WEAK HOMOMORPHISMS AND GRAPH COALGEBRAS

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Abstract. When coalgebras are used to model mathematical structures, such as graphs or topological spaces, standard coalgebra homomorphisms may be too strict. Relaxations of the coalgebra homomorphism concept, in either the upper or lower direction, then yield appropriate maps between the mathematical structures. There are both gains and losses of coalgebraic properties. The main examples of the paper consider coalgebras for the powerset functor, as used in the modeling of graphs. The lower morphisms yield a bicomplete category, while it is shown that the category of upper morphisms is not cocomplete.

1. Introduction

In universal algebra, it can be advantageous to relax the homomorphism concept. Examples are given by order-preserving maps between semilattices, or by convex functions between barycentric algebras [1, 2]. Similarly weakened homomorphism concepts may prove equally advantageous for coalgebras, especially when the coalgebras are used to model mathematical structures. For example, in an undirected graph G, the *neighborhood* of a vertex x is defined as the set of vertices adjacent to x. By using the powerset functor, an undirected graph may be turned into a *graphic coalgebra*, in which the structure map selects the neighborhood of a given vertex. In Section 3, it is observed that a homomorphism between two graphic coalgebras corresponds to a *full* graph homomorphism, preserving edges, such that each edge in the image is induced by some edge in the preimage. On the other hand, graph homomorphisms are just edge-preserving maps. A second example of the excessive strictness of homomorphisms arises when topological spaces are modeled as topological

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coalgebras, using the filter functor [3]. Homomorphisms between two topological coalgebras correspond to maps that are both open and closed. These are more restrictive than the standard homomorphisms between topological spaces, which are just continuous maps.

The purpose of the current paper is to present suitable relaxations of the homomorphism concept for coalgebras that yield correct homomorphisms for mathematical structures like graphs and topological spaces. The standard background for coalgebras is reviewed in Section 2. Section 4 then describes the two kinds of relaxation of the homomorphism concept, *lower* and *upper* morphisms. The classes of all lower and upper morphisms form categories in which each coproduct exists. The key properties of these categories are formalized by the concept of a *weakly closed* class of maps (Definition 4.4).

Section 5 introduces *weak coquasivarieties.* These are subclasses of weakly closed classes that inherit categorical completeness in the presence of surjective-injective factorizations. The topic of Section 6 is the category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ of lower morphisms between coalgebras for the powerset functor $\mathcal{P}$. It is well known that the category $\mathbf{Set}_{\mathcal{P}}$ of standard homomorphisms between coalgebras for the powerset functor does not have a terminal coalgebra [4]. On the other hand, the category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is shown to be bicomplete, each limit and colimit being constructed exactly as in the underlying category of sets. Moreover, the category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ has the surjective-injective factorization property ensuring that its weak coquasivarieties inherit the completeness.

Section 7 shows that lower morphisms between graphic coalgebras coincide with the edge-preserving maps. It is also shown that the category $\mathcal{G}$ of all graphic $\mathcal{P}$-coalgebras is complete, since it forms a weak coquasivariety. Now although the category of standard homomorphisms between coalgebras for a given endofunctor is cocomplete [5], Section 8 shows that the category $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$ of upper morphisms for the powerset functor is not cocomplete.

The paper uses algebraic notation placing maps to the right of their arguments, either in line xf or as a superfix x^f. This minimizes the number of brackets, and makes it easier to follow chains of arrows in diagrams.

2. Preliminaries

Let $\mathbf{Set}$ be the category of sets, and let $F : \mathbf{Set} \to \mathbf{Set}$ be an endofunctor.

Definition 2.1. An *F-coalgebra* is a pair (X, α) consisting of a set X and a map $\alpha : X \to XF$. The set X is called the *base set* (or *state set*), and α is the *structure map* on X.

$$\begin{array}{c} X \\ {\scriptstyle\alpha}\big\downarrow \\ XF \end{array}$$

Definition 2.2. Let (X, α) and (Y, β) be F-coalgebras. An *F-homomorphism* from (X, α) to (Y, β) is a map $f : X \to Y$ for which the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ XF & \xrightarrow[f^F]{} & YF \end{array}$$

The identity map is always an F-homomorphism, and the composition of two F-homomorphisms is again an F-homomorphism. Thus the class of all F-homomorphisms forms a category which we shall denote by $\mathbf{Set}_F$.

Definition 2.3. For the endofunctop F, an F-coalgebra (S, α_S) is called a *subcoalgebra* (or *substructure*) of (X, α_X) if $S \subseteq X$ and the canonical inclusion map $\iota : S \hookrightarrow X$ is an F-homomorphism. We write

$$(S, \alpha_S) \leq (X, \alpha_X)$$

if (S, α_S) is a subcoalgebra of (X, α_X).

Theorem 2.4. [5] *The category* $\mathbf{Set}_F$ *is cocomplete. Each colimit in* $\mathbf{Set}_F$ *is preserved by the underlying set functor.*

Theorem 2.5 ("Lambek's Lemma")**.** [4] *If (P, π) is a terminal coalgebra, then the structure map π is an isomorphism in* $\mathbf{Set}_F$.

A coalgebra (Y, α_Y) is called an *F-homomorphic image of a coalgebra* (X, α_X) if there exists a surjective F-homomorphism $f : X \twoheadrightarrow Y$.

Definition 2.6. Let $\mathcal{K}$ be a class of F-coalgebras. We define the following classes:

(a) $\mathsf{H}(\mathcal{K})$: the class of all F-homomorphic images of objects from $\mathcal{K}$;
(b) $\mathsf{S}(\mathcal{K})$: the class of all those F-coalgebras which are isomorphic to subcoalgebras of objects from $\mathcal{K}$;
(c) $\Sigma(\mathcal{K})$: the class of all F-coalgebras which are isomorphic to coproducts of objects from $\mathcal{K}$.

A class $\mathcal{K}$ is said to be *closed* under H, S, or Σ when the respective inclusions $\mathsf{H}(\mathcal{K}) \subseteq \mathcal{K}$, $\mathsf{S}(\mathcal{K}) \subseteq \mathcal{K}$, or $\Sigma(\mathcal{K}) \subseteq \mathcal{K}$ hold.

Definition 2.7. A *covariety* is a class $\mathcal{K}$ of coalgebras which is closed under H, S, and Σ. A *coquasivariety* is a class closed under H and Σ.

Proposition 2.8. [6] *If* $\mathbf{Set}_F$ *is complete, then so is every coquasivariety of* $\mathbf{Set}_F$.

3. Graphic coalgebras

Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}; (f : X \to Y) \mapsto (2^X \to 2^Y; S \mapsto Sf)$ denote the (covariant) powerset functor.

Definition 3.1. A $\mathcal{P}$-coalgebra (X, α) is said to be *graphic* if

$$\forall\, x, y \in X,\ x \in y\alpha \ \Leftrightarrow\ y \in x\alpha.$$

The class of all graphic $\mathcal{P}$-coalgebras is denoted by $\mathcal{G}$. Note that a graphic $\mathcal{P}$-coalgebra (X, α) specifies an undirected graph $G_{(X,\alpha)}$ (including loops) such that $V(G_{(X,\alpha)}) = X$, and for given $x \in X$, the set of neighbors of x is $x\alpha$. Also, an undirected graph can be turned into a graphic $\mathcal{P}$-coalgebra, where the structure map is defined by the set of neighborhoods for each vertex. In this sense, we may identify graphic $\mathcal{P}$-coalgebras and graphs.

For a map $\varphi : X \to Y$, define $\widetilde{\varphi} : X \to X\varphi$ as the corestriction of φ to its image defined by $x\widetilde{\varphi} = x\varphi$ for $x \in X$. Then the map φ factors as $\varphi = \widetilde{\varphi}\iota$, where $\iota : X\varphi \hookrightarrow Y$ is the natural inclusion. Each $\mathcal{P}$-homomorphism $\varphi : (X, \alpha) \to (Y, \beta)$ in $\mathbf{Set}_{\mathcal{P}}$ induces a unique coalgebra structure γ on $X\varphi$ so that both $\widetilde{\varphi}$ and ι are $\mathcal{P}$-homomorphisms [4]. Indeed, for given $x\varphi \in X\varphi$, the effect $x\varphi\gamma$ of the structure map γ is defined by $x\varphi\beta$. The coalgebra $(X\varphi, \gamma)$ is called the *image* of φ.

Proposition 3.2. *Let f be a $\mathcal{P}$-homomorphism from (X, α) to (Y, β) in $\mathcal{G}$. Then f is a full graph homomorphism for induced graphs, i.e. f preserves edges, and each edge in the image is induced by some edge in the preimage.*

Proof. If $x' \in x\alpha$, then $x'f \in xf\beta$. So f preserves edges. Let $y' \in xf\beta$ for $x \in X$. Since $x\alpha f^{\mathcal{P}} = xf\beta$, there is a vertex $x' \in x\alpha$ such that $x'f^{\mathcal{P}} = y'$. Hence f is a full graph homomorphism. □

Although $\mathcal{P}$-coalgebras give a natural way to express graphs, the concept of $\mathcal{P}$-homomorphism is too strict, since graph homomorphisms are usually defned simply as edge-preserving maps.

4. Weak homomorphisms of coalgebras

4.1. **Lower and upper morphisms.** Let $T : \mathbf{Set} \to \mathbf{Set}$ be an endofunctor on the category of sets, such that for a given set X, the image XT is a set of sets. Suppose further that for a given map $f : X \to Y$, and for given members A, B of XT, the containment $A \subseteq B$ implies $Af^T \subseteq Bf^T$. Then the endofunctor T is described as *monotonic*. For example, T could be the covariant powerset functor, or the filter functor. Monotonic endofunctors are examples of "functors with order" as described in [7].

Definition 4.1. Let T be a monotonic endofunctor. Let (X, α) and (Y, β) be T-coalgebras. Then a *lower T-morphism* from (X, α) to (Y, β) is a map $f : X \to Y$ such that for each $x \in X$, we have the inclusion $x\alpha f^T \subseteq xf\beta$. Similarly, an *upper T-morphism* from (X, α) to (Y, β) is a map $f : X \to Y$ such that for each $x \in X$, the inclusion $x\alpha f^T \supseteq xf\beta$ holds.

With a suitable enrichment of the category of sets, upper/lower morphisms are instances of lax/oplax morphisms, as used in the study of forward and backward simulations (compare [8] or [9], for example). Bearing in mind that the structure map of a coalgebra assigns a packet of information to each state of the system under consideration, we may regard a lower T-morphism as a map preserving this information. Similarly, an upper T-morphism may be considered as a map reflecting the information. This interpretation leads one to expect that edge-preserving maps between graphs might be described as lower morphisms, and continuous maps between topological spaces might be described as upper morphisms. Indeed, it is shown that lower morphisms between graphic coalgebras agree with the edge-preserving maps of graphs in Section 7. Likewise, upper morphisms between topological coalgebras agree with the correct maps of topological spaces, namely continuous maps [10].

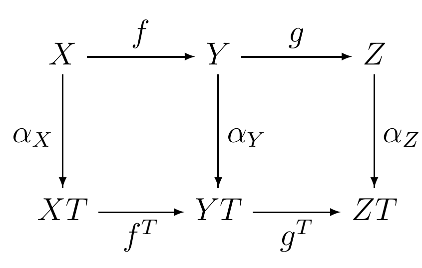
Proposition 4.2. *Let T be a monotonic endofunctor on the category of sets.*

(a) *Each identity map is always a lower and upper T-morphism.*
(b) *The composition of two lower (resp. upper) T-morphisms is again a lower (resp. upper) T-morphism.*

Proof. The first observation is immediate. Now let (X, α_X), (Y, α_Y), and (Z, α_Z) be T-coalgebras. Suppose that $f : X \to Y$ and $g : Y \to Z$ are lower T-morphisms. Consider an element x of X. Since f is a lower T-morphism, $x\alpha_X f^T \subseteq xf\alpha_Y$. Since g is a lower T-morphism, $xf\alpha_Y g^T \subseteq xfg\alpha_Z$. Thus

$$x\alpha_X(fg)^T = x\alpha_X f^T g^T \subseteq xf\alpha_Y g^T \subseteq xfg\alpha_Z\,,$$

and fg is a lower T-morphism.

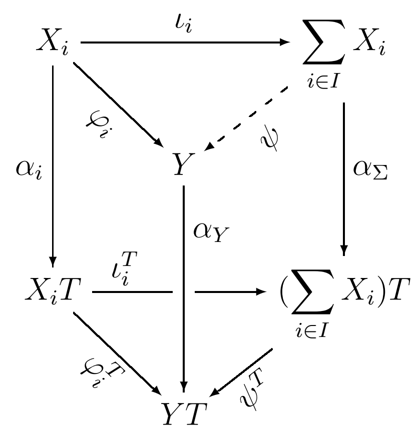


For the case of upper morphisms, we may just reverse the direction of the inclusions in the above proof. □

By Proposition 4.2, the class of all lower (resp. upper) morphisms between T-coalgebras forms a category denoted by $\underrightarrow{\mathbf{Set}}_T$ (resp. $\underleftarrow{\mathbf{Set}}_T$).

Proposition 4.3. *Let $T : \mathbf{Set} \to \mathbf{Set}$ be a monotonic endofunctor. For each family $(X_i, \alpha_i)_{i \in I}$ of T-coalgebras, there exists a sum $\sum_{i\in I}(X_i, \alpha_i)$ in $\underrightarrow{\mathbf{Set}}_T$ (resp. $\underleftarrow{\mathbf{Set}}_T$). The sum is preserved by the underlying set functor, and its structure map α_Σ is given by $x_i\alpha_\Sigma = x_i\alpha_i\iota_i^T$ for $x_i \in X_i$, where $\iota_i : X_i \to \sum X_i$ is the insertion map. In particular, each insertion map is a T-homomorphism.*

Proof. Let (Y, α_Y) be a T-coalgebra. For $i \in I$, let $\varphi_i : X_i \to Y$ be a lower T-morphism. Then there is a unique map $\psi : \sum X_i \to Y$ in $\mathbf{Set}$ with $\iota_i\psi = \varphi_i$.



Since $\varphi_i^T = \iota_i^T\psi^T$,

$$\alpha_i\varphi_i^T = \alpha_i\iota_i^T\psi^T = \iota_i\alpha_\Sigma\psi^T.$$

Since φ_i is a lower morphism, for given $x \in X_i$,

$$x\alpha_i\varphi_i^T \subseteq x\varphi_i\alpha_Y = x\iota_i\psi\alpha_Y.$$

Thus for given $i \in I$ and $x \in X_i$, the inclusion $x\iota_i\alpha_\Sigma\psi^T \subseteq x\iota_i\psi\alpha_Y$ holds, and ψ becomes a lower T-morphism. For the case of upper morphisms, we may just reverse the direction of the inclusions in the above proof. □

4.2. **Weak categories.** To formalize the key properties of categories such as $\underrightarrow{\mathbf{Set}}_T$ and $\underleftarrow{\mathbf{Set}}_T$, we introduce the following concept.

Definition 4.4. Let F be an endofunctor on the category of sets. A class $\mathcal{W}$ of maps between the underlying sets of F-coalgebras is said to be *weakly closed* if the following conditions are satisfied:

(a) $\mathcal{W}$ contains the class of all F-homomorphisms;
(b) $\mathcal{W}$ forms a category in which each coproduct exists;

(c) Each coproduct in $\mathcal{W}$ is preserved by the underlying set functor.

If $\mathcal{W}$ is weakly closed, then a member of $\mathcal{W}$ is called a *weak F-homomorphism*. The corresponding category is called a *weak category of F-coalgebras*, and denoted by $\mathbf{W}$.

Assume that we have a weak category $\mathbf{W}$. Then *subcoalgebras*, *covarieties* and *coquasivarieties* under weak F-homomorphisms are defined as in Section 2, simply replacing F-homomorphisms with weak F-homomorphisms. We write

$$(S, \alpha_S) \leq_w (X, \alpha_X)$$

if (S, α_S) is a subcoalgebra of (X, α_X) over $\mathbf{W}$. Propositions 4.2 and 4.3 may be summarized as follows.

Theorem 4.5. *For a given monotonic endofunctor T on the category of sets, the class of all lower (resp. upper) T-morphisms is weakly closed.*

For the remainder of this paragraph, we consider another example of a weak category. For given sets A and B, denote the symmetric difference between A and B by $A \triangle B$, i.e.

$$A \triangle B = (A \cup B) \smallsetminus (A \cap B)\,.$$

Definition 4.6. Let $T : \mathbf{Set} \to \mathbf{Set}$ be a monotonic endofunctor. Let (X, α) and (Y, β) be T-coalgebras. Then an *almost T-homomorphism* from (X, α) to (Y, β) is a map $f : X \to Y$ such that for each $x \in X$, we have

$$|x\alpha f^T \triangle x f \beta| < \infty\,.$$

For given sets A and B, if $A \subseteq B$ and $|B \smallsetminus A| < \infty$, denote the containment by $A \subseteq_{cofin} B$. Let $T : \mathbf{Set} \to \mathbf{Set}$ be a monotonic endofunctor. Then T is said to be a *strictly monotonic endofunctor* if for a given map $f : X \to Y$ and for given sets A, $B \in XT$, the containment $A \subseteq_{cofin} B$ implies $Af^T \subseteq_{cofin} Bf^T$.

Lemma 4.7. *Let $f : (X, \alpha) \to (Y, \beta)$ be an almost T-homomorphism for a strictly monotonic endofunctor $T : \mathbf{Set} \to \mathbf{Set}$. Then for any $A\,, B \in XT$ with $|A \triangle B| < \infty$, we have $|Af^T \triangle Bf^T| < \infty$.*

Proof. Let $A\,, B \in XT$ with $|A \triangle B| < \infty$. Since T is monotonic, we have

$$Af^T \cup Bf^T \subseteq (A \cup B)f^T$$

and

$$(A \cap B)f^T \subseteq Af^T \cap Bf^T\,.$$

Since $(A \cap B) \subseteq_{cofin} (A \cup B)$ by assumption,

$$(A \cap B)f^T \subseteq_{cofin} (A \cup B)f^T\,.$$

Then

$$|Af^T \triangle bf^T| = |(Af^T \cup Bf^T) \setminus (Af^T \cap Bf^T)| \leq |(A \cup B)f^T \setminus (A \cap B)f^T| < \infty\,.$$

□

The following lemma is immediate.

Lemma 4.8. *Let A, B, and C be sets. Then $|A \triangle B| < \infty$ and $|B \triangle C| < \infty$ imply $|A \triangle C| < \infty$.*

Proposition 4.9. *Let T be a strictly monotonic endofunctor T on the category of sets. Then the composition of two almost T-homomorphisms is again an almost T-homomorphism.*

Proof. Let (X, α_X), (Y, α_Y), and (Z, α_Z) be T-coalgebras. Let $f : X \to Y$ and $g : Y \to Z$ be almost T-homomorphisms.

$$\begin{array}{ccccc} X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_Z} \\ XT & \xrightarrow[f^T]{} & YT & \xrightarrow[g^T]{} & ZT \end{array}$$

For a given $x \in X$, since f is an almost T-homomorphism,

$$|x\alpha_X f^T \triangle x f \alpha_Y| < \infty\,.$$

By Lemma 4.7,

$$|x\alpha_X f^T g^T \triangle x f \alpha_Y g^T| = |x\alpha_X (fg)^T \triangle x f \alpha_Y g^T| < \infty\,.$$

Since g is an almost T-homomorphism, $|x f \alpha_Y g^T \triangle x f g \alpha_Z| < \infty$. So

$$|x\alpha_X (fg)^T \triangle x f g \alpha_Z| < \infty$$

by Lemma 4.8. Thus fg is an almost T-homomorphism. □

Let T be a strictly monotonic endofunctor T on the category of sets. By Proposition 4.9, the class of all almost T-homomorphisms between T-coalgebras forms a category, denoted by $\underset{\longleftrightarrow}{\mathbf{Set}}_T$.

Proposition 4.10. *Let $T : \mathbf{Set} \to \mathbf{Set}$ be a strictly monotonic endofunctor. For each family $(X_i, \alpha_i)_{i \in I}$ of T-coalgebras, the sum $\sum_{i \in I}(X_i, \alpha_i)$ exists in $\underset{\longleftrightarrow}{\mathbf{Set}}_T$. The sum is preserved by the underlying set functor, and its structure map α_Σ is given by $x_i \alpha_\Sigma = x_i \alpha_i \iota_i^T$ for $x_i \in X_i$, where $\iota_i : X_i \to \sum X_i$ is the insertion map.*

Theorem 4.11. *For a given strictly monotonic endofunctor T on the category of sets, the class of all almost T-homomorphisms is weakly closed.*

5. Weak coquasivarieties

Suppose that $\mathbf{W}$ is a weak category of F-coalgebras. Let $\mathcal{K}$ be a subclass of F-coalgebras. We denote the full subcategory of $\mathbf{W}$ with the object class $\mathcal{K}$ by $\mathbf{K}$. In particualr, we denote the full subcategory of $\underrightarrow{\mathbf{Set}}_T$ (resp. $\underleftarrow{\mathbf{Set}}_T$) with the object class $\mathcal{K}$ by $\underrightarrow{\mathbf{K}}$ (resp. $\underleftarrow{\mathbf{K}}$). The concepts of this section produce key subclasses of weakly closed classes that guarantee the inheritance of categorical completeness.

Definition 5.1. Let $\mathbf{W}$ be a weak category of F-coalgebras, and let $\mathcal{K}$ be a subclass of F-coalgebras. A *weak coquasivariety of* $\mathbf{K}$ is a subclass $\mathcal{L}$ of $\mathcal{K}$, closed under Σ over $\mathbf{K}$, and such that for a given surjective morphism $f : (X, \alpha_X) \twoheadrightarrow (Y, \alpha_Y)$ over $\mathbf{K}$ with $(X, \alpha_X) \in \mathcal{L}$, there is a structure map α on Y with the following properties:

(a) $(Y, \alpha) \in \mathcal{L}$;
(b) $(Y, \alpha) \leq_w (Y, \alpha_Y)$;
(c) $f : (X, \alpha_X) \to (Y, \alpha)$ is a morphism over $\mathbf{K}$.

Note that each coquasivariety is a weak coquasivariety. In Section 7, it is shown that although the class $\mathcal{G}$ of graphic coalgebras is not closed under lower $\mathcal{P}$-morphic images, it does forms a weak coquasivariety in the category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$.

Let $\varphi : X \to Y$ be a map. We say that φ *SI-factors through* Z if there is a surjective map $f : X \twoheadrightarrow Z$ and an injective map $g : Z \rightarrowtail Y$ such that $\varphi = fg$. If φ SI-factors through Z, then φ is also said to be *SI-factorizable by* Z. One natural way to SI-factorize φ is to take $\varphi = \widetilde{\varphi}\iota$, where $\widetilde{\varphi}$ is the corestriction of φ to its image, and ι is the natural inclusion.

Definition 5.2. Suppose that $\mathbf{W}$ is a weak category of F-coalgebras. Let $\mathcal{K}$ be a subclass of F-coalgebras. Suppose that $\varphi : (X, \alpha_X) \to (Y, \alpha_Y)$ is a weak F-homomorphism over $\mathbf{K}$. Then φ is said to be *weakly SI-factorizable over* $\mathbf{K}$ if for any set Z, and for a given SI-factorization $f : X \twoheadrightarrow Z$ and $g : Z \rightarrowtail Y$ with $fg = \varphi$, there is a structure map α on Z such that $(Z, \alpha) \in \mathcal{K}$ and both $f : (X, \alpha_X) \twoheadrightarrow (Z, \alpha)$ and $g : (Z, \alpha) \rightarrowtail (Y, \alpha_Y)$ are weak F-homomorphisms over $\mathbf{K}$. The full subcategory $\mathbf{K}$ is called *weakly SI-factorizable* if every weak F-homomorphism over $\mathbf{K}$ is weakly SI-factorizable over $\mathbf{K}$.

Suppose that $\mathbf{W}$ is a weakly SI-factorizable weak category. Then $\mathbf{W}$ is a weak coquasivariety of itself.

Lemma 5.3. *Let* $\mathbf{W}$ *be a weak category of F-coalgebras, and let* $\mathcal{K}$ *be a subclass of F-coalgebras such that* $\mathbf{K}$ *is weakly SI-factorizable. Let* $\mathcal{L}$ *be a weak coquasivariety of* $\mathbf{K}$. *Let* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *be a weak homomorphism over* $\mathbf{K}$ *with* $(X, \alpha_X) \in \mathcal{L}$. *Then there is a structure map* α *on* Xf *such that*

$\widetilde{f} : (X, \alpha_X) \to (Xf, \alpha)$ *is a weak homomorphism over* $\mathbf{K}$*, the image* (Xf, α) *lies in* $\mathcal{L}$*, and* $(Xf, \alpha) \leq_w (Y, \alpha_Y)$*.*

Proof. Since $\mathbf{K}$ is weakly SI-factorizable, there is a structure map α' on Xf such that $(Xf, \alpha') \in \mathcal{K}$, $\widetilde{f} : (X, \alpha_X) \twoheadrightarrow (Xf, \alpha')$ is a weak homomorphism over $\mathbf{K}$, and $(Xf, \alpha') \leq_w (Y, \alpha_Y)$. Since $\mathcal{L}$ is a weak coquasivariety of $\mathbf{K}$, there is a structure map α on Xf such that $\widetilde{f} : (X, \alpha_X) \to (Xf, \alpha)$ is a weak homomorphism over $\mathbf{K}$, $(Xf, \alpha) \in \mathcal{L}$, and $(Xf, \alpha) \leq_w (Xf, \alpha') \leq_w (Y, \alpha_Y)$. ☐

Proposition 5.4. *Let* $\mathbf{W}$ *be a weak category of* F*-coalgebras, and let* $\mathcal{K}$ *be a subclass of* F*-coalgebras such that* $\mathbf{K}$ *is weakly* SI*-factorizable. Let* $\mathcal{L}$ *be a weak coquasivariety of* $\mathbf{K}$*. Then the union of a family of* $\mathcal{L}$*-subcoalgebras of* $(X, \alpha_X) \in \mathcal{K}$ *is an* $\mathcal{L}$*-subcoalgebra of* (X, α_X)*.*

Proof. Consider a given family $(S_i, \alpha_i)_{i \in I}$ of $\mathcal{L}$-subcoalgebras of $(X, \alpha_X) \in \mathcal{K}$. Since $\mathcal{L}$ is a weak coquasivariety, the sum $\sum_{i \in I}(S_i, \alpha_i)$ exists in $\mathcal{L}$, and is preserved by the underlying set functor, with insertion maps $e_i : S_i \to \sum_{i \in I} S_i$. Since for each $i \in I$, the inclusion map $\iota_i : S_i \to X$ is a weak F-homomorphism, there exists a unique weak F-homomorphism $\psi : \sum_{i \in I} S_i \to X$ such that $e_i \psi = \iota_i$. Since $\mathbf{K}$ is weakly SI-factorizable, there is a structure map α on $(\sum_{i \in I} S_i)\psi$ such that $((\sum_{i \in I} S_i)\psi, \alpha) \in \mathcal{L}$ and $((\sum_{i \in I} S_i)\psi, \alpha) \leq_w (X, \alpha_X)$ by Lemma 5.3. Since $(\sum_{i \in I} S_i)\psi = \bigcup S_i$, we have $(\bigcup S_i, \alpha) \in \mathcal{L}$ and $(\bigcup S_i, \alpha) \leq_w (X, \alpha_X)$. ☐

From the proof of Proposition 5.4, we obtain the following.

Corollary 5.5. *Let* $\mathbf{W}$ *be a weak category of* F*-coalgebras, and let* $\mathcal{K}$ *be a subclass of* F*-coalgebras such that* $\mathbf{K}$ *is weakly* SI*-factorizable. Let* $(S_i, \alpha_i)_{i \in I}$ *be a family of* $\mathcal{L}$*-subcoalgebras of* $(X, \alpha_X) \in \mathcal{K}$*. Then* $(S_i, \alpha_i) \leq_w (\bigcup S_i, \alpha)$*, where* α *is the structure map defined on* $\bigcup S_i$ *in the proof of Proposition 5.4.*

Theorem 5.6. *Let* $\mathbf{W}$ *be a weak category of* F*-coalgebras, and let* $\mathcal{K}$ *be a subclass of* F*-coalgebras such that* $\mathbf{K}$ *is weakly* SI*-factorizable. If* $\mathbf{K}$ *is complete, then so is every weak coquasivariety of* $\mathbf{K}$*. Similarly, if* $\mathbf{K}$ *is finitely complete, then so is every weak coquasivariety of* $\mathbf{K}$*.*

Proof. Let $\mathcal{L}$ be a weak coquasivariety of $\mathbf{K}$. Let I be a small category, and let $D : I \to \mathbf{L}$ be a functor. Then since $\mathbf{K}$ is complete, we have the limit $((L, \alpha), (\eta_i)_{i \in I})$ in $\mathbf{K}$. Let $((L', \alpha'), (\eta'_i)_{i \in I})$ be a cone of D in $\mathcal{L}$. Then there is a unique weak homomorphism $\tau : (L', \alpha') \to (L, \alpha)$ such that there is a structure map β' on $L'\tau$ such that $\widetilde{\tau} : (L', \alpha') \to (L'\tau, \beta')$ is a weak homomorphism, $(L'\tau, \beta') \in \mathcal{L}$, and $(L'\tau, \beta') \leq_w (L, \alpha)$ by Lemma 5.3. By Proposition 5.4, we have $(S, \beta) \in \mathcal{L}$, the union of all subcoalgebras in $\mathcal{L}$ of (L, α). Thus the inclusion map $\iota : (L'\tau, \beta') \hookrightarrow (S, \beta)$ is a weak homomorphism. Therefore $\widetilde{\tau}\iota : (L', \alpha') \to (S, \beta)$ is the unique weak homomorphism such that (S, β) is the limit in $\mathbf{L}$. ☐

6. Lower $\mathcal{P}$-morphisms

In this section, it will be shown that the category of lower $\mathcal{P}$-morphisms is bicomplete.

Lemma 6.1. *Consider the underlying set functor $U : \underrightarrow{\mathbf{Set}}_{\mathcal{P}} \to \mathbf{Set}$.*

(a) *If a colimit exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$, then it is preserved by U.*
(b) *If a limit exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$, then it is preserved by U.*

Proof. (a): The underlying set functor has a right adjoint G defined on objects by $G : X \mapsto \big(X : X \mapsto 2^X ; x \mapsto X\big)$. Then as a left adjoint, U is cocontinuous. (b): The underlying set functor has a left adjoint F defined on objects by $F : X \mapsto \big(X : X \mapsto 2^X ; x \mapsto \varnothing\big)$. Then as a right adjoint, U is continuous. □

6.1. **Completeness.** By Theorem 2.5, no terminal coalgebra exists in $\mathbf{Set}_{\mathcal{P}}$. However, a terminal coalgebra exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$. In this paragraph, we show that $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is complete, and that limits are preserved by the underlying set functor.

For a given family $(X_i)_{i\in I}$ of sets, let $\prod_{i\in I} X_i$ denote the set

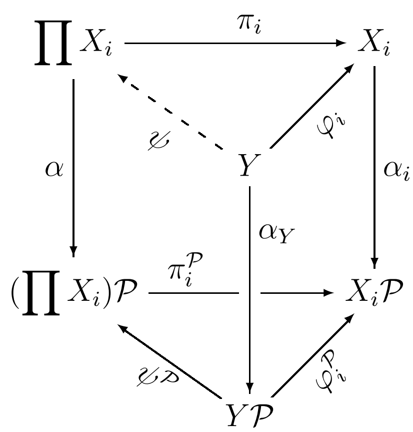
$$\{f : I \to \textstyle\bigcup_{i\in I} X_i \mid \forall\, i \in I,\ if \in X_i\}.$$

A choice function $f \in \prod_{i\in I} X_i$ is written by $\prod_{i\in I} x_i$ if for each $i \in I$, we have $if = x_i$. It is well known that $\big(\prod_{i\in I} X_i, (\pi_i)_{i\in I}\big)$ is the product of $(X_i)_{i\in I}$ in $\mathbf{Set}$, where each *projection map* $\pi_i : \prod X_i \to X_i$ is given by $f\pi_i = if$.

Proposition 6.2. *For each family $(X_i, \alpha_i)_{i\in I}$ of $\mathcal{P}$-coalgebras, there exists a product of $(X_i, \alpha_i)_{i\in I}$ in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$, which is preserved by the underlying set functor. If $I = \varnothing$, then the product is the terminal coalgebra $(\{*\}, \alpha)$ with $*\alpha = \{*\}$. If $I \neq \varnothing$, then its structure map α is given by $(\prod_{i\in I} x_i)\alpha = \prod_{i\in I}(x_i\alpha_i)$, i.e. each projection map π_i is a $\mathcal{P}$-homomorphism.*

Proof. It is easy to check that $(\{*\}, \alpha)$ is the terminal coalgebra. Suppose $I \neq \varnothing$. Let (Y, α_Y) be a $\mathcal{P}$-coalgebra. For each $i \in I$, let $\varphi_i : Y \to X_i$ be a lower $\mathcal{P}$-morphism. Then there is a unique map $\psi : Y \to \prod X_i$ in $\mathbf{Set}$ with

$\psi\pi_i = \varphi_i$.



For given $y \in Y$, we have $y\psi = \prod(y\varphi_i)$ since $\psi\pi_i = \varphi_i$. So $y\psi\alpha = \prod(y\varphi_i\alpha_i)$. Let $\prod b_i \in y\alpha_Y\psi^{\mathcal{P}}$ be given. Then it is enough to show that $\forall\, i \in I,\ b_i \in y\varphi_i\alpha_i$. Note that $b_i \in y\alpha_Y\psi^{\mathcal{P}}\pi_i^{\mathcal{P}} = y\alpha_Y(\psi\pi_i)^{\mathcal{P}}$. Since

$$\begin{aligned} y\alpha_Y(\psi\pi_i)^{\mathcal{P}} &= y\alpha_Y\varphi_i^{\mathcal{P}} \qquad (\mathcal{P} \text{ is a functor and } \psi\pi_i = \varphi_i) \\ &\subseteq y\varphi_i\alpha_i \qquad (\varphi_i \text{ is a lower } \mathcal{P}\text{-morphism}), \end{aligned}$$

$b_i \in y\varphi_i\alpha_i$. Therefore ψ is a lower $\mathcal{P}$-morphism. □

Proposition 6.3. *Let $f : (X, \alpha_X) \to (Z, \alpha_Z)$ and $g : (Y, \alpha_Y) \to (Z, \alpha_Z)$ be two lower $\mathcal{P}$-morphisms. Then there exists a pullback $\big((P, \alpha_P), (\pi_X, \pi_Y)\big)$ in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ which is preserved by the underlying set functor, i.e.*

$$P = \{(x, y) \in X \times Y \mid xf = yg\},$$

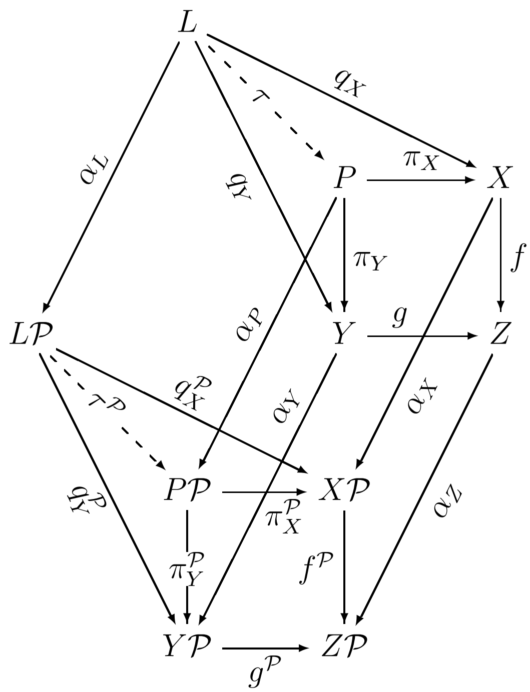
where π_X and π_Y are the projection maps. The structure map α_P is given by

$$(x, y)\alpha_P = P \cap [(x\alpha_X) \times (y\alpha_Y)]$$

for $x \in X$ and $y \in Y$.

Proof. Let (L, α_L) be a $\mathcal{P}$-coalgebra, and let $q_X : L \to X$ and $q_Y : L \to Y$ be lower $\mathcal{P}$-morphisms such that $q_X f = q_Y g$. Let $P = \{(x, y) \in X \times Y \mid xf = yg\}$.

Then there is a unique map $\tau : L \to P$ in **Set** with $\tau\pi_X = q_X$ and $\tau\pi_Y = q_Y$.



For given $l \in L$, note that $l\tau = (lq_X, lq_Y)$ since $\tau\pi_X = q_X$ and $\tau\pi_Y = q_Y$. So

$$l\tau\alpha_P = P \cap [(lq_X\alpha_X) \times (lq_Y\alpha_Y)] .$$

Let $(a, b) \in l\alpha_L\tau^{\mathcal{P}}$ be given. Then $a \in l\alpha_L\tau^{\mathcal{P}}\pi_X^{\mathcal{P}}$ and $b \in l\alpha_L\tau^{\mathcal{P}}\pi_Y^{\mathcal{P}}$. Since

$$\begin{aligned} l\alpha_L\tau^{\mathcal{P}}\pi_X^{\mathcal{P}} &= l\alpha_L q_X^{\mathcal{P}} && (\mathcal{P} \text{ is a functor and } \tau\pi_X = q_X) \\ &\subseteq lq_X\alpha_X && (q_X \text{ is a lower } \mathcal{P}\text{-morphism}), \end{aligned}$$

$a \in lq_X\alpha_X$. Similarly, $b \in lq_Y\alpha_Y$. Therefore τ is a lower $\mathcal{P}$-morphism. □

By Proposition 6.2 and 6.3, we obtain the following.

Theorem 6.4. $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ *is complete.*

Proposition 6.5. $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ *(resp.* $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$*) is weakly SI-factorizable.*

Proof. Let $\varphi : (X, \alpha_X) \to (Y, \alpha_Y)$ be a lower $\mathcal{P}$-morphism. Suppose that $f : X \twoheadrightarrow Z$ and $g : Z \rightarrowtail Y$, for some set Z, give an SI-factorization with $fg = \varphi$. For given $z \in Z$, we define a structure map α on Z by

$$z\alpha = \bigcup_{xf=z} x\alpha_X f^{\mathcal{P}}.$$

Then it is easy to see that both $f : (X, \alpha_X) \twoheadrightarrow (Z, \alpha)$ and $g : (Z, \alpha) \rightarrowtail (Y, \alpha_Y)$ are lower $\mathcal{P}$-morphisms. The case of upper $\mathcal{P}$-morphisms follows on defining a structure map α for Z by

$$z\alpha = \bigcap_{xf=z} x\alpha_X f^{\mathcal{P}}.$$

□

By Theorem 5.6 and Proposition 6.5, we obtain the following.

Corollary 6.6. *Each weak coquasivariety of* $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ *is complete.*

6.2. **Cocompleteness.** In this paragraph, we show that $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is cocomplete. By Proposition 4.3, each sum exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$. So, for the cocompleteness, it suffices to show the existence of pushouts.

Lemma 6.7. *Let* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *and* $g : (X, \alpha_X) \to (Z, \alpha_Z)$ *be two lower* $\mathcal{P}$*-morphisms. Suppose that they have a pushout* $((P, \beta), (p_Y, p_Z))$*. Let* θ *be the smallest equivalence relation on* $Y + Z$ *containing all pairs* (xf, xg) *with* $x \in X$*. Then the natural projection* $\operatorname{nat}\theta : (Y + Z, \alpha_\Sigma) \to ((Y + Z)^\theta, \beta)$ *is a lower* $\mathcal{P}$*-morphism.*

Proof. Since the pushout is preserved by the underlying set functor according to Lemma 6.1, $P = (Y + Z)^\theta$, where θ is the smallest equivalence relation on $Y + Z$ containing all pairs (xf, xg) with $x \in X$. Furthermore, $p_Y = \iota_Y(\operatorname{nat}\theta)$ and $p_Z = \iota_Z(\operatorname{nat}\theta)$, where ι_Y and ι_Z are the insertion maps into the coproduct, and $\operatorname{nat}\theta$ is the natural projection. Note that p_Y and p_Z are lower $\mathcal{P}$-morphisms. By Proposition 4.3, ι_Y and ι_Z are $\mathcal{P}$-homomorphisms.

$$\begin{array}{ccccc}
Y & \xrightarrow{\iota_Y} & Y + Z & \xrightarrow{\operatorname{nat}\theta} & (Y+Z)^\theta \\
\downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_\Sigma} & & \downarrow{\scriptstyle \beta} \\
Y\mathcal{P} & \xrightarrow[\iota_Y^{\mathcal{P}}]{} & (Y+Z)\mathcal{P} & \xrightarrow[(\operatorname{nat}\theta)^{\mathcal{P}}]{} & (Y+Z)^\theta\mathcal{P}
\end{array}$$

For given $y \in Y$,

$$\begin{aligned}
y\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}} &= y\iota_Y\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}} \\
&= y\alpha_Y\iota_Y^{\mathcal{P}}(\operatorname{nat}\theta)^{\mathcal{P}} \qquad (\iota_Y \text{ is a } \mathcal{P}\text{-homomorphism}) \\
&\subseteq y\iota_Y(\operatorname{nat}\theta)\beta \qquad (p_Y = \iota_Y(\operatorname{nat}\theta) \text{ is a lower } \mathcal{P}\text{-morphism}) \\
&= y(\operatorname{nat}\theta)\beta.
\end{aligned}$$

Similarly, for given $z \in Z$, $z\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}} \subseteq z(\operatorname{nat}\theta)\beta$. Thus $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism. □

By Lemma 6.7, it is natural to study the structure maps on A^θ such that $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism. Indeed, for an arbitrary equivalence relation θ on A, we can give a structure on A^θ so that $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism as follows.

Definition 6.8. Let (A, α) be a $\mathcal{P}$-coalgebra, and let θ be an equivalence relation on A. We define a $\mathcal{P}$-coalgebra $(A^\theta, \alpha_\theta)$ by

$$a^\theta \alpha_\theta = \bigcup_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta)^{\mathcal{P}} .$$

It is easy to see that α_θ is well-defined.

Proposition 6.9. *Let (A, α) be a $\mathcal{P}$-coalgebra, and let θ be an equivalence relation on A. Then for each $a^\theta \in A^\theta$, the image $a^\theta\alpha_\theta$ is the smallest subset of A^θ consistent with* $\operatorname{nat}\theta$ *being a lower $\mathcal{P}$-morphism.*

Proof.

$$\begin{array}{ccc} A & \xrightarrow{\operatorname{nat}\theta} & A^\theta \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha_\theta} \\ A\mathcal{P} & \xrightarrow[(\operatorname{nat}\theta)^{\mathcal{P}}]{} & A^\theta\mathcal{P} \end{array}$$

Let $a \in A$ be given. Note that

$$a(\operatorname{nat}\theta)\alpha_\theta = a^\theta\alpha_\theta = \bigcup_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta).$$

Since $a\alpha(\operatorname{nat}\theta)^{\mathcal{P}} \subseteq \bigcup_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta)$, the map $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism. Now suppose that $\operatorname{nat}\theta$ is a lower $\mathcal{P}$-morphism with a structure map β on A^θ. Then for given $a^\theta \in A^\theta$ and for each $b \in a^\theta$, we have

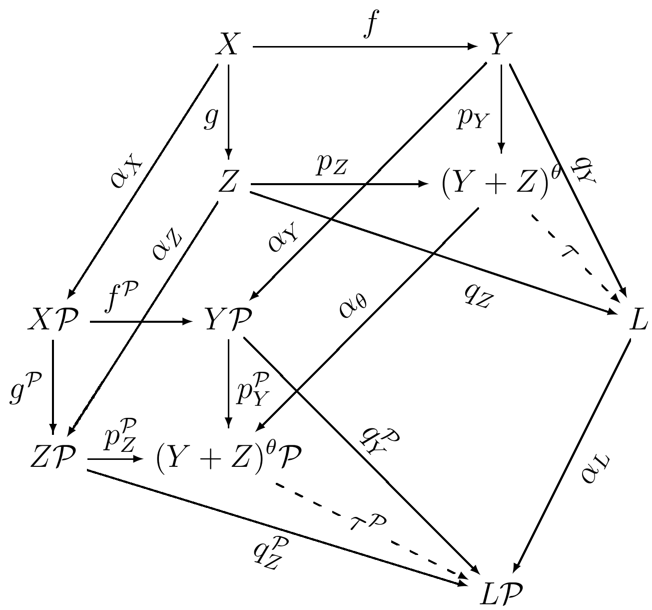
$$b\alpha(\operatorname{nat}\theta)^{\mathcal{P}} \subseteq b(\operatorname{nat}\theta)\beta = a(\operatorname{nat}\theta)\beta = a^\theta\beta.$$

Thus $a^\theta\alpha_\theta \subseteq a^\theta\beta$. □

Proposition 6.10. *Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ and $g : (X, \alpha_X) \to (Z, \alpha_Z)$ be two lower $\mathcal{P}$-morphisms. Then the pushout $\big((P, \alpha_P), (p_Y, p_Z)\big)$ of f and g exists in $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$, and is preserved by the underlying set functor, i.e. $P = (Y + Z)^\theta$ with $p_Y = \iota_Y(\operatorname{nat}\theta)$ and $p_Z = \iota_Z(\operatorname{nat}\theta)$, where θ is the smallest equivalence relation on $Y + Z$ containing all pairs (xf, xg) with $x \in X$. The structure map α_P is α_θ.*

Proof. Let $q_Y : Y \to L$ and $q_Z : Z \to L$ be lower $\mathcal{P}$-morphisms such that $fq_Y = gq_Z$. Then there is a unique map $\tau : (Y + Z)^\theta \to L$ such that $p_Y\tau = q_Y$ and $p_Z\tau = q_Z$. Let α_Σ be the structure map of the sum of (Y, α_Y) and (Z, α_Z), as in Proposition 4.3. Then α_Σ is the sum of α_Y and α_Z. Let $a^\theta \in (Y+Z)^\theta$. Without loss of generality, we may assume that $a \in Y$. Let $b \in a^\theta$. If $b \in Y$, then

$$\begin{aligned} b\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}}\tau^{\mathcal{P}} &= b\alpha_\Sigma p_Y^{\mathcal{P}}\tau^{\mathcal{P}} \\ &= b\alpha_Y q_Y^{\mathcal{P}} \\ &\subseteq bq_Y\alpha_L \qquad (q_Y \text{ is a lower } \mathcal{P}\text{-morphism}) \\ &= aq_Y\alpha_L = a^\theta\tau\alpha_L. \end{aligned}$$



If $b \in Z$, then

$$\begin{aligned} b\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}}\tau^{\mathcal{P}} &= b\alpha_\Sigma p_Z^{\mathcal{P}}\tau^{\mathcal{P}} \\ &= b\alpha_Z q_Z^{\mathcal{P}} \\ &\subseteq bq_Z\alpha_L \qquad (q_Z \text{ is a lower } \mathcal{P}\text{-morphism}) \\ &= aq_Y\alpha_L = a^\theta\tau\alpha_L. \end{aligned}$$

Since $a^\theta\alpha_\theta = \bigcup_{(a,b)\in\theta} b\alpha_\Sigma(\operatorname{nat}\theta)^{\mathcal{P}}$, τ is a lower $\mathcal{P}$-morphism. ☐

By Proposition 4.3 and 6.10, we obtain the following.

Theorem 6.11. $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ *is cocomplete.*

7. Graphic coalgebras with lower morphisms

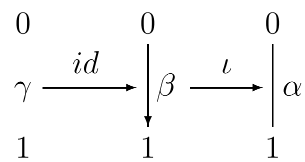
Recall that $\mathcal{G}$ is the class of all graphic $\mathcal{P}$-coalgebras.

Proposition 7.1. *Let f be a lower $\mathcal{P}$-morphism from (X,α) to (Y,β) in $\mathcal{G}$. Then f preserves edges, and thus forms a graph homomorphism between the induced graphs. Conversely, each graph homomorphism is a lower $\mathcal{P}$-morphism between the corresponding graphic $\mathcal{P}$-coalgebras.*

Proof. If $x' \in x\alpha$, then $x'f \in xf\beta$ since $x\alpha f^{\mathcal{P}} \subseteq xf\beta$. So f preserves edges. The other direction is immediate. $\square$

Proposition 7.1 shows that the homomorphism concept has been relaxed to a proper level. We denote the full subcategory of $\overrightarrow{\mathbf{Set}}_{\mathcal{P}}$ with object class $\mathcal{G}$ by $\overrightarrow{\mathbf{Gph}}$. Let **Graph** denote the category of graphs, where Ob(**Graph**) is the class of all undirected graphs including loops, and Mor(**Graph**) is the class of all edge-preserving maps. By Proposition 7.1, the category $\overrightarrow{\mathbf{Gph}}$ is equivalent to **Graph**. Since the completeness of **Graph** is known [11], $\overrightarrow{\mathbf{Gph}}$ is complete. We present a coalgebraic proof of completeness for $\overrightarrow{\mathbf{Gph}}$.

By Corollary 6.6, it suffices to prove that $\mathcal{G}$ is a weak coquasivariety of $\overrightarrow{\mathbf{Set}}_{\mathcal{P}}$. First note that $\mathcal{G}$ is not a coquasivariety of $\overrightarrow{\mathbf{Set}}_{\mathcal{P}}$. Take $(X = \{0,1\}, \alpha) \in \mathcal{G}$ with $0\alpha = \{1\}$ and $1\alpha = \{0\}$. Let (X,β) be the $\mathcal{P}$-coalgebra with $0\beta = \{1\}$ and $1\beta = \varnothing$. Take $(X,\gamma) \in \mathcal{G}$ with $0\gamma = 1\gamma = \varnothing$.



The canonical inclusion map $\iota : (X,\beta) \to (X,\alpha)$ is a lower $\mathcal{P}$-morphism, and the identity map $(X,\gamma) \to (X,\beta)$ is a surjective lower $\mathcal{P}$-morphism. Since $(\{0,1\},\beta) \notin \mathcal{G}$, the class $\mathcal{G}$ is neither a covariety nor a coquasivariety. Note that by Proposition 4.3, each coproduct exists in $\overrightarrow{\mathbf{Set}}_{\mathcal{P}}$.

Proposition 7.2. *The class $\mathcal{G}$ is a weak coquasivariety of $\overrightarrow{\mathbf{Set}}_{\mathcal{P}}$.*

Proof. Clearly, $\mathcal{G}$ is closed under Σ. Let $f : (X,\alpha_X) \to (Y,\alpha_Y)$ be a surjective lower $\mathcal{P}$-morphism with $(X,\alpha_X) \in \mathcal{G}$. We define a structure map α on Y by

$$y\alpha = \bigcup_{xf=y} x\alpha_X f^{\mathcal{P}}$$

for $y \in Y$. For each $x \in X$, we have $x\alpha_X f^{\mathcal{P}} \subseteq \bigcup_{x'f=xf} x'\alpha_X f^{\mathcal{P}} = xf\alpha$, so $f : (X,\alpha_X) \to (Y,\alpha)$ is a lower $\mathcal{P}$-morphism. Now for x in X with $xf = y$, we have $x\alpha_X f^{\mathcal{P}} \subseteq y\alpha_Y$. Thus $y\alpha\iota^{\mathcal{P}} \subseteq y\iota\alpha_Y$ for each $y \in Y$, and $(Y,\alpha) \leq_w (Y,\alpha_Y)$. Now suppose that $y_1 \in y_2\alpha$ with $y_1, y_2 \in Y$. Then $y_1 \in a\alpha_X f^{\mathcal{P}}$ for some $a \in X$ with $af = y_2$, so there is an element b of $a\alpha_X$ such that $bf = y_1$. Since $(X,\alpha_X) \in \mathcal{G}$, we have $a \in b\alpha_X$. Thus $y_2 = af \in b\alpha_X f^{\mathcal{P}}$, and $y_2 \in y_1\alpha$. $\square$

Corollary 7.3. $\underrightarrow{\mathbf{Gph}}$ *is complete.*

8. Upper $\mathcal{P}$-morphisms

In Section 6, it was seen that although the category $\mathbf{Set}_{\mathcal{P}}$ is not complete, the category $\underrightarrow{\mathbf{Set}}_{\mathcal{P}}$ is bicomplete. This is an example of the advantages gained by weak homomorphisms. For upper $\mathcal{P}$-morphisms, the situation is reversed. By Theorem 2.4, $\mathbf{Set}_{\mathcal{P}}$ is cocomplete. We now show that $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$ is not cocomplete.

Lemma 8.1. *If a colimit exists in* $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$*, then it is preserved by the underlying set functor* $U : \underleftarrow{\mathbf{Set}}_{\mathcal{P}} \to \mathbf{Set}$.

Proof. The underlying set functor has a right adjoint G defined on objects by $G : X \mapsto \big(X : X \mapsto 2^X; x \mapsto \varnothing\big)$. Then as a left adjoint, U is cocontinuous. □

Lemma 8.2. *Let* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *and* $g : (X, \alpha_X) \to (Z, \alpha_Z)$ *be two upper* $\mathcal{P}$*-morphisms. Suppose that they have a pushout* $\big((P, \beta), (p_Y, p_Z)\big)$*. Let* θ *be the smallest equivalence relation on* $Y + Z$ *containing all pairs* (xf, xg) *with* $x \in X$*. Then the natural projection* $\operatorname{nat}\theta : (Y + Z, \alpha_\Sigma) \to ((Y + Z)^\theta, \beta)$ *is an upper* $\mathcal{P}$*-morphism.*

Proof. Reverse the direction of the inclusions in Lemma 6.7. □

Definition 8.3. Let (A, α) be a $\mathcal{P}$-coalgebra, and let θ be an equivalence relation on A. We define a $\mathcal{P}$-coalgebra $(A^\theta, \alpha_{u,\theta})$ by

$$a^\theta \alpha_\theta^{\mathrm{u}} = \bigcap_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta)^{\mathcal{P}}.$$

It is easy to see that $\alpha_{u,\theta}$ is well-defined.

Proposition 8.4. *Let* (A, α) *be a* $\mathcal{P}$*-coalgebra, and let* θ *be an equivalence relation on* A*. Then for each element* a^θ *of* A^θ*, the image* $a^\theta\alpha_\theta^{\mathrm{u}}$ *is the largest subset of* A^θ *consistent with* $\operatorname{nat}\theta$ *being an upper* $\mathcal{P}$*-morphism.*

Proof.

$$\begin{array}{ccc} A & \xrightarrow{\operatorname{nat}\theta} & A^\theta \\ \alpha\downarrow & & \downarrow\alpha_\theta^{\mathrm{u}} \\ A\mathcal{P} & \xrightarrow[(\operatorname{nat}\theta)^{\mathcal{P}}]{} & A^\theta\mathcal{P} \end{array}$$

Let $a \in A$ be given. Note that

$$a(\operatorname{nat}\theta)\alpha_\theta^{\mathrm{u}} = a^\theta\alpha_\theta^{\mathrm{u}} = \bigcap_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta)^{\mathcal{P}}.$$

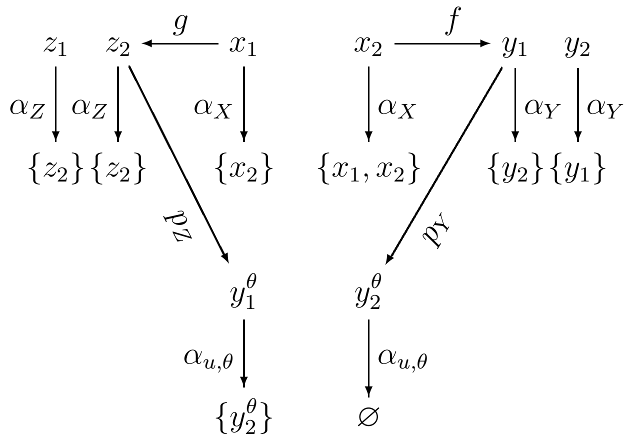
Since $a\alpha(\operatorname{nat}\theta)^{\mathcal{P}} \supseteq \bigcap_{(a,b)\in\theta} b\alpha(\operatorname{nat}\theta)^{\mathcal{P}}$, the map $\operatorname{nat}\theta$ is an upper $\mathcal{P}$-morphism. Now suppose that $\operatorname{nat}\theta$ is an upper $\mathcal{P}$-morphism, with a structure map β on A^θ. Then for given $a^\theta \in A^\theta$ and for each $b \in a^\theta$, we have

$$b\alpha(\operatorname{nat}\theta)^{\mathcal{P}} \supseteq b(\operatorname{nat}\theta)\beta = a(\operatorname{nat}\theta)\beta = a^\theta\beta.$$

Thus $a^\theta\alpha_\theta^{\mathrm{u}} \supseteq a^\theta\beta$. □

Theorem 8.5. *The category* $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$ *is not cocomplete.*

Proof. One counterexample suffices. Let $(X = \{x_1, x_2\}, \alpha_X)$ be a coalgebra such that $x_1\alpha_X = \{x_2\}$ and $x_2\alpha_X = X$. Suppose that $(Y = \{y_1, y_2\}, \alpha_Y)$ and $(Z = \{z_1, z_2\}, \alpha_Z)$ are coalgebras such that $y_1\alpha_Y = \{y_2\}$, $y_2\alpha_Y = \{y_1\}$, $z_1\alpha_Z = \{z_2\}$, and $z_2\alpha_Z = \{z_2\}$. Let $f : X \to Y$ be the map defined by $x_i = y_i$ for $i = 1, 2$, and let $g : X \to Z$ be the map defined by $x_i = z_i$ for $i = 1, 2$. Then f and g are upper $\mathcal{P}$-morphisms.



Now assume that the pushout $((P, \alpha), (p_Y, p_Z))$ of f and g exists in $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$. Then by Lemma 8.1, $P = (Y+Z)^\theta$, where θ is the smallest equivalence relation on $Y+Z$ containing all pairs (xf, xg) with $x \in X$. Furthermore, $p_Y = \iota_Y(\operatorname{nat}\theta)$ and $p_Z = \iota_Z(\operatorname{nat}\theta)$. It can readily be seen that $(Y + Z)^\theta = \{y_1^\theta, y_2^\theta\}$ with $y_1^\theta = \{y_1, z_1\}$ and $y_2^\theta = \{y_2, z_2\}$. Let $(L = \{0\}, \alpha_L)$ be a $\mathcal{P}$-coalgebra such that $0\alpha_L = \{0\}$. Let $q_Y : Y \to L$ and $q_Z : Z \to L$ be the constant functions. Then q_Y and q_Z are upper $\mathcal{P}$-morphisms such that $fq_Y = gq_Z$. There is a unique upper $\mathcal{P}$-morphism $\tau : (Y + Z)^\theta \to L$ such that $p_Y\tau = q_Y$ and $p_Z\tau = q_Z$. By Lemma 8.2 and Proposition 8.4, τ should be an upper $\mathcal{P}$-morphism with structure map $\alpha_{u,\theta}$ on $(Y + Z)^\theta$. However,

$$y_2^\theta\alpha_\theta^{\mathrm{u}}\tau^{\mathcal{P}} = \varnothing \not\supseteq \{0\} = y_2^\theta\tau\alpha_L\,,$$

which is a contradiction. Thus there is no pushout of f and g in $\underleftarrow{\mathbf{Set}}_{\mathcal{P}}$. □

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