphisms. The forgetful functor $\mathbf{Q} \to \mathbf{RQ}; (Q, \cdot, /, \backslash) \mapsto (Q, \cdot, /)$ has a left adjoint [6, §IV.3.4]. If $(Q, \cdot, /)$ is a right quasigroup, the Q-component η_Q of the unit η of the adjunction projects $(Q, \cdot, /)$ onto its largest quasigroup image.

Definition 2.6. Let $(Q, \cdot, /)$ be a right quasigroup. Then the kernel of η_Q is the *quasigroup replica congruence* μ_R on $(Q, \cdot, /)$. If $(Q, \cdot, \backslash)$ is a left quasigroup, then the *quasigroup replica congruence* μ_L on $(Q, \cdot, \backslash)$ is defined dually.

Lemma 2.7. *Let $(Q, \cdot, /)$ be a right quasigroup. Then the kernel* $\ker R$ *of the map*

$$R\colon Q \to \mathrm{RMlt}(Q, \cdot, /); q \mapsto R(q)$$

is contained in the quasigroup replica congruence μ_R.

In Lemma 2.7, note that while μ_R is a right quasigroup congruence, $\ker R$ is generally only an equivalence relation. The following result is dual to Lemma 2.7.

Lemma 2.8. *Let $(Q, \cdot, \backslash)$ be a left quasigroup. Then the kernel* $\ker L$ *of the map*

$$L \colon Q \to \mathrm{LMlt}(Q, \cdot, /); q \mapsto L(q)$$

is contained in the quasigroup replica congruence μ_L.

Proposition 2.9. *Let $(Q, \triangleleft, \triangleright, \swarrow, \searrow)$ be a diquasigroup. Then the relations*

$$\ker L \subseteq \mu_L \subseteq \mu \supseteq \mu_R \supseteq \ker R \tag{2.5}$$

hold in the partition lattice of the set Q.

3. Linear quasigroups and diquasigroups

3.1. Basic definitions.

Definition 3.1. Let S be a unital ring.

(a) A left quasigroup $(A, \triangleright, \searrow)$ is said to be $(S\text{-})$*linear* if there is a unital S-module structure $(A, +, 0)$, with automorphism λ and endomorphism R, such that

$$x \triangleright y = x^R + y^\lambda \quad \text{and} \quad x \searrow y = (y - x^R)^{\lambda^{-1}} \tag{3.1}$$

for x, y in A. It is $(S\text{-})$*affine* if there is a unital S-module structure $(A, +, 0)$, with automorphism λ and endomorphism R, such that

$$x \triangleright y = x^R + y^\lambda - a \quad \text{and} \quad x \searrow y = (y - x^R + a)^{\lambda^{-1}} \tag{3.2}$$

for x, y in A and a fixed element a of A. In this context, R is the *right endomorphism*, λ is *left automorphism*, and the element a is the *left shift*.

(b) A right quasigroup $(A, \triangleleft, \swarrow)$ is said to be $(S\text{-})$*linear* if there is a unital S-module structure $(A, +, 0)$, with endomorphism L and automorphism ρ, such that

$$x \triangleleft y = x^\rho + y^L \quad \text{and} \quad x \swarrow y = (x - y^L)^{\rho^{-1}} \tag{3.3}$$

for x, y in A. It is $(S\text{-})$*affine* if there is a unital S-module structure $(A, +, 0)$, with endomorphism L and automorphism ρ, such that

$$x \triangleleft y = x^\rho + y^L - b \quad \text{and} \quad x \swarrow y = (x - y^L + b)^{\rho^{-1}} \tag{3.4}$$

for x, y in A and a fixed element b of A. In this context, ρ is the *right automorphism*, L is the *left endomorphism*, and the element b is the *right shift*.

(c) A quasigroup $(A, \cdot, /, \backslash)$ is said to be (S-)*linear* if there is a unital S-module structure $(A, +, 0)$, with automorphisms λ and ρ, such that

$$x \cdot y = x^\rho + y^\lambda\,,\ \ x/y = (x - y^\lambda)^{\rho^{-1}} \ \text{ and } \ x\backslash y = (y - x^\rho)^{\lambda^{-1}} \tag{3.5}$$

for x, y in A. It is (S-)*affine* if there is a unital S-module structure $(A, +, 0)$, with automorphisms λ and ρ, such that

$$\begin{aligned} x \cdot y &= x^\rho + y^\lambda - c\,, \\ x/y &= (x - y^\lambda + c)^{\rho^{-1}} \\ \text{and } \ x\backslash y &= (y - x^\rho + c)^{\lambda^{-1}} \end{aligned} \tag{3.6}$$

for x, y in A and a fixed element c of A.

Definition 3.2. Let $(A, +, 0)$ be a module over a unital ring S.

(a) A diquasigroup $(A, \lhd, \rhd, \swarrow, \searrow)$ is (S-)*linear* if $(A, \rhd, \searrow)$ and $(A, \lhd, \swarrow)$ are S-linear.

(b) A diquasigroup $(A, \lhd, \rhd, \swarrow, \searrow)$ is (S-)*affine* if $(A, \rhd, \searrow)$ and $(A, \lhd, \swarrow)$ are S-affine.

3.2. **Left and right shifts.**

Lemma 3.3. *Let A be a unital right module over a unital ring S.*

(a) *Let $(A, \rhd, \searrow)$ be an affine left quasigroup, with operations given by* (3.2)*. The left shift may be expressed as $-(0 \rhd 0)$ or $(0 \searrow 0)^\lambda$.*

(b) *Let $(A, \lhd, \swarrow)$ be an affine right quasigroup, with operations given by* (3.4)*. The right shift may be expressed as $-(0 \lhd 0)$ or $(0 \swarrow 0)^\rho$.*

Proof. (a): Direct substitution in (3.2) yields $0 \rhd 0 = 0 + 0 - a$ and $(0 \searrow 0) = 0\lambda^{-1}$.

(b) is dual to (a). □

3.3. **The internal associative law.**

Proposition 3.4. *Let $(A, \lhd, \rhd, \swarrow, \searrow)$ be an affine diquasigroup, with operations given by* (3.2) *and* (3.4)*. The internal associative law*

$$(x \rhd y) \lhd z = x \rhd (y \lhd z) \tag{3.7}$$

is equivalent to the conjunction of the following properties:

(a) *The automorphism λ fixes $\operatorname{Im} R$ and bS;*

(b) *The automorphism ρ fixes $\operatorname{Im} L$ and aS;*

(c) *The automorphisms ρ and λ commute.*

Proof. ?????????????

□

3.4. Entropic diquasigroups.

Theorem 3.5. *A diquasigroup* $(A, \lhd, \rhd, \swarrow, \searrow)$ *is entropic if and only if the following conditions hold:*

(a) *The diquasigroup* $(A, \lhd, \rhd, \swarrow, \searrow)$ *is affine;*
(b) *The left and right automorphisms commute;*
(c) ...

Proof. ????????????? $\square$

4. Abelian groups

The most elementary examples of nontrivial affine diquasigroups are provided by paired abelian group structures, as discussed within the current section. In this context, the apparent chirality merely serves as a label to distinguish the two groups.

4.1. Abelian groups as affine quasigroups.

Definition 4.1. Suppose that $(A, +, 0)$ is an abelian group. Consider a $\mathbb{Z}$-affine left quasigroup $(A, \rhd, \searrow)$ as in (3.2), with $R = \lambda = 1_A$, and a $\mathbb{Z}$-affine right quasigroup $(A, \lhd, \swarrow)$ as in (3.4), with $\rho = L = 1_A$. Then $(A, \lhd, \rhd, \swarrow, \searrow)$ is said to be an *abelian diquasigroup*.

Proposition 4.2. *Let* $(A, \lhd, \rhd, \swarrow, \searrow)$ *be an abelian diquasigroup.*

(a) *The internal associative law* (3.7) *is satisfied.*
(b) *The* $\mathbb{Z}$*-affine left quasigroup* $(A, \rhd, \searrow)$ *is an abelian group, with the left shift as identity element. Similarly, the* $\mathbb{Z}$*-affine right quasigroup* $(A, \lhd, \swarrow)$ *is an abelian group, with the right shift as identity element.*

Proof. (a): Apply Proposition 3.4.

(b): Use the notation of Definition 4.1, referring to (3.2) and (3.4). The multiplication $\rhd$ is commutative. Since $x \rhd a = x + a - a = x$, the multiplication $\rhd$ has a as an identity element. Setting $a = b$ yields a quasigroup $(A, \rhd, \swarrow, \searrow)$. By (a), this quasigroup is associative. Thus $(A, \rhd, \searrow)$ is an abelian group, with the left shift as identity element. Treatment of $(A, \lhd, \swarrow)$ is dual. $\square$

4.2. Undirected replicas of abelian diquasigroups.

The following proposition shows that the undirected replica of an abelian diquasigroup is obtained simply by identifying the left and right shifts.

Proposition 4.3. *Suppose that* $(A, \triangleleft, \triangleright, \swarrow, \searrow)$ *is an abelian diquasigroup, with left shift* a *and right shift* b. *Then the undirected replica of* $(A, \triangleleft, \triangleright, \swarrow, \searrow)$ *is the quotient of* A *by* $(a-b)\mathbb{Z}$.

Proof. ???????????????? □

Corollary 4.4. *With the exception of* μ, *all the relations of* (2.5) *on* $(A, \triangleleft, \triangleright, \swarrow, \searrow)$ *are trivial.*

Proof. ???????????????? □

5. Endomorphic diquasigroups

Definition 5.1. Suppose that $(A, +, 0)$ is a unital right module over a unital ring S. Consider an S-linear left quasigroup $(A, \triangleright, \searrow)$ as in (3.1), with $\lambda = 1_A$, and an S-linear right quasigroup $(A, \triangleleft, \swarrow)$ as in (3.3), with $\rho = 1_A$. Then $(A, \triangleleft, \triangleright, \swarrow, \searrow)$ is said to be an *endomorphic linear diquasigroup*.

Proposition 5.2. *Suppose that* $(A, \triangleleft, \triangleright, \swarrow, \searrow)$ *is an endomorphic diquasigroup, with left endomorphism* L *and right endomorphism* R. *Then the undirected replica of* $(A, \triangleleft, \triangleright, \swarrow, \searrow)$ *is the quotient of* A *by* $\mathrm{Ker}(1-L) + \mathrm{Ker}(1-R)$.

Proof. ???????????????? □

Corollary 5.3. *The relations of* (2.5) *on* $(A, \triangleleft, \triangleright, \swarrow, \searrow)$ *take the form* ??????????????????

Proof. ???????????????? □

Example 5.4. Let c and d be elements of a principal ideal domain S. Define

$$x \triangleright y = x(1+c) + y\,, \qquad x \searrow y = y - x(1+c)\,,$$
$$x \triangleleft y = x + y(1+d)\,, \qquad x \swarrow y = x - y(1+d)$$

for elements x, y of S. Then $(S, \triangleleft, \triangleright, \swarrow, \searrow)$ is an entropic endomorphic diquasigroup. Its undirected replica is the quotient of S by $\gcd(c, d)S$. While the one-sided quasigroups $(S, \triangleright, \searrow)$ and $(S, \triangleleft, \swarrow)$ are neither commutative nor associative in general, the undirected replica is the abelian group of residues modulo $\gcd(c, d)S$ under addition.

6. Linear diquasigroups

6.1. **Undirected replicas.**

Theorem 6.1. *Suppose that* $(A, \lhd, \rhd, \swarrow, \searrow)$ *is an* S*-affine diquasigroup, with operations given by* (3.2) *and* (3.4)*. Then the undirected replica of* $(A, \lhd, \rhd, \swarrow, \searrow)$ *is realized as the quotient of* A *by the submodule* $\mathrm{Ker}(\lambda - L) + \mathrm{Ker}(\rho - R) + (a - b)S$*.*

Proof. ?????????????????????? □

6.2. **Relations.** (2.5) in context of Theorem 6.1

7. Linear Bol diquasigroups

Definition 7.1. A magma $(A, \cdot)$ is said to satisfy the *right Bol* identity if

$$x(yz \cdot y) = (xy \cdot z)x \tag{7.1}$$

for elements x, y, z of A.

Proposition 7.2. *Suppose that a linear right quasigroup* (3.3) *satisfies the right Bol identity. Then:*

(a) *The right automorphism satisfies the equation* $\rho^3 = \rho$*, i.e.,* ρ *is involutary on* $\mathrm{Im}(\rho)$*.*
(b) *The right automorphism and left endomorphism together satisfy the equation* $L\rho L = L\rho$*, i.e.,* L *is identical on* $\mathrm{Im}(L\rho)$*.*
(c) *The right automorphism and left endomorphism together satisfy the equation* $\rho^2 L + L^2 = L\rho + L$*.*

Proof. (a): Set $y = z = 0$ in (7.1).

(b): Set $x = y = 0$ in (7.1).

(c): Set $x = z = 0$ in (7.1), and apply (b). □

QUESTION: Can we get a nontrivial instance of $\rho^3 = \rho$, $L\rho L = L\rho$, $\rho^2 L + L^2 = L\rho + L$, ideally in abelian groups? So take ρ and L as $r \times r$ matrices over $\mathbb{Z}$ or $\mathbb{Q}$ (with ρ invertible), or possibly in some non-trivial characteristic.

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[1] Department of Mathematics, Chonnam National University, Gwang-ju 500-757, Republic of Korea.

[2] Zespół Matematyki i Fizyki, Politechnika Warszawska Filia w Płocku, Płock, Poland.

[3] Department of Mathematics, Iowa State University, Ames, Iowa 50011, U.S.A.

Email address: [1] bim@jnu.ac.kr
Email address: [2] matczak.katarzyna@gmail.com
Email address: [3] jdhsmith@iastate.edu
URL: [3] https://jdhsmith.math.iastate.edu/