GROUPS, TRIALITY, AND HYPERQUASIGROUPS

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Abstract. Hyperquasigroups were recently introduced to provide a more symmetrical approach to quasigroups, and a far-reaching implementation of triality (S_3-action). In the current paper, various connections between hyperquasigroups and groups are examined, on the basis of established connections between quasigroups and groups. A new graph-theoretical characterization of hyperquasigroups is exhibited. Torsors are recognized as hyperquasigroups, and group representations are shown to be equivalent to linear hyperquasigroups. The concept of orthant structure is introduced, as a tool for recovering classical information from a hyperquasigroup.

1. Introduction

This paper is concerned with the recently introduced algebraic structures known as hyperquasigroups, which are intended as refinements of quasigroups. A quasigroup $(Q, \cdot)$ was first understood as a set Q with a binary multiplication, denoted by $\cdot$ or mere juxtaposition, such that in the equation

$$x \cdot y = z \ ,$$

knowledge of any two of x, y, z specifies the third uniquely. To make a distinction with subsequent concepts, it is convenient to describe a quasigroup in this original sense as a *combinatorial quasigroup* $(Q, \cdot)$. A *loop* $(Q, \cdot, e)$ is a combinatorial quasigroup $(Q, \cdot)$ with an *identity* element e such that $e \cdot x = x = x \cdot e$ for all x in Q. The body of the multiplication table of a combinatorial quasigroup of finite order n is a *Latin square* of order n, an $n \times n$ square array in which each row and each column contains each element of an n-element set exactly once. Conversely, each Latin square becomes the body of the multiplication table of a combinatorial quasigroup if distinct labels from the square are attached to the rows and columns of the square. In the guise of Latin squares, quasigroups form one of the oldest topics of algebra, dating back at least as far as Euler [23]. Now in Bourbaki's terminology [4, Ch. 1, §1.1], a set with a binary operation is a *magma*[1], so a combinatorial quasigroup is a special kind of magma. For each element q of a magma $(Q, \cdot)$, define the *left multiplication*

$$L(q) : Q \to Q; x \mapsto q \cdot x \tag{1.1}$$

and *right multiplication*

$$R(q) : Q \to Q; x \mapsto x \cdot q \, . \tag{1.2}$$

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[1]Øystein Ore's earlier term "groupoid" is now often applied to denote categories in which all the arrows are invertible.

If $(Q,\cdot)$ is a combinatorial quasigroup, then the right and left multiplications are bijections of the underlying set Q. Indeed, a magma $(Q,\cdot)$ is a combinatorial quasigroup if and only if the left multiplication $L(q)$ and right multiplication $R(q)$ are bijective for each element q of Q.

Unfortunately, the combinatorial specification of a quasigroup does not admit the use of deeper algebraic techniques. For example, there is a combinatorial quasigroup $(Q,\cdot)$ admitting a magma homomorphism $f\colon (Q,\cdot)\to(P,\cdot)$ whose image is not a combinatorial quasigroup [27, Ch. I, Ex. 2.2.1]. In 1949, Evans [11] redefined quasigroups as *equational quasigroups*, sets $(Q,\cdot,/,\backslash)$ equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

(IL)	$\mathsf{v}\backslash(\mathsf{v}\cdot\mathsf{w})=\mathsf{w}\,;$	(IR)	$\mathsf{w}=(\mathsf{w}\cdot\mathsf{v})/\mathsf{v}\,;$
(SL)	$\mathsf{v}\cdot(\mathsf{v}\backslash\mathsf{w})=\mathsf{w}\,;$	(SR)	$\mathsf{w}=(\mathsf{w}/\mathsf{v})\cdot\mathsf{v}\,.$

The identities (IL), (IR) give the injectivity of the left and right multiplications, while (SL), (SR) give their surjectivity. An equational quasigroup $(Q,\cdot,/,\backslash)$ yields a combinatorial quasigroup $(Q,\cdot)$. Conversely, a combinatorial quasigroup $(Q,\cdot)$ yields an equational quasigroup $(Q,\cdot,/,\backslash)$ with divisions $x/y=xR(y)^{-1}$ and $x\backslash y=yL(x)^{-1}$. Equational quasigroups form a variety in the sense of universal algebra, so homomorphic images of equational quasigroups are equational quasigroups [27, p. 314]. Evans' reformulation of the quasigroup concept opened up combinatorial questions about quasigroups and Latin squares to analysis with algebraic techniques [14].

The theory of equational quasigroups has an explicit twofold symmetry, given by a chiral duality or S_2-action between left and right. For example, reflection of the identities (IL) and (SL) in the vertical line above yields the respective identities (IR) and (SR)—especially given the symmetrical sans-serif typeface used for the variables in these identities! The chiral symmetry interchanges the right and left divisions, and reverses the order of the arguments in the binary operations. In 1951, Evans solved the word problem for equational quasigroups (and loops) [12]. The chiral duality led to some simplification of the solution, but still left many separate cases to consider, at least in principle. Recently, however, Evans' solution of the word problem was drastically simplified by the explicit use of a stronger triality symmetry or S_3-action[2] that interchanges all three equational quasigroup operations and their opposites [25]. Although this symmetry is already implicit in the theory of equational quasigroups, and its presence has long been recognized, the choice of specific operations in the equational theory has been an impediment to its use in practice. Indeed, implementation of the symmetry entails the introduction of a new approach to quasigroups, by means of the concept of a hyperquasigroup [24, 26]. In the historical context, hyperquasigroups may thus be considered as a further step beyond the progression from combinatorial quasigroups to equational quasigroups.

The interplay between quasigroups and groups has long been a fundamental theme of quasigroup theory, for example working with the (combinatorial) multiplication group of a quasigroup (compare Section 2), or obtaining quasigroups from transversals to subgroups of groups (compare Section 10). The aim of the current

[2]The triality symmetry under consideration in the current paper was identified in [25] as *syntactic triality*. That paper also discussed *semantic triality*, which is more closely related to the triality symmetry of Moufang loops and the Coxeter-Dynkin diagram of type D_4 (compare [9]).

paper is to establish comparable connections between hyperquasigroups and groups. Following some background on universal multiplication groups (Section 2) and linear quasigroups (Section 3), hyperquasigroups themselves are defined in Section 4, along with the auxiliary concept of a reflexion-inversion space. Section 5 presents a new class of examples, showing how torsors (principal homogeneous spaces) may be recognized as hyperquasigroups. The key concept of the Cayley graph of a hyperquasigroup is introduced in Section 6. Within the Cayley graph, (6.5) provides a striking geometrical characterization of hyperquasigroups. Sections 7 – 9 describe various groups associated with a hyperquasigroup, modeled on the combinatorial and universal multiplication groups of a quasigroup. In Section 10, it is shown how hyperquasigroups are encoded within these groups by a certain system of transversals.

The final two sections focus on the role of triality in hyperquasigroups, taking advantage of an orthant structure within the reflexion-inversion space to recover "classical" quasigroup information from a hyperquasigroup. Section 11 introduces the class of linear hyperquasigroups, indicating how their orthant structure leads to an equivalence with group representations. From this point of view, hyperquasigroups emerge as a broad extension of group representations. Section 12 uses orthant structure to recover the combinatorial multiplication group of a quasigroup, and the family of orthogonal Latin squares in a desarguean projective plane, from corresponding hyperquasigroups. Although it has been possible to identify and apply the orthant structure in these examples, it should be noted that no universal definition of the concept is currently available. Indeed, the search for such a definition is one of the open problems of the subject.

Readers seeking further background in quasigroup theory are referred to [23]. Generally speaking, notational and other conventions used in the paper follow [27]. In particular, algebraic notation (with functions following their arguments as in the usages x^2 and $n!$) is taken as the default option in primal situations.

2. Multiplication groups

The (*combinatorial*) *multiplication group* $\mathrm{Mlt}(Q,\cdot)$ or $\mathrm{Mlt}\,Q$ of a quasigroup $(Q,\cdot)$ is defined to be the subgroup

$$\langle L(q), R(q) \mid q \in Q\rangle_{Q!} \tag{2.1}$$

of the symmetric group $Q!$ of all bijections of the set Q that is generated by all the left and right multiplications. For example, if Q is a group with center $Z(Q)$, then the multiplication group G of Q is specified by the exact sequence

$$1 \to Z(Q) \xrightarrow{\Delta} Q \times Q \xrightarrow{T} G \to 1$$

with $\Delta\colon z \mapsto (z,z)$ and $T : (q_1,q_2) \mapsto L(q_1)^{-1}R(q_2)$. Note that the image of a diagonal pair (q,q) in $Q\times Q$ under T is the inner automorphism of Q given by conjugation with q.

If $f\colon (Q_1,\cdot,/,\backslash) \to (Q_2,\cdot,/,\backslash)$ is a surjective (equational) quasigroup homomorphism, then a group homomorphism $\mathrm{Mlt}\,f\colon \mathrm{Mlt}\,Q_1 \to \mathrm{Mlt}\,Q_2$ is defined by $L(q) \mapsto L(qf)$ and $R(q) \mapsto R(qf)$ for q in Q_1. This definition may break down if f is not surjective. Consider the symmetric group embedding $f : \{0,1\}! \hookrightarrow \{0,1,2\}!$ as an example. The right and left multiplications by the transposition $(0\ 1)$ coincide in $\mathrm{Mlt}\{0,1\}!$, but in $\mathrm{Mlt}\{0,1,2\}!$ they become distinct.

In order to overcome the ill-definition problem for $\mathrm{Mlt}\, f \colon \mathrm{Mlt}\, Q_1 \to \mathrm{Mlt}\, Q_2$ when $f \colon Q_1 \to Q_2$ is not surjective, one considers the so-called "universal multiplication groups." If Q is a subquasigroup of a quasigroup P, then the *relative multiplication group* $\mathrm{Mlt}_P Q$ of Q in P is the subgroup

$$\langle L_P(q), R_P(q) \mid q \in Q \rangle_{P!}$$

of $P!$ generated by all the left multiplications $L_P(q) \colon P \to P; x \mapsto qx$ and right multiplications $R_P(q) \colon P \to P; x \mapsto xq$ on P by elements q of Q. Suppose that Q is a member of a variety or equationally-defined class $\mathbf{V}$ of (equational) quasigroups, for example the class $\mathbf{G}$ of associative quasigroups (the class comprising groups and the empty quasigroup). Let $\widetilde{Q}$ or $Q_{\mathbf{V}}[X]$ be the coproduct (or "free product") in $\mathbf{V}$ of Q and the free $\mathbf{V}$-quasigroup on a single generator X [23, §2.7][27, IV.2.2]. In other words, $Q_{\mathbf{V}}[X]$ contains an element X such that for each element x of a $\mathbf{V}$-quasigroup Q', and for each homomorphism $f \colon Q \to Q'$, there is a unique homomorphism $\widetilde{f} \colon \widetilde{Q} \to Q'$ extending f, and taking the "indeterminate" X to x. Then the *universal multiplication group* $U(Q; \mathbf{V})$ of Q in $\mathbf{V}$ is defined to be the relative multiplication group of Q in $Q_{\mathbf{V}}[X]$. For each homomorphism $f \colon Q_1 \to Q_2$ of $\mathbf{V}$-quasigroups, a group homomorphism $U(f; \mathbf{V}) \colon U(Q_1; \mathbf{V}) \to U(Q_2; \mathbf{V})$ is well-defined by sending $L_{\widetilde{Q}}(q)$ to $L_{\widetilde{Q}}(qf)$ and $R_{\widetilde{Q}}(q)$ to $R_{\widetilde{Q}}(qf)$ for each element q of Q. If $\mathbf{Q}$ is the class of all quasigroups, then $U(Q; \mathbf{Q})$ is free on the disjoint union $L(Q) + R(Q)$. If Q is a group, then $U(Q; \mathbf{G})$ is $Q \times Q$, realized as the product of the (regular permutation) group of left multiplications with the (regular permutation) group of right multiplications. The group $U(Q; \mathbf{G})$ is sometimes called the *diagonal group* [7].

3. Linear quasigroups

Linear quasigroups constitute one of the most important classes of quasigroups. Let A be a group of automorphisms of an abelian group (or right A-module) M. Suppose that A is generated by elements R and S. (Recall, for instance, that all finite simple groups are known to be 2-generated.) Then a combinatorial quasigroup structure $(M, \circ)$ or equational quasigroup structure $(M, \circ, /\!/, \backslash\!\backslash)$ is defined by

$$x \circ y = xR + yS \tag{3.1}$$

for x, y in M. Such a quasigroup $(M, \circ)$ is known as a *linear quasigroup*[3]. It may be pointed by the zero element 0 of the module M. Given such a pointed linear quasigroup $(M, \circ, 0)$, the abelian group $(M, +)$ and group A may be recovered. Indeed $x + y = (x /\!/ 0) \circ (y \backslash\!\backslash 0)$ for x, y in M, while $R = R(0)$ and $S = L(0)$ in $(M, \circ)$.

Example 3.1 (Dihedral groups)**.** Given $1 < d \in \mathbb{Z}$, let M be the additive group $\mathbb{Z}/_d$ of integers modulo d. Suppose that R is the trivial automorphism, and that S is negation. Then the corresponding linear quasigroup (3.1) is the quasigroup $(\mathbb{Z}/_d, -)$ of integers modulo d under subtraction. The multiplication group $\mathrm{Mlt}(\mathbb{Z}/_d, -)$ is the dihedral group D_d of degree d. In this particular case, the generated subgroup (2.1) is just the generating set $\{L(q), R(q) \mid q \in Q\}$ itself.

Example 3.2 (Nonassociative integers)**.** [23, Th. 11.1] Let $A = \langle R, S \rangle$ be the free (nonabelian) group on two generators R and S. Let $M = \mathbb{Z}\langle R, S \rangle$ be the integral group algebra of A. Then the submagma $\langle 1 \rangle_\circ$ of $(M, \circ)$ generated by 1 is

[3]Compare [28], where the group M is not required to be abelian.

the free magma on that generator. The elements of $\langle 1\rangle_\circ$, *index θ-polynomials* in the terminology of Minc [19], may be regarded as nonassociative analogues of the positive integers, since they serve to index nonassociative powers. (Compare [13], which even includes a proof of Fermat's Last Theorem for nonassociative integers.)

Example 3.3 (Skein polynomials)**.** [22] Let A be the free abelian group on distinct generators R and S. Let $M = \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables l and m. If R is multiplication by $-m^{-1}l$, and S is multiplication by $-m^{-1}l^{-1}$, then A acts faithfully on M. Recall that in knot theory, a *surgery triple* is an ordered triple (K_R, K_l, K_0) of oriented links having presentations (planar diagrams) that coincide outside a ball, within which the representative presentations are as displayed in Figure 1. Associated with each oriented link is a

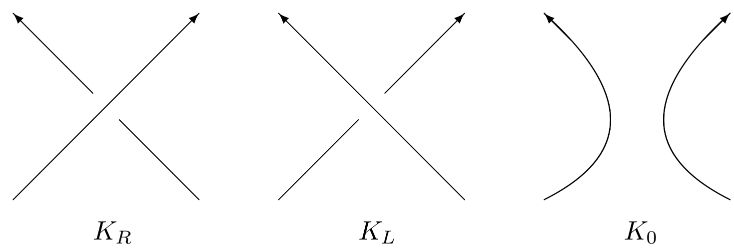


FIGURE 1. A surgery triple

homotopy-invariant element of M, the *skein polynomial* (otherwise known as the "oriented polynomial," "P-polynomial," "HOMFLY polynomial," etc.). Identifying oriented links with their skein polynomials, the surgery triple satisfies the very simple relation

$$K_R \circ K_L = K_0$$

in the linear quasigroup $(M, \circ, /\!/, \backslash\!\backslash)$. For a positive integer c, write U^c for (the skein polynomial of) c copies of an unlink (oriented circle). Let S be the subalgebra of $(M, /\!/, \backslash\!\backslash)$ generated by $\{U^c \mid 0 < c \in \mathbb{Z}\}$. Then the skein polynomials of arbitrary oriented links all lie in S.

Despite their evident usefulness, linear quasigroups are inherently limited by the requirement that the automorphism group A of the module M be 2-generated. In Section 11 it will be shown how a hyperquasigroup construction, linear hyperquasigroups, may be used to circumvent this limitation.

4. Hyperquasigroups

Hyperquasigroups involve the auxiliary concept of a reflexion-inversion space:

Definition 4.1. A *reflexion-inversion space* (Ω, σ, τ) is a set Ω equipped with two involutive actions, a *reflexion*

$$\sigma : \Omega \to \Omega; \omega \mapsto \sigma\omega \tag{4.1}$$

and an *inversion*

$$\tau : \Omega \to \Omega; \omega \mapsto \tau\omega\,. \tag{4.2}$$

Example 4.2. The most basic reflexion-inversion space is the symmetric group $S_3 = \{1,2,3\}!$, with reflexion and inversion implemented as left multiplications by the respective transpositions (1 2) and (2 3). In this context, it is convenient to identify each element of S_3 as the image of the identity permutation under the left action of a series of reflexions and inversions.

Remark 4.3. Although many natural reflexion-inversion spaces may be equipped with a topology (say in Example 4.6 below), the considerations of the current paper will remain discrete. At this level, the only relations among the points of Ω are those given by the orbit structure of the action of the infinite dihedral group generated by the reflexion and inversion.

Definition 4.4. A *hyperquasigroup* (Q, Ω) is a pair consisting of a set Q and a reflexion-inversion space Ω, together with a binary action

$$Q^2 \times \Omega \to Q; (x, y, \omega) \mapsto xy\,\underline{\omega} \tag{4.3}$$

of Ω on Q, such that the *hypercommutative law*

$$xy\,\underline{\sigma\omega} = yx\,\underline{\omega} \tag{4.4}$$

and the *hypercancellation law*

$$x\,(xy\,\underline{\omega}\,)\,\underline{\tau\omega} = y \tag{4.5}$$

are satisfied for all x, y in Q and ω in Ω.

Remark 4.5. Readers with concerns about an algebra (Q, Ω) where the operation set Ω itself has some algebra structure may wish to contemplate the example of a vector space $(V, +, F)$, where the set F of scalar multiplications carries the structure of a field that features in axioms such as the mixed associative law $(vf)f' = v(ff')$ for elements v of V and f, f' of F.

The following new example of hyperquasigroups has a role to play later in the paper. For further examples, see [24, 26] and the final sections of this paper.

Example 4.6. Suppose that A is a group of automorphisms of an abelian group (or right A-module) M. The set $\Omega = A \times A$ becomes a reflexion-inversion space, with reflexion

$$\sigma : \Omega \to \Omega; (a, b) \mapsto (b, a)$$

and inversion

$$\tau : \Omega \to \Omega; (a, b) \mapsto (ab^{-1}, b^{-1})\,.$$

Define

$$xy\,\underline{(a, b)} = -xa - yb$$

for (a, b) in Ω and x, y in M. Then (M, Ω) is a hyperquasigroup.

Remark 4.7. Hyperquasigroups are heterogeneous algebras, with operations (4.1), (4.2) and (4.3) — compare [16, 17]. The theory of heterogeneous algebras provides a homomorphism concept for hyperquasigroups [17, p. 146]. Nevertheless, a full treatment of hyperquasigroups, including a further development of the ideas of Section 10 below, somewhat along the lines of [1], would also require an analysis of the corresponding homotopies. This topic is deferred to a later paper.

The most immediate connections between hyperquasigroups and quasigroups are obtained as follows.

Proposition 4.8. [24, Prop. 5.2] *Let $(Q,\cdot,/,\backslash)$ be an equational quasigroup. Let Ω be the symmetric group S_3, interpreted as a reflexion-inversion space according to Example 4.2. Setting*

$$xy\,\underline{(1)} = x\cdot y\,,\quad xy\,\underline{(13)} = x/y\,,\quad xy\,\underline{(23)} = x\backslash y\,,$$
$$xy\,\underline{(12)} = y\cdot x\,,\quad xy\,\underline{(123)} = y/x\,,\quad xy\,\underline{(132)} = y\backslash x\,,$$

the pair (Q,Ω) becomes a hyperquasigroup.

Theorem 4.9. [24, Th. 6.1] *Let (Q,Ω) be a hyperquasigroup. Then for each element ω of the reflexion-inversion space Ω, there is an equational quasigroup $(Q,\underline{\sigma\omega}\,,\underline{\sigma\tau\omega}\,,\underline{\tau\sigma\omega}\,)$.*

Corollary 4.10. [24, Cor. 6.2] *Let (Q,Ω) be a hyperquasigroup. Then for each element ω of the reflexion-inversion space Ω, there is a combinatorial quasigroup $(Q,\underline{\omega})$.*

5. Torsor hyperquasigroups

This section shows how torsors (principal homogeneous spaces) may be construed as hyperquasigroups. Compare [2] for an elementary discussion of torsors and their application in physics, or [3], [10, §4.5], [15] for their role in cohomology. Proposition 7.7 will make use of the construction presented in this section.

For a set Q, let $\Omega = Q\times S_3$ be the free left S_3-set on the underlying set Q. Thus $\varpi(q,\pi) = (q,\varpi\pi)$ for ϖ, π in S_3 and q in Q. The component at Q of the unit of the free left S_3-set adjunction is

$$\eta : Q\to\Omega; q\mapsto (q,1)\,.$$

As in Example 4.2, set $\sigma = (1\ 2)$ and $\tau = (2\ 3)$. Then the reflexion $\sigma(q,\pi) = (q,\sigma\pi)$ and inversion $\tau(q,\pi) = (q,\tau\pi)$ for q in Q and π in S_3 make the free left S_3-set Ω a reflexion-inversion space. Now suppose that $(Q,\cdot,e)$ is a group. Define a binary action of Ω on Q by

$$x_{2\pi}x_{1\pi}\,\underline{(x_{3\pi},\pi)} = x_2x_3^{-1}x_1 \tag{5.1}$$

for x_1, x_2, x_3 in Q and π in S_3. Then (Q,Ω) is a hyperquasigroup, known as a *torsor hyperquasigroup* for reasons that will become apparent below. Note that (5.1) is a Mal'tsev operation $(x_2,x_3,x_1)P$ in the sense of [21]. Also note that $\underline{(e,1)}$ recovers the original group multiplication. Indeed, if π is an element of S_3, then the operations $\underline{\pi}$ of Proposition 4.8 correspond to the operations $\underline{(e,\pi)}$ in (5.1).

The torsor hyperquasigroup construction offers an interpretation of torsors or principal homogeneous spaces construed as ternary algebras — Prüfer's *flocks* or Baer's *abstract cosets* — within the language of hyperquasigroups (compare [6, §II.6]).

Theorem 5.1. *Let (Q,Ω) be a hyperquasigroup, in which Q is nonempty. Then (Q,Ω) is a torsor hyperquasigroup if and only if the following conditions are satisfied:*

(a) *The reflexion-inversion space Ω is the free left S_3-set on Q, with*

$$\sigma\omega = (1\ 2)\omega \quad \textit{and} \quad \tau\omega = (2\ 3)\omega$$

for ω in Ω;

(b) *For all x, y in Q, the hyperidentity*

$$yx\,\underline{y^\eta} = x = xy\underline{y^\eta}$$

holds;

(c) *For all u, v, w, x, y in Q, the hyperidentity*

$$(uw\,\underline{v^\eta}\,)\,y\underline{x^\eta} = u\,(wy\,\underline{x^\eta}\,)\,\underline{v^\eta}$$

holds.

Proof. First, suppose that (Q, Ω) is a torsor hyperquasigroup. Condition (a) then holds directly by definition. By (5.1), condition (b) reduces to $yy^{-1}x = x = xy^{-1}y$ in the group Q. Similarly, condition (c) reduces to $\left(uv^{-1}w\right)x^{-1}y = uv^{-1}\left(wx^{-1}y\right)$ in the group Q.

Conversely, suppose that a hyperquasigroup (Q, Ω), with Q nonempty, satsfies the conditions (a)–(c) of the theorem. Using (a), a ternary operation may be defined on Q by

$$(x, y, z)P = xz\,\underline{y^\eta}\,.$$

The conditions (b) and (c) then translate to

$$(y, y, x)P = x = (x, y, y)P \tag{5.2}$$

and

$$((u, v, w)P, x, y)\,P = (u, v, (w, x, y)P)\,P$$

respectively, the identities [6, (6.10)] and [6, (6.9)] for flocks or abstract cosets. In particular, (5.2) shows that P is a Mal'tsev operation on Q. Choosing an element e of Q, it follows that $xy = (x, e, y)P$ defines a group multiplication on Q with e as the identity element, with inversion given by $x^{-1} = (e, x, e)P$, and more generally with $xy^{-1}z = (x, y, z)P$ [6, (6.11)]. Thus $x_2x_3^{-1}x_1 = x_2x_1\,\underline{(x_3, 1)}$. The remaining cases of (5.1) are then recovered using hypercommutativity and hypercancellation. □

6. The Cayley graph

Let (Q, Ω) be a hyperquasigroup. For elements x of Q and ω of Ω, define the (*left*) *translation*

$$L_\omega(x) : Q \to Q; y \mapsto xy\underline{\omega} \tag{6.1}$$

and *right translation*

$$R_\omega(x) : Q \to Q; y \mapsto yx\underline{\omega} \tag{6.2}$$

by analogy with (1.1) and (1.2). Note that

$$R_\omega(x) = L_{\sigma\omega}(x) \tag{6.3}$$

by hypercommutativity, and

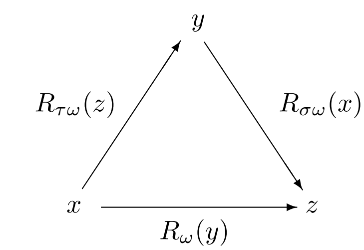
$$L_\omega(x)^{-1} = L_{\tau\omega}(x) \tag{6.4}$$

by hypercancellativity. (The latter relation may be viewed as justification for the use of the term "inversion" to describe τ.) The relation

$$R_\omega(x)^{-1} = R_{\sigma\tau\sigma\omega}(x)$$

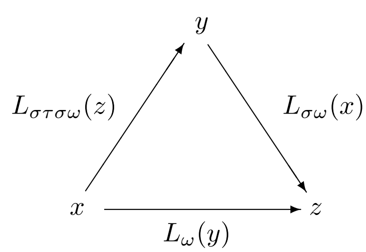
follows from (6.3) and (6.4).

The defining requirements of hypercommutativity and hypercancellativity may be summarized by the diagram



(6.5)

in terms of the right translations, or by the diagram



(6.6)

using left translations, with ω in Ω and x, y, z in Q. For example, the bottom line of (6.5) gives $z = xy\underline{\omega}$. The right leg then yields the hypercommutativity (4.4), while the left leg yields the hypercancellativity (4.5). The equivalence of (6.5) with (6.6) follows by replacement of ω with $\sigma\omega$, and use of (6.3).

The (*left*) *Cayley graph* of the hyperquasigroup (Q, Ω) is defined as the directed graph with vertex set Q, such that for each ordered pair (x, y) of elements of Q, and for each element ω of Ω, there is a unique directed edge from x to y labeled by L_ω or $L_\omega\big(xy\,\underline{\tau\sigma\omega}\big)$. Note that (6.6) may be viewed as a fragment of the Cayley graph of (Q, Ω). The *right Cayley graph* is defined similarly, using edges labeled R_ω or $R_\omega\big(xy\,\underline{\tau\omega}\big)$ from x to y instead. Thus (6.5) represents a fragment of the right Cayley graph.

7. Translation groups

Let (Q, Ω) be a hyperquasigroup. For x in Q and ω in Ω, the translations $R_\omega(x)$ and $L_\omega(x)$ are permutations (bijective self-maps) of Q — compare (6.3) and (6.4).

Definition 7.1. For a hyperquasigroup (Q, Ω), the *translation group* or *mapping group* $T(Q, \Omega)$ is the subgroup

$$\langle R_\omega(x), L_\omega(x) \mid x \in Q\,,\ \omega \in \Omega\rangle_{Q!} \tag{7.1}$$

of the group $Q!$ of permutations of Q generated by all the right and left translations of the hyperquasigroup (Q, Ω).

Proposition 7.2. *Suppose that (Q, Ω) is a hyperquasigroup in which the reflexion-inversion space Ω is nonempty. Then:*

(a) *The translation group acts transitively on Q;*
(b) *For each element ω of Ω, the combinatorial multiplication group* $\mathrm{Mlt}(Q, \underline{\omega})$ *of the quasigroup $(Q, \underline{\omega})$ is a subgroup of $T(Q, \Omega)$.*

Proof. (a): For elements x and y of Q, consider $z = xy\underline{\omega}$ for some element ω of Ω. Then

$$xR_{\tau\omega}(z) = x\big(xy\underline{\omega}\big)\,\underline{\tau\omega} = y$$

by hypercancellativity.
(b): The generating set $\{R_\omega(x), L_\omega(x) \mid x \in Q\}$ for $\mathrm{Mlt}(Q, \underline{\omega})$ in $Q!$ is a subset of the generating set for $T(Q, \Omega)$ from (7.1). □

Remark 7.3. Given a quasigroup Q, one may use Proposition 4.8 to build a hyperquasigroup (Q, S_3). By Proposition 7.2(b), the combinatorial multiplication group $\mathrm{Mlt}\,Q$ of Q is a subgroup of the translation group $T(Q, S_3)$. In general, $\mathrm{Mlt}\,Q$ may be a proper subgroup of $T(Q, S_3)$. Recovery of $\mathrm{Mlt}\,Q$ from the hyperquasigroup (Q, S_3) requires use of the orthant structure discussed in Example 12.1 below.

Definition 7.4. The *inner mapping group* $I(Q, \Omega)$ of a hyperquasigroup (Q, Ω) is defined as an abstract group isomorphic to a point stabilizer in the permutation action of $T(Q, \Omega)$ on Q.

Proposition 7.2(a) shows that the inner mapping group of a hyperquasigroup (Q, Ω) is well-defined. Note that $I(Q, \Omega)$ is trivial if either of Q or Ω is empty. If Q and Ω are nonempty, it is convenient to identify $I(Q, \Omega)$ with any one of the conjugate point stabilizers $T(Q, \Omega)_x$ of an element x of Q in the permutation group $T(Q, \Omega)$.

Example 7.5. Let Q be a group, construed as a hyperquasigroup (Q, S_3) using Proposition 4.8. Then the inner mapping group $I(Q, S_3)$ is the direct product $\mathrm{Inn}\,Q \times \langle J \rangle$ of the inner automorphism group $\mathrm{Inn}\,Q$ of Q with the group generated by the inversion mapping $J : Q \to Q; x \mapsto x^{-1}$.

The following result gives a simple illustration of the way that mapping groups may sometimes be used to determine the structure of a hyperquasigroup.

Theorem 7.6. *Let (Q, Ω) be a hyperquasigroup, for which the inner mapping group is a normal subgroup of the translation group. Then precisely one of the following holds:*

(a) *At least one of Q and Ω is empty;*
(b) *For each element ω of Ω, the quasigroup $(Q, \underline{\omega})$ is an elementary abelian group of exponent 2, coinciding with $(Q, \underline{\sigma\omega})$ and $(Q, \underline{\tau\omega})$.*

Proof. If (a) holds, then $I(Q, \Omega)$ is trivial. Suppose that Q and Ω are nonempty, with respective elements e and ω. By Proposition 7.2(a), the translation group $T(Q, \Omega)$ acts transitively on Q, so the stabilizers $T(Q, \Omega)_x$ of points x of Q are conjugate in $T(Q, \Omega)$. Since $T(Q, \Omega)$ acts faithfully on Q, the normal generic point stabilizer $I(Q, \Omega)$ is trivial. By Proposition 7.2(b), the point stabilizer $\mathrm{Mlt}(Q, \underline{\omega})_e$, as a subgroup of the trivial stabilizer $T(Q, \Omega)_e$, is also trivial. It follows that the quasigroup $(Q, \underline{\omega})$ is (an) abelian (group) [23, Prop. 3.16]. As noted in Example 7.5, the inversion mapping in this group yields an element of the inner mapping group $I(Q, \Omega)$. Since that group is trivial, the group $(Q, \underline{\omega})$ has exponent 2, thereby coinciding with $(Q, \underline{\sigma\omega})$ and $(Q, \underline{\tau\omega})$. □

It is quite common for distinct (nonisomorphic, or even nonisotopic) quasigroups to share the same combinatorial multiplication group. As a comparison between quasigroup and hyperquasigroup behaviors, the following result illustrates a different kind of phenomenon in which distinct hyperquasigroups may share the same translation group. The group $\mathbb{Z}/_2$ is realized multiplicatively as $(\{\pm 1\}, \cdot)$.

Proposition 7.7. *Let Q be a group. Let $\mathbb{Z}/_2$ act on $Q \times Q$ by*

$$-1 : (q_1, q_2) \mapsto (q_2, q_1)\,.$$

Consider the hyperquasigroup $(Q, Q \times S_3)$ given by (5.1), and the hyperquasigroup (Q, S_3) given by Proposition 4.8. Then the translation group of each is the group G appearing in the exact sequence

$$1 \longrightarrow Z(Q) \xrightarrow{\Delta} (Q \times Q) \rtimes \mathbb{Z}/_2 \xrightarrow{t} G \longrightarrow 1$$

with $\Delta : z \mapsto (z, z, 1)$ and $t : (q_1, q_2, \varepsilon) \mapsto \big(x \mapsto (q_1^{-1} x q_2)^{\varepsilon}\big)$.

8. Universal groups

Universal multiplication groups of quasigroups (in the variety of all quasigroups) were described briefly in Section 2. This section introduces corresponding groups associated with hyperquasigroups.

Definition 8.1. Let (Q, Ω) be a hyperquasigroup.

(a) The *fully universal group* or *fully universal mapping group* $V(Q, \Omega)$ is the group with generating set $Q \times \Omega$ and relations

$$(x, \tau\omega) = (x, \omega)^{-1}$$

for x in Q and ω in Ω.

(b) The *universal group* or *universal mapping group* $U(Q, \Omega)$ is the group with generating set $Q \times \Omega$ and relations

$$(x, \tau\omega) = (x, \omega)^{-1} \quad \text{and} \quad (x, \sigma\tau\sigma\omega) = (x, \tau\sigma\tau\omega)$$

for x in Q and ω in Ω.

There are two useful ways to consider the fully universal group.

Proposition 8.2. *Let (Q, Ω) be a hyperquasigroup.*

(a) *Let $\tau\backslash\Omega$ denote the set of orbits of the group $\{1, \tau\}$ on the reflexion-inversion space Ω. Then $V(Q, \Omega)$ is the free group on $Q \times (\tau\backslash\Omega)$.*

(b) *The monoid reduct $\big(V(Q, \Omega), \cdot, 1\big)$ of the group $\big(V(Q, \Omega), \cdot, {}^{-1}, 1\big)$ is generated by the set $Q \times \Omega$ subject to the relations $(x, \tau\omega)(x, \omega) = 1$ for x in Q and ω in Ω.*

Proof. Part (a) just moves between pairs of words $\{(x, \tau\omega), (x, \omega)\}$ under the relation $(x, \tau\omega) = (x, \omega)^{-1}$ and a single word indexed by the pair. The monoid relation $(x, \tau\omega)(x, \omega) = 1$ in Part (b) just restates the group relation $(x, \tau\omega) = (x, \omega)^{-1}$. □

Proposition 8.2(b) means that $V(Q, \Omega)$ is the set of those words in the alphabet $Q \times \Omega$ for which no letter (x, ω) is immediately followed by the letter $(x, \tau\omega)$.

Proposition 8.3. *Let (Q, Ω) be a hyperquasigroup.*

(a) *There is a surjective group homomorphism*

$$V(Q,\Omega) \to T(Q,\Omega); (x,\omega) \mapsto L_\omega(x) \tag{8.1}$$

providing a permutation representation of $V(Q,\Omega)$ *on* Q.

(b) *The homomorphism* (8.1) *factors through a surjective group homomorphism*

$$U(Q,\Omega) \to T(Q,\Omega); (x,\omega) \mapsto L_\omega(x) \tag{8.2}$$

providing a permutation representation of $U(Q,\Omega)$ *on* Q.

Proof. Part (a) follows directly from (6.3) and (6.4), while part (b) is a consequence of the fact that

$$xy\,\underline{\sigma\tau\sigma\omega} = xy\,\underline{\tau\sigma\tau\omega}$$

for x, y in Q and ω in Ω [24, Prop. 6.6]. □

The following result describes the universal property of $U(Q,\Omega)$.

Proposition 8.4. *Let* (Q,Ω) *and* (Q',Ω) *be hyperquasigroups. Suppose that for each element* ω *of* Ω*, the quasigroup* $(Q,\underline{\omega})$ *is a subquasigroup of* $(Q',\underline{\omega})$*. Then the subgroup* $T'(Q,\Omega)$ *of* $T(Q',\Omega)$ *generated by*

$$\{L_\omega(q) : Q' \to Q' \mid q \in Q\}$$

is a homomorphic image of the universal group $U(Q,\Omega)$.

Proof. There is a well-defined group monomorphism given by

$$j : U(Q,\Omega) \to U(Q',\Omega); (q,\omega) \mapsto (q,\omega)\,.$$

The group $T'(Q,\Omega)$ then appears as the image of $U(Q,\Omega)$ under the composite of j with the permutation representation

$$U(Q',\Omega) \to T(Q',\Omega); (x,\omega) \mapsto L_\omega(x)$$

given by (8.2) for (Q',Ω). □

9. Universal stabilizers

For nonempty Ω, Proposition 7.2(a) shows that the permutation representations of Proposition 8.3 are transitive. To describe the *fully universal stabilizers* $V(Q,\Omega)_x$ and *universal stabilizers* $U(Q,\Omega)_x$ for x in Q, the respective point stabilizers of these representations, it is helpful to define certain elements of the (fully) universal group. When considering words in the (fully) universal groups $V(Q,\Omega)$ and $U(Q,\Omega)$ of a hyperquasigroup (Q,Ω), it is generally convenient to write $L_\omega(x)$ in place of (x,ω) for x in Q and ω in Ω.

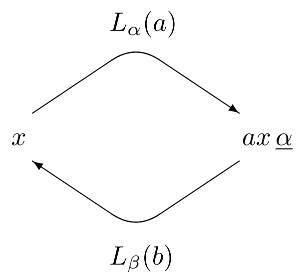
For α, β in Ω and x, a in Q, define

$$\Gamma_x(\alpha,a,\beta) = L_\alpha(a)L_\beta(b)$$

with

$$b = ax\,\underline{\alpha}\,x\,\underline{\tau\sigma\beta}\,.$$

Then $\Gamma_x(\alpha, a, \beta)$ represents the following circuit in the left Cayley graph of (Q, Ω):



(9.1)

Example 9.1 (Group conjugations)**.** Let $(Q, \cdot, 1)$ be a group, construed as a hyperquasigroup (Q, S_3) by Proposition 4.8. Under the permutation representations of Proposition 8.3, the circuit $\Gamma_1(q, \sigma, \tau)$ yields conjugation by an element q of Q.

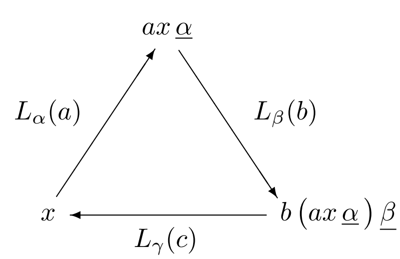
For α, β, γ in Ω and x, a, b in Q, define

$$\Gamma_x(\alpha, a, \beta, b, \gamma) = L_\alpha(a) L_\beta(b) L_\gamma(c)$$

with

$$c = b\, ax\, \underline{\alpha}\, \underline{\beta}\, x\, \underline{\tau\sigma\gamma}\,.$$

Thus $\Gamma_x(\alpha, a, \beta, b, \gamma)$ represents the following circuit in the left Cayley graph of (Q, Ω):



(9.2)

Example 9.2. Let x and y be elements of a loop $(Q, \cdot, 1)$, construed as a hyperquasigroup (Q, S_3) by Proposition 4.8. Under the permutation representations of Proposition 8.3, the circuit $\Gamma_1(1, x, 1, y, \tau)$ yields the mapping $L(x, y)$ introduced by Bruck [6, IV, (1.5)].

Theorem 9.3. *Let (Q, Ω) be a hyperquasigroup. For each element x of Q, the (fully) universal stabilizer of x is generated by the set*

$$\{\Gamma_x(\alpha, a, \beta, b, \gamma), \Gamma_x(\alpha, a, \beta) \mid a, b \in Q\,,\ \alpha, \beta, \gamma \in \Omega\} \tag{9.3}$$

as a subgroup of $U(Q, \Omega)$.

Proof. First note that the elements of the set (9.3) fix x, by definition. Conversely, suppose that an element w of $V(Q, \Omega)$ or $U(Q, \Omega)$ fixes x. By Proposition 8.2(b), the element w is represented by a word $w = L_{\alpha_1}(a_1) \dots L_{\alpha_n}(a_n)$ in the alphabet $\{L_\alpha(a) \mid a \in Q\,,\ \alpha \in \Omega\}$. Induction on the length n of w will be used to show that

w lies in the subgroup Γ_x generated by (9.3). If $n = 0$, the result is immediate. If $n = 1$, then w may be written as the element

$$L_{\alpha_1}(xx\,\underline{\tau\sigma\alpha_1})L_{\alpha_1}(xx\,\underline{\tau\sigma\alpha_1})^{-1}L_{\alpha_1}(xx\,\underline{\tau\sigma\alpha_1}) = \Gamma_x(\alpha_1, x, \tau\alpha_1, x, \alpha_1)$$

of (9.3). If $n = 2$, then w is the element

$$L_{\alpha_1}(a_1)L_{\alpha_2}(a_2) = \Gamma_x(\alpha_1, a_1, \alpha_2)$$

of (9.3). If $n = 3$, then w is the element

$$L_{\alpha_1}(a_1)L_{\alpha_2}(a_2)L_{\alpha_3}(a_3) = \Gamma_x(\alpha_1, a_1, \alpha_2, a_2, \alpha_3)$$

of (9.3). Now suppose that $n > 3$. Write

$$u = L_{\alpha_1}(a_1)\dots L_{\alpha_{n-2}}(a_{n-2})$$

and $a = xu$. Since $aL_{\alpha_1}(ax\,\underline{\tau\sigma\alpha_1}) = x$, the word

$$v = L_{\alpha_1}(a_1)\dots L_{\alpha_{n-2}}(a_{n-2})L_{\alpha_1}(ax\,\underline{\tau\sigma\alpha_1})$$

of length $n-1$ fixes x. By the induction hypothesis, v lies in Γ_x. But then so does

$$\begin{aligned} w &= vL_{\alpha_1}(ax\,\underline{\tau\sigma\alpha_1})^{-1}L_{\alpha_{n-1}}(a_{n-1})L_{\alpha_n}(a_n) \\ &= vL_{\tau\alpha_1}(ax\,\underline{\tau\sigma\alpha_1})L_{\alpha_{n-1}}(a_{n-1})L_{\alpha_n}(a_n) \\ &= v\Gamma_x(\tau\alpha_1, ax\,\underline{\tau\sigma\alpha_1}, \alpha_{n-1}, a_{n-1}, \alpha_n)\,, \end{aligned}$$

as required to complete the induction. □

Remark 9.4. The generating set (9.3) admits an interpretation as its image in $T(Q, \Omega)$ under the permutation representations of Proposition 8.3. It then provides a generating set for $I(Q, \Omega)$ as the point stabilizer of x in $T(Q, \Omega)$. Theorem 9.3 may thus be traced back to Bruck's classical determination of inner mapping groups of loops [5, Th. I.3B]. For an alternative approach, with the particular advantage of avoiding redundancy in the generating sets for the respective universal stabilizers, one might apply the Schreier Subgroup Theorem [18, Th. 2.9]. This application would be simplified by using Proposition 8.2(a) to give free generators for $V(Q, \Omega)$.

10. Hyperquasigroup transversals

Recall that a *transversal* L to a subgroup H of a group G is a complete set of representatives for the cosets of H in G, so that G is the disjoint union $\sum_{l \in L} Hl$. Now if $G = \mathrm{Mlt}(Q, \cdot)$ for a quasigroup $(Q, \cdot)$, and x is an element of Q, then

$$L = \{L(q) \mid q \in Q\} \tag{10.1}$$

is a transversal in G to the stabilizer G_x of x in G. Indeed, an element g of G lies in the coset $G_xL(xg/x)$, while $G_xL(y) = G_xL(z)$ for elements y, z of Q implies $y \cdot x = xL(y) = xL(z) = z \cdot x$, so that $y = z$. Conversely, a quasigroup structure $(Q, *_x)$ is defined on Q by the containment $L(z)L(y) \in G_xL(y *_x z)$. In particular, for a loop $(Q, \cdot, e)$, one has $(Q, \cdot) = (Q, *_e)$, so the loop is recovered from the transversal (10.1). This section examines comparable transversals to point stabilizers in translation or universal groups of hyperquasigroups, and the recovery of the hyperquasigroup structure from such transversals.

Let G be the translation group or (fully) universal group of a hyperquasigroup (Q, Ω). For each element x of Q, consider the stabilizer G_x of x in the permutation representation of G on Q. For the translation group, the permutation representation is the defining representation $T(Q,\Omega) \hookrightarrow Q!$, while for the (fully) universal group, it is given by Proposition 8.3.

Proposition 10.1. *For elements x of Q and ω of Ω, the set*

$$L_\omega(Q) = \{L_\omega(q) \mid q \in Q\} \tag{10.2}$$

is a transversal to G_x in G.

Proof. Each element g of G has an expression of the form

$$g = \Big(gL_\omega\big(xgL_{\tau\sigma\omega}(x)\big)^{-1}\Big) \cdot L_\omega\big(xgL_{\tau\sigma\omega}(x)\big)$$

with $gL_\omega\big(xgL_{\tau\sigma\omega}(x)\big)^{-1}$ in G_x, since

$$xL_\omega\big(xgL_{\tau\sigma\omega}(x)\big) = \big(x(xg)\,\underline{\tau\sigma\omega}\big)x\,\underline{\omega} = x\big(x(xg)\,\underline{\tau\sigma\omega}\big)\,\underline{\sigma\omega} = xg$$

by hypercommutativity and hypercancellativity. Then

$$\begin{aligned} & G_xL_\omega(y) = G_xL_\omega(z) \\ \Rightarrow \quad & xy\,\underline{\sigma\omega} = xL_\omega(y) = xL_\omega(z) = xz\,\underline{\sigma\omega} \\ \Rightarrow \quad & y = x(xy\,\underline{\sigma\omega})\,\underline{\tau\sigma\omega} = x(xz\,\underline{\sigma\omega})\,\underline{\tau\sigma\omega} = z \end{aligned}$$

by hypercommutativity and hypercancellativity, so each coset representative in (10.2) is unique. □

Recall that a transversal to a subgroup is said to be *normalized* if the identity element represents the subgroup in the transversal. For example, a nonempty quasigroup is a loop if and only if the corresponding transversal (10.1) is normalized.

Proposition 10.2. *Let x be an element of Q, and let ω be an element of Ω. Then:*

(a) *The sets $L_{\tau\omega}(x)L_\omega(Q)$ and $L_{\tau\sigma\omega}\big(xL_{\tau\omega}(x)\big)L_{\sigma\omega}(Q)$ both form normalized transversals to G_x in G;*

(b) *The transversals of* (a) *are G_x-connected in the sense of* [20]*, so that*

$$[L_{\tau\omega}(x)L_\omega(Q), L_{\tau\sigma\omega}\big(xL_{\tau\omega}(x)\big)L_{\sigma\omega}(Q)] \subseteq G_x\,. \tag{10.3}$$

Proof. By Corollary 4.10 and [23, Cor. 2.2], the operation

$$y + z = \big(yL_{\tau\sigma\omega}\big(xL_{\tau\omega}(x)\big)\big)\,(zL_{\tau\omega}(x))\,\underline{\omega}$$

defines a loop on Q with identity element x. The sets of (a) are the transversals given by the respective left and right multiplications in the loop. The commutator condition (10.3) expresses the slight associativity [23, (1.14)] holding in the loop. □

Definition 10.3. The system

$$L_\Omega(Q) = \{L_\omega(Q) \mid \omega \in \Omega\}$$

of transversals is known as the *hyperquasigroup transversal* to G_x in G.

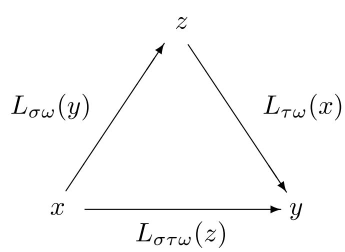
The hyperquasigroup structure (Q,Ω) is recovered from the hyperquasigroup transversal as follows.

Theorem 10.4. *Let (Q,Ω) be a hyperquasigroup. Let x and y be elements of Q. Let ω be an element of Ω. Then the result $xy\,\underline{\omega}$ of the action* (4.3) *of ω on the pair (x,y) is given as the unique element z of Q for which the equation*

$$G_xL_{\sigma\omega}(y)L_{\tau\omega}(x) = G_xL_{\sigma\tau\omega}(z) \tag{10.4}$$

holds within G.

Proof. Consider the fragment



(10.5)

of the Cayley graph of (Q, Ω), obtained by replacing ω with $\sigma\tau\omega$ and interchanging y with z in (6.6). The left leg of (10.5) gives $z = xL_{\sigma\omega}(y) = xy\,\underline{\omega}$. Since

$$xL_{\sigma\omega}(y)L_{\tau\omega}(x)L_{\sigma\tau\omega}(z)^{-1} = x\,,$$

the cosets on each side of (10.4) coincide. By Proposition 10.1, the equation (10.4) uniquely specifies the element z. □

11. Linear hyperquasigroups

Let A be an arbitrary group of automorphisms of an abelian group (or right A-module) M. Define sets

$$\begin{cases} \Omega_A^{+,+} = A \times A\,, \\ \Omega_A^{-,+} = (-A) \times A\,, \\ \Omega_A^{+,-} = A \times (-A)\,, \end{cases} \tag{11.1}$$

known respectively as the *positive cone* or *first* or 2^0-*th orthant*, the *second* or 2^1-*st orthant*, and the the *fourth* or 2^2-*nd orthant*. The notation is motivated by the case where M is the real line, and A is the group of positive scalars (see Figure 2.) (The

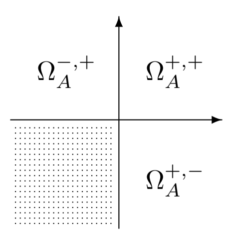


Figure 2. Orthant structure

"missing" third orthant is reprsented by Example 4.6.) Define

$$\Omega_A = \Omega_A^{+,+} \cup \Omega_A^{-,+} \cup \Omega_A^{+,-}\,. \tag{11.2}$$

Note that the unions need not be disjoint, for example if M has exponent 2, or in the case where M is the real line and A is the group of all nonzero real scalars. The following proposition describes such situations.

Proposition 11.1. *Let A be a group of automorphisms of an abelian group M. Suppose that the orthants* (11.1) *are not disjoint. Then:*

(a) *The orthants* (11.1) *coincide;*
(b) *If M is not of exponent* 2*, the group A contains a central involution.*

Proof. If the orthants are not disjoint, there are elements a and b of A such that $a = -b$. Then $A = aA = -bA = -A$, so the orthants coincide, and A contains the automorphism $-1 = ab^{-1}$. If M is not of exponent 2, the automorphism -1 is a central involution. □

Corollary 11.2. *If M is not of exponent* 2*, and A is simple or of odd order, then the orthants* (11.1) *are disjoint.*

Remark 11.3. Consider M as the additive group $\mathbb{C}$ of complex numbers, and A as the 2-element group generated by complex conjugation. Then the orthants (11.1) are disjoint, even though M is not of exponent 2, and the group A contains a central involution.

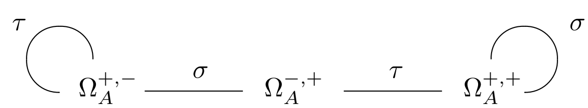
Now define a reflexion

$$\sigma : \Omega_A \to \Omega_A ; (r, s) \mapsto (s, r) \tag{11.3}$$

and an inversion

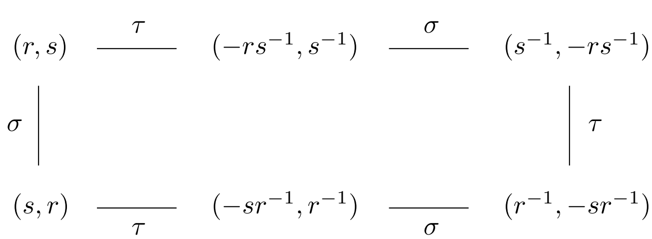
$$\tau : \Omega_A \to \Omega_A ; (r, s) \mapsto (-rs^{-1}, s^{-1}) \tag{11.4}$$

to make Ω_A a reflexion-inversion space. The actions of the reflexion and inversion on the orthants are given by the following Cayley diagram:



(11.5)

The inherent triality symmetry is given explicitly here by the elements σ and τ generating S_3. At the elementary level, the Cayley diagram appears as follows:



For (r, s) in Ω_A, define a binary action on M by

$$xy\,\underline{(r, s)} = xr + ys \tag{11.6}$$

for x, y in M. Note the similarity of (11.6) with (3.1).

Definition 11.4. A hyperquasigroup is said to be *linear* if it has the form (M, Ω_A), with structure given by (11.3)–(11.6), for a group A of automorphisms of an abelian group M. It is *pointed* if A is pointed by 0 and Ω_A is pointed by $(1, 1)$.

The main result of this section demonstrates the equivalence of pointed linear hyperquasigroups with group representations (as automorphisms of an abelian group), by showing how the abelian group M, automorphism group A, and action of A on M are all recovered from the pointed linear hyperquasigroup structure (M, Ω_A).

Theorem 11.5. *Let (M, Ω_A) be a pointed linear hyperquasigroup. Then:*

(a) *The addition and subtraction in the abelian group M are given by*

$$x + y = xy\,\underline{(1,1)}$$

and

$$x - y = xy\,\underline{(1,-1)} = xy\,\underline{\sigma\tau(1,1)}$$

for x, y in M;

(b) *The zero element of M is given as*

$$0 = xx\,\underline{\sigma\tau(1,1)}$$

for any element x of M;

(c) *The set*

$$P = \{L_{(r,s)}(x) \mid (r,s) \in \Omega_A^{+,+}\,,\ x \in M\}$$

of left translations from the positive cone is a subgroup of $T(M, \Omega_A)$;

(d) *The group A is the stabilizer P_0 of 0 in the action of P on M;*

(e) *For m in M and a in A, the equation*

$$ma = 0m\,\underline{(1,a)}$$

gives the action of a on m.

Proof. For the less immediate parts (c) and (d), it is convenient to implement the split extension $M \rtimes \operatorname{Aut} M$ of M by the automorphism group $\operatorname{Aut} M$ as the set of matrices

$$\begin{bmatrix} \alpha & 0 \\ m & 1 \end{bmatrix}$$

with m in M and α in $\operatorname{Aut} M$. The map

$$\mu : T(M, \Omega_A) \to M \rtimes (\pm A);\, L_{(r,s)}(x) \mapsto \begin{bmatrix} s & 0 \\ xr & 1 \end{bmatrix} \tag{11.7}$$

is part of a similarity between the action of $T(M, \Omega_A)$ on M and the action of $M \rtimes (\pm A)$ on the set of vectors $\begin{bmatrix} m & 1 \end{bmatrix}$ with m in M. Indeed, note that

$$\begin{bmatrix} yL_{(r,s)}(x) & 1 \end{bmatrix} = \begin{bmatrix} xr + ys & 1 \end{bmatrix} = \begin{bmatrix} y & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ xr & 1 \end{bmatrix}$$

for x, y in M and (r, s) in Ω_A. The image of P under the matrix representation μ of (11.7) is the subgroup

$$\left\{ \begin{bmatrix} a & 0 \\ m & 1 \end{bmatrix} \,\middle|\, a \in A\,,\ m \in M \right\}$$

of $M \rtimes (\pm A)$, so (c) holds. Finally, since

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ m & 1 \end{bmatrix} = \begin{bmatrix} m & 1 \end{bmatrix}$$

for a in A and m in M, one has

$$P_0\mu = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \,\middle|\, a \in A \right\},$$

yielding (d). □

12. Orthant structure

In Theorem 11.5, the orthant structure on the reflexion-inversion space Ω_A of a linear hyperquasigroup (M, Ω_A) ensured recovery of the complete module structure from the hyperquasigroup. This section examines two further cases where it is possible to identify an orthant structure for the retrieval of "classical" information from a hyperquasigroup. The first case uses the positive cone, while the other uses the second and fourth orthants.

Example 12.1. Let $(Q, \cdot)$ be a quasigroup, interpreted as a hyperquasigroup (Q, S_3) according to Proposition 4.8. An orthant structure on $\Omega = S_3$ is given by the respective left cosets

$$\begin{cases} \Omega^{+,+} = \{1, \sigma\}\,, \\ \Omega^{-,+} = \{\tau, \tau\sigma\}\,, \\ \Omega^{+,-} = \{\sigma\tau, \sigma\tau\sigma\} \end{cases}$$

of the positive cone $\Omega^{+,+}$. Standard coset enumeration [8, Ch. 2 and §3.7] provides a triality diagram analogous to (11.5). In this case, the set P of left translations determined by points of the positive cone is the generating set

$$\{L(q), R(q) \mid q \in Q\}$$

for the (combinatorial) multiplication group $\mathrm{Mlt}(Q, \cdot)$ of the quasigroup Q.

Now let F be a field, with group F^* of nonzero elements. Define an orthant structure

$$\begin{cases} \Omega_F^{+,+} = \{(m, -m) \mid m \in F^*\}\,, \\ \Omega_F^{-,+} = \{(1, m) \mid m \in F^*\}\,, \\ \Omega_F^{+,-} = \{(m, 1) \mid m \in F^*\} \end{cases} \tag{12.1}$$

on the reflexion-inversion space

$$\Omega_F = \Omega_F^{+,+} \cup \Omega_F^{-,+} \cup \Omega_F^{+,-} \tag{12.2}$$

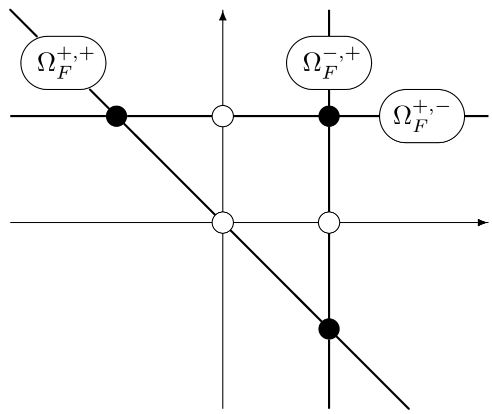
with reflexion

$$\sigma : \Omega_F \to \Omega_F; (r, s) \mapsto (s, r) \tag{12.3}$$

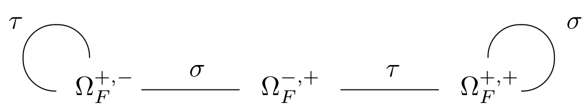
and inversion

$$\tau : \Omega_F \to \Omega_F; (r, s) \mapsto (-rs^{-1}, s^{-1})\,. \tag{12.4}$$

For the case $F = \mathbb{R}$, the geometry of the reflexion-inversion space is illustrated as:



One obtains a triality diagram



analogous to (11.5). Interpreting F^* as the group of scalar multiplications of the additive group $(F, +, 0)$ of the field F, the construction of Section 11 yields a linear hyperquasigroup (F, Ω_{F^*}). Within this hyperquasigroup, the reflexion-inversion space Ω_F of (12.2) furnishes a subhyperquasigroup (F, Ω_F). (Compare [17, II 2.1(3)] for the concept of substructure in a heterogeneous algebra.)

It is the second and fourth orthants which yield the classical information in this case, maximal families of mutually orthogonal quasigroups that define the desarguean projective plane structure coordinatized by the field F. Recall that combinatorial quasigroups $(Q, \cdot)$ and $(Q, *)$ are *orthogonal* if the map

$$Q \times Q \to Q \times Q; (x, y) \mapsto (x \cdot y, x * y)$$

is invertible.

Theorem 12.2. *Let F be a field. Consider the hyperquasigroup (F, Ω_F). Then the sets*

$$\{(F, \underline{\omega}\,) \mid \omega \in \Omega_F^{-,+}\} \tag{12.5}$$

and

$$\{(F, \underline{\omega}\,) \mid \omega \in \Omega_F^{+,-}\} \tag{12.6}$$

each form maximal families of mutually orthoganal quasigroups on F.

Proof. Consider distinct elements $(1, m)$ and $(1, m')$ of $\Omega_F^{-,+}$. Then

$$\begin{vmatrix} 1 & m \\ 1 & m' \end{vmatrix} = m' - m \neq 0$$

implies the orthogonality of the quasigroups $\big(F, \underline{(1, m)}\big)$ and $\big(F, \underline{(1, m')}\big)$. The same argument applies for $\Omega_F^{+,-}$.

The elements $(1, m)$ of $\Omega_F^{-,+}$ and $(m^{-1}, 1)$ of $\Omega_F^{+,-}$ determine combinatorial quasigroups $\big(F, \underline{(1, m)}\big)$ and $\big(F, \underline{(m^{-1}, 1)}\big)$ which are not orthogonal. The element $(1, -1)$ of $\Omega_F^{-,+}$ determines a quasigroup $\big(F, \underline{(1, -1)}\big)$ which is orthogonal to none of the quasigroups $(Q, \underline{\omega})$ with ω taken from $\Omega_F^{+,+}$. Similarly, none of these quasigroups is orthogonal to the combinatorial quasigroup $\big(F, \underline{(-1, 1)}\big)$ determined by the element $(-1, 1)$ of $\Omega_F^{+,-}$. Thus the two families (12.5) and (12.6) of mutually orthogonal quasigroups are maximal. □

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