Algebra univers. 38 (1997) 175–184
0002–5240/97/020175–10 $ 1.50 + 0.20/0

Algebra Universalis

Homotopy and semisymmetry of quasigroups

J. D. H. Smith

Abstract. The study of quasigroup homotopies reduces to the study of homomorphisms between semisymmetric quasigroups. In particular, the study of homotopies between central quasigroups reduces to the study of homomorphisms between entropic semisymmetric quasigroups.

1. Introduction

In many applications of quasigroups (e.g., the specification of Latin squares in combinatorics, or the coordinatization of nets in geometry, cf. [1]), isotopies and homotopies are more important than isomorphisms and homomorphisms. This fact has been the source of some discomfort to those algebraists who regard the homomorphism concept as fundamental. One approach has been to consider homotopies between quasigroups as homomorphisms between heterogeneous algebras [5]. However, heterogeneous algebras may occasion almost as much discomfort as homotopies themselves. The purpose of the current paper is to propose a technique for the reduction of homotopies between quasigroups to homomorphisms between (single-sorted) quasigroups. A surprising outcome of the technique is its reduction of homotopies between arbitrary quasigroups, including homomorphisms between arbitrary quasigroups, to homomorphisms between semisymmetric quasigroups. For this reason, the technique is known as "semisymmetrization". It singles out semisymmetric quasigroups as forming a class especially worthy of attention.

The second section records the basic identities for quasigroups. The category $\underline{\underline{\mathrm{Qtp}}}$ of homotopies between quasigroups is introduced in the third section. The fourth section defines the two functors $\Sigma\colon \underline{\underline{\mathrm{P}}} \to \underline{\underline{\mathrm{Qtp}}}$ and $\Delta\colon \underline{\underline{\mathrm{Qtp}}} \to \underline{\underline{\mathrm{P}}}$ connecting $\underline{\underline{\mathrm{Qtp}}}$ with the category $\underline{\underline{\mathrm{P}}}$ of semisymmetric quasigroups and homomorphisms. The following section presents the main result, Theorem 5.2, stating that the semisymmetrization functor Δ is right adjoint to the forgetful functor Σ. The final section specializes the results of §5, showing that the semisymmetrization of a central quasigroup is entropic.

Presented by R. W. Quackenbush.
Received December 20, 1996; accepted in final form September 17, 1997.

Since quasigroups are non-associative, brackets proliferate. They are kept under control by writing functions and functors to the right of their arguments (enabling one to read compositions in natural order from left to right). Another device is to write the membership of an ordered pair (x, y) in an equivalence relation θ as $x\theta y$.

2. Quasigroups

A *quasigroup* Q or $(Q, \cdot, /, \backslash)$ is a set Q equipped with binary operations xy or $x \cdot y$ of *multiplication*, x/y of *right division*, and $x\backslash y$ of *left division*, such that the identities

$$\begin{cases} IL: y\backslash(y \cdot x) = x; & IR: x = (x \cdot y)/y \\ SL: y \cdot (y\backslash x) = x; & SR: x = (x/y) \cdot y \end{cases}$$

are satisfied. As consequences of the identities (2.1), one obtains further identities

$$DL: y/(x\backslash y) = x; \qquad DR: x = (y/x)\backslash y. \tag{2.2}$$

The variety of all quasigroups is denoted by $\underline{\underline{Q}}$. Recall that a set Q with a binary multiplication $x \cdot y$ admits a quasigroup structure $(Q, \cdot, /, \backslash)$ in the sense of (2.1) iff knowledge of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. (Cf. [12] and [15, §1.1] for comparisons of the two approaches.)

In the current paper, quasigroups satisfying the *semisymmetric identity* [2, (2.4)]

$$(yx)y = x \tag{2.3}$$

play an important role. Such quasigroups are also called "3-cyclic". They have been studied previously by Osborn [11], Mendelsohn [8] [9], Grätzer and Padmanabhan [4], Mitschke and Werner [10], and DiPaola and Nemeth [3]. The variety of semisymmetric quasigroups is denoted by $\underline{\underline{P}}$.

PROPOSITION 2.1. *The following quasigroup identities are equivalent*:
(a) $(yx)y = x$;
(b) $y(xy) = x$;
(c) $x\backslash y = yx$;
(d) $x/y = yx$.
In particular, each holds in $\underline{\underline{P}}$.

Proof. (a) $\Rightarrow$ (b): $y(xy) = ((xy)x)(xy) = x$.
(b) $\Rightarrow$ (c): $yx = x\backslash(x(yx)) = x\backslash y$.
(c) $\Rightarrow$ (b): $x = y(y\backslash x) = y(xy)$.
(b) $\Rightarrow$ (a): $(yx)y = (yx)(x(yx)) = x$.
(a) $\Rightarrow$ (d): $yx = ((yx)y)/y = x/y$.
(d) $\Rightarrow$ (a): $x = (x/y)y = (yx)y$.
Note that (a) holds in $\underline{\underline{\mathrm{P}}}$, by definition. □

COROLLARY 2.2. *Let* $(Q, .)$ *be a set with a binary multiplication satisfying* (*a*). *Defining a right division* / *by* (*d*) *and a left division* \ *by* (*c*) *then yields a semisymmetric quasigroup* $(Q, ., /, \backslash)$.

Proof. First note that $(Q, .)$ satisfies (b), by the above proof of (a) $\Rightarrow$ (b). One then verifies:
IL: $y\backslash(yx) = (yx)y = x$;
SL: $y(y\backslash x) = y(xy) = x$;
IR: $(xy)/y = y(xy) = x$;
SR: $(x/y)y = (yx)y = x$.
Thus $(Q, ., /, \backslash)$ is a quasigroup. □

3. Homotopies

A function $f: Q \to P$ between (the underlying sets of) two quasigroups $(Q, ., /, \backslash)$ and $(P, ., /, \backslash)$ is a *homomorphism* iff

$$\forall x, y \in Q, \qquad xf \cdot yf = (xy)f. \tag{3.1}$$

Note that (3.1) implies $xf/yf = (x/y)f$ and $xf\backslash yf = (x\backslash y)f$ for all x, y in Q. A variety of quasigroups, such as $\underline{\underline{\mathrm{P}}}$ or $\underline{\underline{\mathrm{Q}}}$, is considered as a category by taking the homomorphisms between its members (objects) to be the morphisms of the category. The isomorphisms in such categories [6, Definition 5.13] are then just quasigroup isomorphisms in the usual algebraic sense.

A triple $(f_1, f_2, f_3): Q \to P$ of functions between (the underlying sets of) two quasigroups $(Q, ., /, \backslash)$ and $(P, ., /, \backslash)$ is a *homotopy* iff

$$\forall x, y \in Q, \qquad xf_1 \cdot yf_2 = (xy)f_3. \tag{3.2}$$

Note that (3.2) implies

$$\forall x, y \in Q, \qquad xf_3/yf_2 = (x/y)f_1 \tag{3.3}$$

and

$$\forall x, y \in Q, \qquad xf_1\backslash yf_3 = (x\backslash y)f_2. \tag{3.4}$$

The maps f_1, f_2, f_3 are known as the *first*, *second* and *third components* respectively of the homotopy (f_1, f_2, f_3): $Q \to P$. The *category* $\underline{\underline{\text{Qtp}}}$ *of quasigroup homomorphisms* has the class $\underline{\underline{\text{Q}}}$ of all quasigroups as its object class. For given quasigroups Q and P, the class $\underline{\underline{\text{Qtp}}}(Q, P)$ of $\underline{\underline{\text{Qtp}}}$-morphisms from Q to P is the set of homotopies from Q to P. Given (f_1, f_2, f_3) in $\underline{\underline{\text{Qtp}}}(Q, P)$ and (g_1, g_2, g_3) in $\underline{\underline{\text{Qtp}}}(P, N)$, the composite $(f_1, f_2, f_3)(g_1, g_2, g_3)$ in $\underline{\underline{\text{Qtp}}}(Q, N)$ is just (f_1g_1, f_2g_2, f_3g_3). Isomorphisms in the category $\underline{\underline{\text{Qtp}}}$ are called *isotopies*. They are just the homotopies having each component bijective. The identity at an object Q of $\underline{\underline{\text{Qtp}}}$ is the isotopy $(1_Q, 1_Q, 1_Q)$: $Q \to Q$, each of whose components is the identity map $1_Q: Q \to Q$; $q \mapsto q$.

4. Functors

Comparing (3.1) and (3.2), it is immediate that there is a functor $\Sigma: \underline{\underline{\text{Q}}} \to \underline{\underline{\text{Qtp}}}$ whose morphism part sends a quasigroup homomorphism $f: Q \to P$ to the quasigroup homotopy $(f, f, f): Q \to P$ with equal components. The restriction of this functor to the full subcategory $\underline{\underline{\text{P}}}$ of $\underline{\underline{\text{Q}}}$ is also denoted by

$$\Sigma: \underline{\underline{\text{P}}} \to \underline{\underline{\text{Qtp}}}. \tag{4.1}$$

This functor Σ is "forgetful" in two senses. It "forgets" that its objects were semisymmetric, and it "forgets" that its morphisms had equal components.

Now let Q or $(Q, \cdot, /, \backslash)$ be a quasigroup. Define a multiplication on the direct cube Q^3 of the set Q by

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (y_3/x_2, y_1\backslash x_3, x_1y_2). \tag{4.2}$$

Then $((y_1, y_2, y_3)(x_1, x_2, x_3))(y_1, y_2, y_3)$

$$\begin{aligned} &= (x_3/y_2, x_1\backslash y_3, y_1x_2)(y_1, y_2, y_3) \\ &= (y_3/(x_1\backslash y_3), y_1\backslash(y_1x_2), (x_3/y_2)y_2) \\ &= (x_1, x_2, x_3), \end{aligned}$$

the last equality following by DL, IL, and SR. By Corollary 2.2, one thus obtains a semisymmetric quasigroup

$$Q\Delta := (Q^3, \cdot, /, \backslash) \tag{4.3}$$

and the object part $Q \mapsto Q\Delta$ of a functor $\Delta : \underline{\underline{\mathrm{Qtp}}} \to \underline{\underline{\mathrm{P}}}$. The morphism part of this functor sends a quasigroup homotopy $(f_1, f_2, f_3): Q \to P$ to the map

$$Q\Delta \to P\Delta\,;\,(x_1, x_2, x_3) \mapsto (x_1 f_1, x_2 f_2, x_3 f_3). \tag{4.4}$$

Indeed for (x_1, x_2, x_3) and (y_1, y_2, y_3) in $Q\Delta$ one has $(x_1 f_1, x_2 f_2, x_3 f_3)(y_1 f_1, y_2 f_2, y_3 f_3)$

$$= (y_3 f_3 / x_2 f_2,\ y_1 f_1 \backslash x_3 f_3,\ x_1 f_1 \cdot y_2 f_2)$$

$$= ((y_3/x_2) f_1,\ (y_1 \backslash x_3) f_2,\ (x_1 y_2) f_3)$$

$$= ((x_1, x_2, x_3)(y_1, y_2, y_3))(f_1, f_2, f_3)^{\Delta},$$

the penultimate equality holding by (3.3), (3.4), and (3.2) in its respective components. Thus $(f_1, f_2, f_3)\Delta : Q\Delta \to P\Delta$ is a quasigroup homomorphism. It is then routine to complete the proof of

THEOREM 4.1. *There is a functor* $\Delta : \underline{\underline{\mathrm{Qtp}}} \to \underline{\underline{\mathrm{P}}}$ *with object part* (4.3) *and morphism part* (4.4). □

The functor of Theorem 4.1 is called the *semisymmetrization functor*. The image (4.3) of a quasigroup Q under Δ is called the *semisymmetrization* of Q.

5. Adjointness

It will now be shown that there is an adjunction

$$\underline{\underline{\mathrm{Qtp}}}(P\Sigma, Q) \cong \underline{\underline{\mathrm{P}}}(P, Q\Delta) \tag{5.1}$$

between the functors $\Sigma : \underline{\underline{\mathrm{P}}} \to \underline{\underline{\mathrm{Qtp}}}$ of (4.1) and $\Delta : \underline{\underline{\mathrm{Qtp}}} \to \underline{\underline{\mathrm{P}}}$ of Theorem 4.1.

REMARK 5.1. For some readers, it may be helpful to compare the adjunction (5.1) with the more familiar adjunction

$$\underline{\underline{Q}}(XF, Q) \cong \underline{\underline{Set}}(X, QU) \tag{5.2}$$

between the free quasigroup functor $F: \underline{\underline{Set}} \to \underline{\underline{Q}}$ and the underlying set functor $U: \underline{\underline{Q}} \to \underline{\underline{Set}}$. Recall that the unit $\eta: 1 \to FU$ of the adjunction (5.2) is a natural transformation from the identity functor on $\underline{\underline{Set}}$ to the composite functor FU on $\underline{\underline{Set}}$. The component $\eta_X: X \to XFU$ of the unit at a set X inserts the elements of X as generators into the underlying set of the free quasigroup on X. Dually, the counit $\varepsilon: UF \to 1$ of the adjunction (5.2) is a natural transformation from the composite functor $UF: \underline{\underline{Q}} \to \underline{\underline{Q}}$ to the identity functor on $\underline{\underline{Q}}$. The component $\varepsilon_Q: QUF \to Q$ of the counit at a quasigroup Q is the unique homomorphism from the free quasigroup QUF to Q that extends the identity function $1_Q: Q \to Q$. □

For a semisymmetric quasigroup P, consider the function

$$\eta_P: P \to P\Sigma\Delta;\, x \mapsto (x, x, x) \tag{5.3}$$

from P to (the underlying set P^3 of) the semisymmetric quasigroup $P\Sigma\Delta$. For x, y in P, one has

$$\begin{aligned} x\eta_P \cdot y\eta_P &= (x, x, x) \cdot (y, y, y) \\ &= (y/x, y\backslash x, xy), \\ &= (xy, xy, xy) = (xy)\eta_P, \end{aligned}$$

the second equality holding by (4.2) and the third by Proposition 2.1 (d), (c). Thus η_P is actually a quasigroup homomorphism between semisymmetric quasigroups, i.e. a $\underline{\underline{P}}$-morphism. It is straightforward to verify that η_P is the component at P of a natural transformation $\eta: 1 \to \Sigma\Delta$. Dually, for a quasigroup Q, consider the triple

$$(\pi_1, \pi_2, \pi_3): Q\Delta\Sigma \to Q \tag{5.4}$$

of projections $\pi_i: Q^3 \to Q$; $(x_1, x_2, x_3) \mapsto x_i$. Now for (x_1, x_2, x_3) and (y_1, y_2, y_3) in $Q\Delta\Sigma$, one has $(x_1, x_2, x_3)\pi_1 \cdot (y_1, y_2, y_3)\pi_2 = x_1 y_2 = (y_3/x_2, y_1\backslash x_3, x_1 y_2)\pi_3 = ((x_1, x_2, x_3) \cdot (y_1, y_2, y_3))\pi_3$ by (4.2). Thus (5.4) is a homotopy or $\underline{\underline{Qtp}}$-morphism ε_Q. It is again straightforward to verify that ε_Q is the component at Q of a natural transformation $\varepsilon: \Delta\Sigma \to 1$.

THEOREM 5.2. *The functor* $\Delta : \underline{\underline{\mathrm{Qtp}}} \to \underline{\underline{\mathrm{P}}}$ *of Theorem* 4.1 *is right adjoint to the functor* $\Sigma : \underline{\underline{\mathrm{P}}} \to \underline{\underline{\mathrm{Qtp}}}$ *of* (4.1).

Proof. For a quasigroup Q,

$$\eta_{Q\Delta} \cdot \varepsilon_Q^{\Delta} = 1_{Q\Delta}. \tag{5.5}$$

Indeed, for (x_1, x_2, x_3) in $Q\Delta$, one has $(x_1, x_2, x_3)\eta_{Q\Delta}\varepsilon_Q^{\Delta} = ((x_1, x_2, x_3), (x_1, x_2, x_3), (x_1, x_2, x_3))\varepsilon_Q^{\Delta} = ((x_1, x_2, x_3)\pi_1, (x_1, x_2, x_3)\pi_2, (x_1, x_2, x_3)\pi_3) = (x_1, x_2, x_3)$. For a semisymmetric quasigroup P,

$$\eta_P^{\Sigma} \cdot \varepsilon_{P\Sigma} = 1_{P\Sigma}. \tag{5.6}$$

Indeed, for $1 \le i \le 3$, the i-th component of the composite homotopy on the left hand side of (5.6) maps an element x of P or $P\Sigma$ to $(x, x, x)\pi_i = x$. The equations (5.5) and (5.6) are the "triangular identities" [7, IV.1 (9)] exhibiting the required adjointness [7, Theorem IV.1.2. (v)], cf. [6, 27.1]. □

COROLLARY 5.3. *The category* $\underline{\underline{\mathrm{Qtp}}}$ *is isomorphic to a category of homomorphisms between semisymmetric quasigroups.*

Proof. Let $U : \underline{\underline{\mathrm{P}}} \to \underline{\underline{\mathrm{Set}}}$ be the restriction to $\underline{\underline{\mathrm{P}}}$ of the faithful functor $U : \underline{\underline{\mathrm{Q}}} \to \underline{\underline{\mathrm{Set}}}$ of Remark 5.1. There is then a faithful functor $G : \underline{\underline{\mathrm{Qtp}}} \to \underline{\underline{\mathrm{Set}}}$ such that the diagram

$$\begin{array}{ccc} \underline{\underline{\mathrm{Qtp}}} & \xrightarrow{\Delta} & \underline{\underline{\mathrm{P}}} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle U} \\ \underline{\underline{\mathrm{Set}}} & = & \underline{\underline{\mathrm{Set}}} \end{array} \tag{5.7}$$

commutes. The image of $\underline{\underline{\mathrm{Qtp}}}$ under Δ is readily verified to be a subcategory $\underline{\underline{\mathrm{Ptp}}}$ of $\underline{\underline{\mathrm{P}}}$. The inverse of the corestriction $\Delta : \underline{\underline{\mathrm{Qtp}}} \to \underline{\underline{\mathrm{Ptp}}}$ of the semisymmetrization functor is the *cube root functor* $K : \underline{\underline{\mathrm{Ptp}}} \to \underline{\underline{\mathrm{Qtp}}}$ making the diagram

$$\begin{array}{ccccc} \underline{\underline{\mathrm{Qtp}}} & \xrightarrow{\Delta} & \underline{\underline{\mathrm{Ptp}}} & \xrightarrow{K} & \underline{\underline{\mathrm{Qtp}}} \\ {\scriptstyle G}\downarrow & & {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle G} \\ \underline{\underline{\mathrm{Set}}} & = & \underline{\underline{\mathrm{Set}}} & = & \underline{\underline{\mathrm{Set}}} \end{array} \tag{5.8}$$

commute. Thus for an object P of $\underline{\underline{\mathrm{Ptp}}}$, the underlying set PKU of PK has the underlying set PU of P as its direct cube. The quasigroup structure on the set PKU is obtained by the homotopy (5.4). □

6. Central and entropic quasigroups

A quasigroup Q is said to be *central* if the diagonal $\widehat{Q} = \{(q, q) \mid q \in Q\}$ is a normal subquasigroup of the direct square Q^2. In other words there is a *centering* congruence V on Q^2 having $\widehat{Q}$ as a congruence class. The class $\underline{\underline{Z}}$ of central quasigroups forms a variety [6.1, Prop. III.3.12]. Let $\underline{\underline{Ztp}}$ denote the category of central quasigroups and homotopies between them.

A quasigroup Q is said to be *entropic* if the multiplication $Q^2 \to Q$; $(x, y) \mapsto xy$ is a homomorphism. In other words, the quasigroup satisfies the identity

$$(xy)(zt) = (xz)(yt). \tag{6.1}$$

Each non-empty subquasigroup H of an entropic quasigroup Q is normal. (Indeed, (6.1) shows that coset multiplication $Hy \cdot Ht = H \cdot yt$ is well-defined, and then H is a class of the kernel congruence of the homomorphism $y \mapsto Hy$.) Thus entropic quasigroups are central [13, 234]. On the other hand, if M is a faithful module over the free group on a two-element set $\{L, R\}$, then the multiplication $xy = xR + yL$ on M yields a central quasigroup that is not entropic. The variety of entropic semisymmetric quasigroups, as well as the category of homomorphisms between them, are denoted by $\underline{\underline{P}} \otimes \underline{\underline{P}}$. (The notation reflects the fact that entropic semisymmetric quasigroups are just $\underline{\underline{P}}$-algebras in the category $\underline{\underline{P}}$.)

REMARK 6.1. In the universal algebra literature, central algebras have often been described as "abelian" or "affine". However, these names are especially confusing when applied to quasigroups. Many synonyms for "entropic" have been used, including "abelian" (cf. [14, §6]). The term "entropic" refers to the "inner turning" of y and z in (6.1). It is also worth noting the connection with the information-theoretic concept of entropy [17]. □

The proof of Theorem 6.3 below uses some identities satisfied by central quasigroups.

PROPOSITION 6.2. *Central quasigroups satisfy*:
(a) $(zt)/(y\backslash x) = (yt)/(z\backslash x)$;
(b) $(t/z)\backslash(xy) = (t/y)\backslash(xz)$;
(c) $(y/x)(t\backslash z) = (z/x)(t\backslash y)$.

Proof. Let Q be central, with centering congruence V on Q^2.

(a) By reflexivity of V, one has $(y, z)V(y, z)$. Since $\widehat{Q}$ is a V-class, one has $(t, t)V(y\backslash z, y\backslash z)$. On multiplying and using SL of (2.1), one obtains

$$(yt, zt)V(z, z(y\backslash z)). \tag{6.2}$$

Again, one has $(z, y)V(z, y)$ and $(x, x)V(z, z)$. Using left division, one obtains

$$(z\backslash x, y\backslash x)V(z\backslash z, y\backslash z). \tag{6.3}$$

Finally, right division of (6.2) by (6.3), with the use of DL and IR, yields

$$((yt)/(z\backslash x), (zt)/(y\backslash x))V(z, z). \tag{6.4}$$

Since $\widehat{Q}$ is a V-class, the desired equality (a) follows.

(b) is obtained from (a) by the "flipping argument" [15, p. 14], i.e. interchange of right and left.

(c) Right division of $(y, z)V(y, z)$ by $(x, x)V(z\backslash y, z\backslash y)$, along with DL, gives

$$(y/x, z/x)V(z, z/(z\backslash y)). \tag{6.5}$$

Left division of $(t, t)V(z, z)$ into $(z, y)V(z, y)$ gives

$$(t\backslash z, t\backslash y)V(z\backslash z, z\backslash y). \tag{6.6}$$

Finally, multiplication of (6.5) by (6.6), with the use of SL and SR, yields

$$((y/x)(t\backslash z), (z/x)(t\backslash y))V(z, z). \tag{6.7}$$

Since $\widehat{Q}$ is a V-class, the desired equality (c) follows. □

THEOREM 6.3. *The functors of* §4 *restrict to a functor* $\Sigma\colon \underline{\underline{\mathrm{P}}} \otimes \underline{\underline{\mathrm{P}}} \to \underline{\underline{\mathrm{Ztp}}}$ *left adjoint to a functor* $\Delta\colon \underline{\underline{\mathrm{Ztp}}} \to \underline{\underline{\mathrm{P}}} \otimes \underline{\underline{\mathrm{P}}}$. *In particular, the semisymmetrization of a central quasigroup is entropic.*

Proof. As noted above, entropic quasigroups are central. Thus Σ restricts to Σ: $\underline{\underline{\mathrm{P}}} \otimes \underline{\underline{\mathrm{P}}} \to \underline{\underline{\mathrm{Ztp}}}$. It remains to be shown that the semisymmetrization $Q\Delta$ of a central quasigroup Q is entropic. Consider elements x, y, z, t of Q^3 with $x\pi_i = x_i$, etc., for $1 \leq i \leq 3$. Using (4.2), the entropic law (6.1) for $Q\Delta$ reduces to the equality between the triples

$$((z_1 t_2)/(y_1\backslash x_3), (t_3/z_2)\backslash(x_1 y_2), (y_3/x_2)(t_1\backslash z_3)) \tag{6.8}$$

and

$$((y_1 t_2)/(z_1\backslash x_3), (t_3/y_2)\backslash(x_1 z_2)(z_3/x_2)(t_1\backslash y_3)). \tag{6.9}$$

The equality follows on applying (a), (b) and (c) of Proposition 6.2 to the respective components of (6.8) and (6.9). □

Restricting the proof of Corollary 5.3, one obtains

COROLLARY 6.4. *The category* $\underline{\underline{\mathrm{Ztp}}}$ *of homotopies between central quasigroups is isomorphic to a category of homomorphisms between entropic semisymmetric quasigroups*. □

REFERENCES

[1] BRUCK, R. H., *What is a loop?* in *Studies in Modern Algebra*, A. A. Albert (ed.), Prentice-Hall, Englewood Cliffs, NJ, 1963, pp. 59–99.
[2] CHEIN, O., *Examples and methods of construction*, in *Quasigroups and Loops: Theory and Applications*, O. Chein et al. (eds.), Heldermann, Berlin, 1990, pp. 27–93.
[3] DIPAOLA, J. W. and NEMETH, E., *Generalized triple systems and medial quasigroups*, Proc. 7th S.E. Conf. on Comb., Graph Th. and Comp., 1973, Conf. No. 17, Util. Math. Publ., Winnipeg, 1976, pp. 289–306.
[4] GRÄTZER, G. and PADMANABHAN, R., *On idempotent commutative and nonassociative groupoids*, Proc. Amer. Math. Soc. *29* (1973), 249–264.
[5] GVARAMIYA, A. A. and PLOTKIN, B. I., *The homotopies of quasigroups and universal algebras*, in *Universal Algebra and Quasigroup Theory*, A. Romanowska and J. D. H. Smith (eds.), Heldermann, Berlin, 1992, pp. 89–99.
[6] HERRLICH, H. and STRECKER, G. E., *Category Theory*, Allyn and Bacon, Boston, MA, 1973.
[7] MAC LANE, S., *Categories for the Working Mathematician*, Springer, New York, NY, 1971.
[8] MENDELSOHN, N. S., *Orthogonal Steiner systems*, Aequationes Math. *5* (1970), 268–272.
[9] MENDELSOHN, N. S., *A natural generalization of Steiner triple systems*, in *Proc. Sci. Res. Council Atlas Symp. on Computers in Number Theory 2 Oxford, 1969*, Academic Press, London, 1971, pp. 323–338.
[10] MITSCHKE, A. and WERNER, H., *On groupoids representable by vector spaces over finite fields*, Arch. der Math. *24* (1973), 14–20.
[11] OSBORN, J. M., *New loops from old geometries*, Amer. Math. Monthly *68* (1961), 103–107.
[12] QUACKENBUSH, R. W., *Quasigroups, universal algebra and combinatorics*, in *Universal Algebra and Quasigroup Theory*, A. Romanowska and J. D. H. Smith (eds.), Heldermann, Berlin, 1992, pp. 159–171.
[13] SMITH, J. D. H., *Mal'cev Varieties*, Springer, Berlin, 1976.
[14] SMITH, J. D. H., *Finite equationally complete entropic quasigroups*, in *Contributions to General Algebra (Proc. Klagenfurt Conf., Klagenfurt, 1978)*, Heyn, Klagenfurt, 1979, pp. 345–356.
[15] SMITH, J. D. H., *Representation Theory of Infinite Groups and Finite Quasigroups*, Les Presses de l'Université de Montréal, Montréal, 1986.
[16] SMITH, J. D. H., *Centrality*, in *Quasigroups and Loops: Theory and Applications*, O. Chein et al. (eds.), Heldermann, Berlin, 1990, pp. 95–114.
[17] SMITH, J. D. H., *Entropy, character theory and centrality of finite quasigroups*, Math. Proc. Camb. Phil. Soc. *108* (1990), 435–443.

Department of Mathematics
Iowa State University
Ames, IA 50011
U.S.A.
e-mail: jdhsmith@math.iastate.edu