QUASILATTICES AND COMPLEX CONCEPT ANALYSIS

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Abstract. Quasilattices are algebraic structures that comprise a semilattice-ordered system of lattices. In this paper, certain quasilattices (that are characterized abstractly by a local completeness property) provide an extension of Wille's concept analysis to the study of complex systems that function on a number of distinct levels. In an important special case, a chain semilattice serves to represent a time series governing the evolution of a single system.

Natural set representations of locally complete quasilattices have opposed set inclusions describing order relations within a complete lattice, and parallel set inclusions tracking homomorphisms that connect distinct lattice fibers. In the time series model, the sets that appear within the set representation accumulate successive layers at each time point, establishing a mathematical model for historical phenomena.

1. Introduction

The algebras $(X, \times, +)$ now known as *quasilattices* comprise two semilattice structures, namely a meet semilattice $(X, \times)$ and a join semilattice $(X, +)$, satisfying the identities

$$
\begin{aligned}
&[(x + y) \times z] + [y \times z] = (x + y) \times z \quad \text{and} \\
&[(x \times y) + z] \times [y + z] = (x \times y) + z
\end{aligned}
$$

(Definition 2.7). They were first introduced by Płonka [12], under the additional assumption that each semilattice operation distributes over the other, and then studied in full generality by Padmanabhan [11]. Their structure (summarized in §2.6) is such that they decompose into a semilattice-ordered system of lattices, which are mutually connected by lattice homomorphisms. Furthermore, quasilattices are completely determined by the features of their semilattice decomposition.

A *polarity* is a relation between two sets, for example the fixed-point relation between a well-behaved extension field and its Galois

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group (§3.1). The polarity induces a *Galois connection* (essentially, an adjointness relation) between the respective power sets, and then a bijective *Galois correspondence* between the respective subsets of closed sets, which form complete lattices (§3.2). In particular, this process underlies the *MacNeille completion* of a poset, the general version of the completion by Dedekind cuts of the ordered set of rational numbers to the ordered set of extended reals (§3.4).

Under the title of *concept analysis*, Wille introduced a data-science interpretation of the passage from a polarity to a complete lattice (§3.3). The polarity is described as a *context*. In our slightly modified English version of the terminology, the context attributes *properties* to *objects*. Pairs of object sets and property sets related by the Galois correspondence are known as *concepts*. The object set is the *extent* of the concept, and the property set is its *intent*. The complete lattice formed by the concepts is the *concept lattice*. Its order relation is given both by direct inclusion of extents, and by reversed inclusion of intents. Concept analysis is *static* — the context is fixed *a priori*, and *flat* — there is no hierarchical structure partitioning the objects or properties into distinct levels.

The fundamental goal of the current paper is to demonstrate how concept analysis may be extended to complex systems that function at various different levels indexed by a semilattice. Systems of this type pervade mathematical biology. In [15, Example 9.11.2], a simple but informative toy model was presented, where a stage-structured species competes for limited resources with an unstructured species. The interaction between the stages of the stage-structured species lies at a lower, demographic level, while the competition between the two species occurs at a higher, ecological level.

In a different interpretation of the same mathematical content, one may consider a single flat system evolving over a series of discrete time points. The arrow of time furnishes the time series with a semilattice structure, namely a chain. In this fashion, we will obtain a dynamic version of concept analysis. Going further, one might consider complex systems that are evolving over time, possibly even with a varying semilattice of complex levels at different time points. We will not discuss all these diverse possibilities within the present paper. Indeed, because of the space limitations that constrain our desire to exhibit explicitly detailed illustrations such as Figure 1, our examples will all be taken over the two-element semilattice (2.5) encoding a basic time step.

In concept analysis, a single context gave rise to a complete lattice, the concept lattice. In particular, the MacNeille completion allowed each complete lattice to be represented as a concept lattice (§3.4). The complex systems we study carry the data structure of a *quasicontext* or *complex polarity* (Definition 5.1). The analogue of the concept lattice for these data structures is the *concept quasilattice* (Definition 5.3). Concept quasilattices have the abstract property of *local completeness* (Definition 2.16). Theorem 5.4 then shows that each locally complete quasilattice is the concept quasilattice of a complex polarity.

The single order relation on a concept lattice was given equally by two different set representations: the direct inclusion of extents, or the reversed inclusion of intents. In a concept quasilattice, there are two separate order relations, a meet relation and a join relation. We thus consider set representations of concept quasilattices in which the meet relation is given by the direct inclusion of extents, and the join relation is given by the reversed inclusion of intents. These representations are illustrated by Figure 1 and (5.3).

In order to realize these set representations, the extents and intents at successively higher levels of the hierarchy have to be constructed as accumulations of so-called *layers of history*. The layers are the extents and intents of the concepts appearing in the context at each lower (or "earlier") location within the hierarchy. The accumulation, illustrated using slightly different conventions in Figure 1 and in (5.3), provides a mathematical model for the cumulative effect of history that is observed quite explicitly in fields such as geology or paleontology, and to a more subtle but nonetheless pervasive degree in biology.

The plan of the paper is as follows. The abstract theory of quasilattices is summarized in Section 2. The discussion of completeness, local completeness, and fiber finiteness in §§2.8–2.10 appears to be new. Section 3 reviews classical concept analysis. With the exception of notational features that were specifically chosen for the purposes of the present work, nothing here is original. Section 4 then covers categorical aspects of classical concept analysis, introducing notions briefly previewed in [10], and casting fresh light on aspects of concept analysis that were considered previously for different purposes (such as subdirect product constructions). Here, our treatment is similar to the approach that is summarized and attributed to an unpublished preprint of Moshier in [7], although we do stay closer to the terminology of [3, Def'n. 69].

Complex concepts and their concept quasilattices are then treated in Section 5. Since one of our primary goals is to examine concrete set representations such as those illustrated in Figure 1 and (5.3), with their juxtaposition of parallel and reversed set inclusions, we restrict our examples to the case where the underlying time or level ordering is the two-element semilattice. Thus, we are deferring the analysis of more elaborate, naturally occuring examples to subsequent consideration. Another direction for future research raised by the current paper would be to replace our use of concepts and complete lattices by approximable concepts, Chu spaces, and complete, algebraic lattices, as discussed in [5, 20].

While algebraic notation (arguments followed by functions) is taken as the basic default option, we do occasionally use Eulerian notation with functions on the left of their arguments. Readers are referred to [18] for definitions and conventions that are not otherwise given explicitly in the paper.

2. Quasilattices

2.1. **Semilattices.**

2.1.1. *Semilattices as algebras and posets.* As an algebra, a *semilattice* is a commutative, idempotent semigroup. (In [6], these algebras were described as "protosemilattices".) Semigroup homomorphisms between semilattices are described as *semilattice homomorphisms.*

Consider a semilattice $(H, \times)$. It is a *meet semilattice* $(H, \leq_\times)$ when equipped with the order relation

$$x \leq_\times y \;\Leftrightarrow\; x = x \times y\,. \tag{2.1}$$

This relationship may be summarized by the slogan "common is lower": The element x, which is common to both sides of the equation in (2.1), is the lower of the two arguments x, y in the order relationship of (2.1).

A semilattice $(H, +)$ is a *join semilattice* $(H, \leq_+)$ when equipped with the order relation

$$x \leq_+ y \;\Leftrightarrow\; x + y = y\,.$$

This relationship may be summarized as "common is upper."

Order-theoretically, a meet semilattice is a poset in which each subset $\{x, y\}$ has a greatest lower bound $x \times y$, while a join semilattice is a poset in which each subset $\{x, y\}$ has a least upper bound $x + y$. Thus in each of these cases, the semigroup operation is specified by the order structure.

2.1.2. *Semilattices as categories.* A small category $\mathbf{P}$ is said to be a *poset category* if for each pair x, y of objects of $\mathbf{P}$, one has

$$|\mathbf{P}(x, y) \cup \mathbf{P}(y, x)| \in \{0, 1\}\,.$$

The relation

$$x \leq y \;\Leftrightarrow\; |\mathbf{P}(x, y)| = 1$$

then creates a poset $(\mathbf{P}_0, \leq)$ on the object set $\mathbf{P}_0$ of $\mathbf{P}$. Note that in a poset category $\mathbf{P}$, the only isomorphisms are the identity morphisms at objects of $\mathbf{P}$.

Remark 2.1. A meet semilattice $(H, \leq_\times)$ may be construed as a poset category, where $x \times y$ is the product of x and y. Dually, a join semilattice $(H, \leq_+)$ may be construed as a poset category where $x + y$ is the coproduct of x and y.

2.1.3. *Representations of semilattices.*

Definition 2.2. Suppose that $(H, \leq_\times)$ is a meet semilattice and $\mathbf{C}$ is a category.

(a) A $\mathbf{C}$-*representation* of $(H, \leq_\times)$, or a *representation* of $(H, \leq_\times)$ in the category $\mathbf{C}$ (or "$\mathbf{C}$-valued presheaf"), is a contravariant functor

$$R \colon H \to \mathbf{C} \tag{2.2}$$

from H (considered as a poset category in accordance with §2.1.2) to the category $\mathbf{C}$.

(b) A *morphism of* $\mathbf{C}$-*representations* of $(H, \leq_\times)$ is defined to be a natural transformation between functors of the form (2.2).

(c) The functor category

$$\widehat{H} = \mathbf{C}^{H^{\mathsf{op}}} \tag{2.3}$$

is the *category of* $\mathbf{C}$-*representations* of $(H, \leq_\times)$.

2.2. **Bisemilattices.**

Definition 2.3. A *bisemilattice* $(Q, \times, +)$ is an algebra equipped with binary operations $\times$ of *meet* and $+$ of *join*, such that $(Q, \times)$ and $(Q, +)$ are semilattices. Then a function $(Q, \times, +) \to (Q', \times, +)$ between bisemilattices is a *bisemilattice homomorphism* if it entails semilattice homomorphisms $(Q, \times) \to (Q', \times)$ and $(Q, +) \to (Q', +)$.

Definition 2.4. A bisemilattice $(Q, \times, +)$ is (*globally*) *complete* if for each subset S of Q, the meets $\prod_{x \in S} x$ and joins $\sum_{x \in S} x$ exist in Q. Then a function $(Q, \times, +) \to (Q', \times, +)$ between complete bisemilattices is a *complete bisemilattice homomorphism* if it preserves all the meets and joins.

Remark 2.5. Our choice of notation for the two binary operations of a bisemilattice in Definition 2.3, and thus for a quasilattice, is governed by Remark 2.1. This choice then allows us to use the general product and sum notations, as in Definition 2.4, for a complete bisemilattice.

Lemma 2.6. *Each finite bisemilattice is complete.*

We will be considering three particular kinds of bisemilattice, namely lattices (§2.3), quasilattices (§2.4), and (duplicated) semilattices (§2.5). The concepts of completeness that are introduced in Definition 2.4 for bisemilattices and homomorphisms restrict to the usual completeness definitions for lattices [18, p. 221]. Indeed, a lattice homomorphism between complete lattices is a *complete* (*lattice*) *homomorphism* if it preserves arbitrary meets and joins. On the other hand, for the case of semilattices, Definition 2.4 does not restrict to [2, Defn. O.2.11], although finite semilattices are complete in that sense as well.

2.3. **Lattices.** For current purposes, it proves convenient to define a *lattice* $(L, \times, +)$ as a bisemilattice consisting of a meet semilattice $(L, \leq_\times)$ and a join semilattice $(L, \leq_+)$, where $\leq_\times$ and $\leq_+$ are equal, say to $\leq$ as a neutral notation. The agreement of $\leq_\times$ and $\leq_+$ is then expressed algebraically by either of the two identities

$$x + (x \times y) = x \quad \text{ or } \quad x \times (x + y) = x$$

known as *absorption*. A *lattice homomorphism* $(L, \times, +) \to (L', \times, +)$ is a bisemilattice homomorphism between lattices.

A lattice $(L, \times, +)$ is:

- *bounded above* if the semigroup $(L, \times)$ forms a monoid $(L, \times, \top)$;
- *bounded below* if the semigroup $(L, +)$ forms a monoid $(L, +, \bot)$;
- *bounded* if it is bounded both above and below.

Here, the unit $\top$ is the *upper bound*, while the unit $\bot$ is the *lower bound*.

The category **BLat** of bounded lattices has the class of bounded lattices as its object class, while the morphisms, the so-called *bounded lattice homomorphisms*, are lattice homomorphisms preserving upper and lower bounds. Write **CLat** for the category of complete lattice homomorphisms between complete lattices [8, §I.4.3].

2.4. **Quasilattices.**

Definition 2.7. [11, Lemma 1] A bisemilattice $(Q, \times, +)$ is said to be a *quasilattice* if the identities

$$\begin{aligned} [(x + y) \times z] + [y \times z] &= (x + y) \times z \quad \text{and} \\ [(x \times y) + z] \times [y + z] &= (x \times y) + z \end{aligned}$$

are satisfied.

For the following, see [11, p.184, Footnote (3)].

Proposition 2.8. *The class of lattices coincides with the class of quasilattices in which the identity*

$$x + (x \times y) = x$$

is satisfied.

2.5. **Duplicated semilattices.** A *duplicated semilattice* $(Q, \cdot, \cdot)$ is a bisemilattice, each of whose operations is the multiplication of a given semilattice $(Q, \cdot)$ [14, 327].

Lemma 2.9. *The class of (duplicated) semilattices coincides with the class of quasilattices in which the identity*

$$x + y = x \times y$$

is satisfied.

The following characterization of duplicated semilattices may be compared with the description of lattices given in §2.3.

Corollary 2.10. *Duplicated semilattices are equivalent to quasilattices $(Q, \times, +)$ in which the meet semilattice order $\leq_\times$ and join semilattice order $\leq_+$ are dual.*

2.6. **The structure of quasilattices.** Consider the categories **SLat** and **QLat** of bisemilattice homomorphisms, whose respective object classes consist of semilattices and quasilattices. Then the nameless inclusion functor $\mathbf{SLat} \hookrightarrow \mathbf{QLat}$ has a left adjoint $P \colon \mathbf{QLat} \to \mathbf{SLat}$, the *reflection* or *replica* functor. The image Q^P of a quasilattice Q is its *semilattice replica*. Write $\pi \colon Q \to Q^P$ for the unit of the adjunction, interpreted as an analogue of a fiber bundle over a base space Q^P, with Q as the total space of the bundle. (This analogy is made more explicit in [16, §5.1.3].) Then for each point h in $H = Q^P$, the *Płonka fiber* $h^R = \pi^{-1}\{h\}$, as a subquasilattice of Q, is actually a lattice. If $h \leq_\times k$ in Q^P, there is a uniquely defined lattice homomorphism

$$(h \leq_\times k)^R \colon \pi^{-1}\{k\} \to \pi^{-1}\{h\}; x \mapsto x + (x \times y) = \varphi_{hk}(x)$$

that is independent of the choice of an arbitrary element y of $\pi^{-1}\{h\}$. These lattice homomorphisms are called the *Płonka homomorphisms* of the quasilattice $(Q, \times, +)$. Altogether, the quasilattice Q specifies a contravariant functor $R \colon (H, \leq_\times) \to \mathbf{Lat}$ from the poset category $(H, \leq_\times)$, the semilattice replica of Q, to the category **Lat** of lattices. In the terminology of §2.1.3, we have a **Lat**-representation of $(H, \leq_\times)$.

Conversely, consider a **Lat**-representation $R \colon (H, \leq_\times) \to \mathbf{Lat}$ of $(H, \leq_\times)$. A quasilattice structure is defined on the disjoint union

$\biguplus_{h\in H} h^R$ by

$$x \times y = x(h \times k \leq_\times h)^R \times y(h \times k \leq_\times k)^R$$

and

$$x + y = x(h \times k \leq_\times h)^R + y(h \times k \leq_\times k)^R$$

for $h, k \in H$, along with $x \in h^R$ and $y \in k^R$. These two constructions provide an equivalence between the categories $\mathbf{QLat}$ and $\mathbf{Lat}^{H^{\mathsf{op}}}$ [11, 13], [15, Th. 4.3.2], [16, §6.1]. In the literature, contravariant functors from $(H, \leq_\times)$ are sometimes taken in the form of covariant functors from the dual join semilattice $(H, \leq_+)$.

Example 2.11. Consider the lattice homomorphism $\varphi\colon \mathbf{3} \to \mathbf{2} \times \mathbf{2}$ from the three-element chain $\{0 < \frac{1}{2} < 1\}$ to the direct square of the two-element lattice $\mathbf{2} = \{0 \leq 1\}$, displayed as

(2.4)
$$\begin{tikzcd}
1 \arrow[rrr, mapsto, "\varphi"] & & & 11 & \\
\frac{1}{2} \arrow[u] \arrow[urrr, mapsto, "\varphi"] & & 01 \arrow[ur] & & 10 \arrow[ul] \\
0 \arrow[u] \arrow[rrr, mapsto, "\varphi"] & & & 00 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

using a convention where the elements of the direct square are written simply as binary strings of length two. Note that the lattice structures are presented by their *Hasse diagrams*, the directed graphs recording the covering relations in the lattices. Thus the full order relations on the lattices are the reflexive, transitive closures of the covering relations displayed in the Hasse diagram.

Now consider the two-element meet semilattice

(2.5)
$$H = \{\texttt{early} \to \texttt{late}\}$$

which will often be interpreted as an encoding of a time interval. Take the representation $R\colon H \to \mathbf{Lat}$ with $\texttt{early}^R = \mathbf{2} \times \mathbf{2}$, $\texttt{late}^R = \mathbf{3}$, and $(\texttt{early} \to \texttt{late})^R = \varphi$. The construction of this section then yields a quasilattice $(Q, \times, +)$ with

(2.6)
$$\begin{tikzcd}
1 & & & 11 \arrow[lll] \arrow[dlll] & \\
\frac{1}{2} \arrow[u] & & 01 \arrow[ur] & & 10 \arrow[ul] \\
0 \arrow[u] & & & 00 \arrow[lll] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

as the Hasse diagram of $(Q, \leq_\times)$ and

(2.7)
$$\begin{tikzcd}
1 \arrow[rr] & & 11 & \\
\frac{1}{2} \arrow[u] \arrow[urr] & 01 \arrow[ur] & & 10 \arrow[ul] \\
0 \arrow[u] \arrow[rr] & & 00 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

as the Hasse diagram of $(Q, \leq_+)$. These diagrams illustrate features common to all quasilattices. For example, two elements lie in the same lattice fiber if and only if they have the same order relationship in each diagram. On the other hand, an order relationship under $\leq_+$ records the action of a lattice homomorphism if the dual relationship holds under $\leq_\times$.

2.7. **Small quasilattices.** Here, we will consider the three 3-element quasilattices which are neither lattices nor (duplicated) semilattices, and one 4-element quasilattice. Their Płonka representations are given below. Each has the two-element semilattice replica (2.5). Each of the quasilattices is specified by its non-trivial Płonka homomorphism, which is respectively:

- a complete lattice homomorphism,
- a complete meet semilattice homomorphism (thus preserving arbitrary meets),
- a complete join semilattice homomorphism (namely: preserving arbitrary joins), or
- a lattice homomorphism which preserves neither the upper nor the lower bounds.

2.7.1. *The complete lattice homomorphism.* Here, consider the Płonka representation

(2.8)
$$\begin{tikzcd}
 & 1 \arrow[dl, mapsto] \\
\infty & \\
 & 0 \arrow[ul, mapsto] \arrow[uu]
\end{tikzcd}$$

with the singleton $\{\infty\}$ as the early fiber, and the two-element lattice $\{0 \to 1\}$ as the late fiber.

2.7.2. *The complete meet semilattice homomorphism.* Here, the Płonka representation is

$$\begin{tikzcd} 1 & \infty \arrow[l, mapsto] \\ 0 \arrow[u] & \end{tikzcd}$$

with the two-element lattice $\{0 \to 1\}$ as the early fiber; the singleton $\{\infty\}$ is the late fiber. Recall that the empty meet in a complete semilattice is the upper bound, preserved by the Płonka homomorphism in this case.

2.7.3. *The complete join semilattice homomorphism.* Here, the Płonka representation is

$$\begin{tikzcd} 1 & \\ 0 \arrow[u] & \infty \arrow[l, mapsto] \end{tikzcd}$$

with the two-element lattice $\{0 \to 1\}$ as the early fiber; the singleton $\{\infty\}$ is the late fiber. Recall that the empty join in a complete semilattice is the lower bound, preserved by the Płonka homomorphism in this case.

2.7.4. *No bounds preserved.* Here, the Płonka representation is

$$\begin{tikzcd} 1 & \\ \frac{1}{2} \arrow[u] & \infty \arrow[l, mapsto] \\ 0 \arrow[u] & \end{tikzcd}$$

with the three-element lattice $\{0 \to \frac{1}{2} \to 1\}$ as the early fiber; the singleton $\{\infty\}$ is the late fiber. Neither upper nor lower bounds are preserved by the Płonka homomorphism in this case.

2.8. **Complete quasilattices.** The quasilattices that we will be using as the analogues of concept lattices for complex systems will have a property of local completeness, as introduced in §2.9 below. To put that property into context, we begin by examining certain properties of globally complete quasilattices. The first example shows that a Płonka fiber of a complete quasilattice may not necessarily form a complete lattice.

Example 2.12. Consider the following **Lat**-representation of the two-element meet semilattice $\{\texttt{early} \to \texttt{late}\}$:

- The image of the object `early` is the complete closed-interval sublattice $B = [0,1]$ of $(\mathbb{R}, \leq)$;
- The image of the object `late` is the open-interval sublattice $A =]0,1[$ of $(\mathbb{R}, \leq)$;
- The image of the morphism $\texttt{early} \to \texttt{late}$ is the embedding $j\colon A \hookrightarrow B$ of the open unit interval A into the closed unit interval B.

The quasilattice Q represented in this way is complete. Consider, for instance, the subset $S = \{1/n \mid 1 < n \in \mathbb{Z}\}$ of A. Then $\sum_{x \in S} x = 1/2$ in A, while $\prod_{x \in S} x = 0$ in B, since $0 \times \prod_{x \in S} x = 0 \times \prod_{x \in S} j(x) = 0$. On the other hand, the Płonka fiber A is not a complete lattice.

The remainder of this section focusses on quasilattices where each Płonka fiber is a complete lattice.

Lemma 2.13. *Let $(Q, \times, +)$ be a quasilattice where each Płonka fiber is a complete lattice. Let $\pi\colon Q \to H$ be the projection from Q onto its semilattice replica. For each element h of H, set $s_h = \prod_{q \in \pi^{-1}\{h\}} q$.*

(a) *The equivalent relationships*

$$r \times s_h = s_h \quad \textit{or} \quad s_h \leq_\times r$$

hold for each element r of $\pi^{-1}\{h\}$.

(b) *One has*

$$k \leq_\times h \textit{ in } (H, \leq_\times) \quad \Leftrightarrow \quad s_k \leq_\times s_h \textit{ in } Q$$

for elements h, k of H.

Proof. (a) Note

$$r \times s_h = r \times \prod_{q \in \pi^{-1}\{h\}} q = \prod_{q \in \pi^{-1}\{h\}} q = s_h$$

for $r \in \pi^{-1}\{h\}$.

(b) Suppose $k \leq_\times h$ in $(H, \leq_\times)$. Consider the Płonka homomorphism

$$\varphi_{kh}\colon \pi^{-1}\{h\} \to \pi^{-1}\{k\}; q \mapsto \varphi_{kh}(q)\,.$$

Then $s_h \times s_k = \varphi_{kh}(s_h) \times s_k = s_k$ by (a), so that $s_k \leq_\times s_h$ in $(Q, \times, +)$.

Conversely, suppose $s_k \leq_\times s_h$ in Q, so that $s_h \times s_k = s_k$. Applying the replica homomorphism π, one then has $h \times k = k$, so that $k \leq_\times h$ in $(H, \leq_\times)$. $\square$

Proposition 2.14. *If $(Q, \times, +)$ is a complete quasilattice where each Płonka fiber is a complete lattice, then the semilattice replica of Q is complete.*

Proof. Let $\pi\colon Q \to H$ be the projection onto the semilattice replica H of Q. For each element h of H, set $s_h = \prod_{q \in \pi^{-1}\{h\}} q$, as in Lemma 2.13.

Let I be a subset of the semilattice replica H of $(Q, \times, +)$. Suppose that $\prod_{j \in I} s_j = x$ in $(Q, \times)$. Then $s_i \times x = s_i \times \prod_{j \in I} s_j = \prod_{j \in I} s_j = x$ for each i in I. Application of the replica homomorphism π yields $i \times x\pi = x\pi$, so that $x\pi \leq_\times i$ in $(H, \leq_\times)$. Thus $x\pi$ is a lower bound for I in $(H, \leq_\times)$.

Now suppose that l is a lower bound for I in $(H, \leq_\times)$. Then by Lemma 2.13, $s_l \times s_i = s_l$ for $i \in I$. Thus $s_l \times x = s_l \times \prod_{j \in I} s_j = s_l$, whence $l \leq_\times x\pi$ in $(H, \leq_\times)$. It follows that $x\pi$ is the greatest lower bound for I in $(H, \leq_\times)$. $\square$

Remark 2.15. Proposition 2.14 contrasts with the fact that a quotient of a complete lattice need not be complete [1, Ex. V.1.9]. In other words, a complete quasilattice may have a quotient which is not a complete quasilattice.

2.9. **Locally complete quasilattices.**

Definition 2.16. A quasilattice $(Q, \times, +)$ is said to be *locally complete* if its Płonka fibers are complete lattices and its Płonka homomorphisms are complete lattice homomorphisms.

Example 2.17. The three-element quasilattice of §2.7.1 forms a locally complete quasilattice.

Example 2.18. Each complete lattice forms a locally complete quasilattice.

Since semilattices need not be complete, the following example shows that local completeness does not imply global completeness.

Example 2.19. Each duplicated semilattice forms a locally complete quasilattice.

In terms of the representing functors to which they are equivalent, locally complete quasilattices may be characterized as follows.

Proposition 2.20. *Each locally complete quasilattice is represented by a contravariant functor $R\colon H \to$ **CLat** from a semilattice to the category of complete lattice homomorphisms. Conversely, the Płonka sum of each such representation is a locally complete quasilattice.*

Along with Examples 2.12 and 2.19, the following proposition serves to demonstrate the independence of the two concepts of global and local completeness.

Proposition 2.21. *A globally complete quasilattice having complete Płonka fibers need not be locally complete.*

Proof. Consider the three quasilattices of the respective paragraphs 2.7.2–2.7.4. Since the quasilattices and their Płonka fibers are finite, they are complete. On the other hand, they are not locally complete, since their Płonka homomorphisms are not full bounded lattice homomorphisms. □

2.10. **Fiber finite quasilattices.**

Definition 2.22. A quasilattice $(Q, \times, +)$ is said to be *fiber finite* if its Płonka fibers are finite lattices.

Lemma 2.23. *There is a left adjoint $E\colon \mathbf{Lat} \to \mathbf{BLat}$ to the forgetful functor $V\colon \mathbf{BLat} \to \mathbf{Lat}$ from the category of bounded lattices (and their homomorphisms) to the category of lattices. Then the left adjoint restricts and corestricts to a functor $E\colon \mathbf{FinLat} \to \mathbf{FinBLat}$ between the corresponding categories of finite lattices.*

Proof. The existence and nature of the left adjoint is guaranteed on general grounds by [18, Th. IV.3.4.4]. □

For a finite lattice L, the unit $L \to LEW$ of the adjunction in Lemma 2.23 embeds L into the disjoint union of L with $\{\bot, \top\}$, where $\bot$ is an added lower bound and $\top$ is an added upper bound. Lemma 2.23 may be contrasted with the following observation.

Proposition 2.24. *The forgetful functor $W\colon \mathbf{CLat} \to \mathbf{Lat}$, from the category of complete lattices to the category of lattices, does not have a left adjoint.*

Proof. Suppose such a left adjoint, say $D\colon \mathbf{Lat} \to \mathbf{CLat}$, existed. Now the underlying set functor $U\colon \mathbf{Lat} \to \mathbf{Set}$, from the category of lattices to the category of sets, has the free lattice functor $F\colon \mathbf{Set} \to \mathbf{Lat}$ as a left adjoint. Then the composite left adjoint $FD\colon \mathbf{Set} \to \mathbf{CLat}$ would yield a free complete lattice $\{x, y, z\}FD$ on a three-element set $\{x, y, z\}$. However, no such object exists [8, §I.4.7]. □

Theorem 2.25. *Fiber finite quasilattices embed into locally complete quasilattices.*

Proof. Consider a fiber finite quasilattice $(Q, \times, +)$, represented by a contravariant Płonka functor $R\colon H \to \mathbf{FinLat}$ from a semilattice to the category of finite lattice homomorphisms. Then Q embeds into the quasilattice $\overline{Q}$ that is represented by the contravariant functor $RE\colon H \to \mathbf{FinBLat}$ from the semilattice H to the category of finite bounded lattices (using the restriction $E\colon \mathbf{FinLat} \to \mathbf{FinBLat}$ of the left adjoint functor $E\colon \mathbf{Lat} \to \mathbf{BLat}$ of Lemma 2.23). Now since the lattice fibers of $\overline{Q}$ are still finite, the functor RE may be extended to the composite

$$H \xrightarrow{R} \mathbf{FinLat} \xrightarrow{E} \mathbf{FinBLat} \hookrightarrow \mathbf{CLat}$$

that represents $\overline{Q}$ (according to Proposition 2.20) as a locally complete quasilattice containing Q. $\square$

An application of Theorem 2.25 is discussed in Section 5.3.

3. Classical concept analysis

This section recalls the basic features of classical concept analysis, with special emphasis on those aspects that turn out to be important for the subsequent extension to complex concept analysis.

3.1. **Polarities.** Suppose that Ω is a set. Its elements are described as *objects*. Let Π be a set, whose elements are described as *properties*. Let α be a subset of $\Omega \times \Pi$, considered as a relation of *attribution* between Ω and Π. Thus if $(x, p) \in \alpha$ (or equivalently $x\ \alpha\ p$), we say that property p is attributed to (or is an attribute of) object x. The triple (Ω, Π, α) is called a *polarity* [1, §V.7] or *NK-structure* [4] or *context* [3, 19].

Example 3.1. We collect a number of standard instances of polarities.

(1) For a set X, take $(X, X, \neq)$ [19].
(2) For a poset H, take $(H, H, \leq)$.
(3) For a field extension $F \hookrightarrow E$, suppose that $[F, E]$ is the set of all intermediate fields. Let G be the *Galois group* of the extension, the group of automorphisms of E that fix F. For $K \in [F, E]$ and $g \in G$, write $K\ \alpha\ g$ if g fixes each element of K. Then $([F, E], G, \alpha)$ is a polarity.

3.2. **The Galois connection.** For a set X of objects, define

$$X^r = \{p \in \Pi \mid \forall\ x \in X\,,\ x\ \alpha\ p\} \tag{3.1}$$

as the set of properties common to all objects in X. Dually, for a set P of properties, define

$$P^s = \{x \in \Omega \mid \forall\ p \in P\,,\ x\ \alpha\ p\} \tag{3.2}$$

as the set of objects attributed to all properties in P. The specifications (3.1) and (3.2) yield a pair

$$\left(2^{\Omega}, \subseteq\right) \underset{s}{\overset{r}{\rightleftarrows}} \left(2^{\Pi}, \supseteq\right) \tag{3.3}$$

of order-preserving functions, an adjoint pair of functors between poset categories [18, III §3.3]. Following Example 3.1(3), the pair (3.3) is known as a *Galois connection*. It restricts to a pair

$$\left(2^{\Pi}s, \subseteq\right) \underset{s}{\overset{r}{\rightleftarrows}} \left(2^{\Omega}r, \supseteq\right) \tag{3.4}$$

of mutually inverse order-preserving functions, a *Galois correspondence*, between the respective images $2^{\Omega}r$ and $2^{\Pi}s$ of the functions r and s from (3.3).

3.3. **Classical concept lattices.** We will now recall and augment the terminology of formal concept analysis [3, 19], using a notation that reflects our English nomenclature.

Definition 3.2. Consider a polarity or context (Ω, Π, α), with Galois correspondence (3.4).

(a) Elements of $2^{\Omega}r$ and $2^{\Pi}s$ are described as *closed*.
(b) A *concept* of the context is an ordered pair

$$(A|B) \tag{3.5}$$

in $2^{\Pi}s \times 2^{\Omega}r$ (written using a vertical line as the separator), with $A^r = B$ and $B^s = A$.
(c) The closed set A is the *extent* of the concept (3.5).
(d) The closed set B is the *intent* of the concept (3.5).
(e) The *conceptualization* of an object $x \in \Omega$ is the concept $x^{\varepsilon} = (\{x\}^{rs}|\{x\}^r)$.
(f) The *conceptualization* of a property $p \in \Pi$ is the concept $p^{\eta} = (\{p\}^s|\{p\}^{sr})$.
(g) As a set, the *concept lattice* $\mathcal{L}(\Omega, \Pi, \alpha)$ is the set of all concepts of the context (Ω, Π, α).

While the following lemma is standard, it deserves special emphasis, since it will assume an extra significance in the extension from classical to complex concept analysis.

Lemma 3.3. *Given concepts $(A_1|B_1)$, $(A_2|B_2)$ of a context (Ω, Π, α), one has $A_1 \subseteq A_2$ if and only if $B_1 \supseteq B_2$.*

Proof. If $A_1 \subseteq A_2$, the order-preserving property of r from (3.3) implies $B_1 = A_1^r \supseteq A_2^r = B_2$. The "if" claim is dual. $\square$

Definition 3.4. As a bisemilattice, the *concept lattice*

$$\big(\mathcal{L}(\Omega,\Pi,\alpha),\times,+\big) \tag{3.6}$$

of the context (Ω,Π,α) is specified by the order relations

$$(A_1|B_1)\le_\times(A_2|B_2)\quad\Leftrightarrow\quad A_1\subseteq A_2 \tag{3.7}$$

and

$$(A_1|B_1)\le_+(A_2|B_2)\quad\Leftrightarrow\quad B_1\supseteq B_2 \tag{3.8}$$

for $(A_1|B_1),(A_2|B_2)\in\mathcal{L}(\Omega,\Pi,\alpha)$. These order relations correspond respectively to

$$(A_1|B_1)\times(A_2|B_2)=\big(A_1\cap A_2\big|(A_1\cap A_2)^r\big) \tag{3.9}$$

and

$$(A_1|B_1)+(A_2|B_2)=\big((B_1\cap B_2)^s\big|B_1\cap B_2\big) \tag{3.10}$$

as semilattice operations. Write $\le$ for the equal relations $\le_\times$ and $\le_+$ in a concept lattice.

The following lemma shows how a polarity may be recovered from its concept lattice (compare the proof of [3, Satz/Th. 3]).

Lemma 3.5. *Let* (Ω,Π,α) *be a context. Then for* $x\in\Omega$ *and* $p\in\Pi$*, the following are equivalent:*

(a) $(x,p)\in\alpha$;
(b) $\{x\}^{rs}\subseteq\{p\}^s$;
(c) $\{x\}^r\supseteq\{p\}^{sr}$;
(d) $(\{x\}^{rs}|\{x\}^r)\le(\{p\}^s|\{p\}^{sr})$;
(e) $x^\varepsilon\le p^\eta$.

Theorem 3.6. [1, V Th.19][3, Satz/Th. 3] *The concept lattice* (3.6) *is a complete lattice.*

Proof. By Lemma 3.3, the two orders (3.7) and (3.8) agree, so that (3.6) is a lattice. Then for a subset $\{(A_i|B_i)\mid i\in I\}$ of $\mathcal{L}(\Omega,\Pi,\alpha)$, one has

$$\prod_{i\in I}(A_i,B_i)=\Big(\bigcap_{i\in I}A_i\Big|\big(\bigcap_{i\in I}A_i\big)^r\Big) \tag{3.11}$$

and

$$\sum_{i\in I}(A_i,B_i)=\Big(\big(\bigcap_{i\in I}B_i\big)^s\Big|\bigcap_{i\in I}B_i\Big) \tag{3.12}$$

in $\mathcal{L}(\Omega,\Pi,\alpha)$. □

Remark 3.7. In the concept lattice meet operations (3.9), (3.11) and join operations (3.10), (3.12), one side is "natural" in the sense of only involving intersections, while the other side is "unnatural" in that it also involves one of the two functions r, s from (3.4). A similar split may be observed in various applications of lattices, where one of the two lattice operations has a lower complexity than the other [17].

3.4. **Concept lattices and MacNeille completions.** The following result provides a converse to Theorem 3.6.

Theorem 3.8. [3, Satz/Th. 3] *Let L be a complete lattice. Then L is isomorphic to the concept lattice $\mathcal{L}(L, L, \leq)$ of the polarity $(L, L, \leq)$.*

In this context, it is instructive to recall the original definition of the (Dedekind-)MacNeille completion [9].

Definition 3.9. Let $(P, \leq)$ be a partially ordered set. The *MacNeille completion* of $(P, \leq)$ is the complete lattice $\mathcal{L}(P, P, \leq)$,

Example 3.10. Let (Ω, Π, α) be a context. Define a partial order $(P, \leq)$ on the disjoint union P of Ω and Π by setting $x < p$, for x in (the embedded version of) Ω and p in (the embedded version of) Π, if and only if $(x, p) \in \alpha$. Then $\mathcal{L}(P, P, \leq) = \mathcal{L}(\Omega, \Pi, \alpha)$.

Proposition 3.11. [9, §11] *Let $(P, \leq)$ be a partially ordered set. Then the conceptualization*

$$\eta \colon P \to \mathcal{L}(P, P, \leq); p \mapsto (\{p\}^s | \{p\}^{sr}) \tag{3.13}$$

of Definition 3.2(f) is an order-preserving embedding of the partially ordered set $(P, \leq)$ into $\big(\mathcal{L}(P, P, \leq), \leq\big)$.

Remark 3.12. In the context of Proposition 3.11, one should note that $p^\eta = (p^{\geq} | p^{\leq}) = p^\varepsilon$ for $p \in P$.

Example 3.13. Consider the two-element antichain $\{a, b\}$, described by the following poset context:

$$\begin{array}{c|cc} \leq & a & b \\ \hline a & \times & \\ b & & \times \end{array} \,. \tag{3.14}$$

Abstractly, its MacNeille completion is isomorphic to the lattice $\mathbf{2}\times\mathbf{2}$, the direct square of a two-element chain. Here,

$$
\begin{tikzcd}
 & (ab|\varnothing) \arrow[dl, hook, shift left] \arrow[dr, hook, shift left] & \\
(a|a) \arrow[ur, hook, shift left] \arrow[dr, hook, shift left] & & (b|b) \arrow[ul, hook, shift left] \arrow[dl, hook, shift left] \\
 & (\varnothing|ab) \arrow[ul, hook, shift left] \arrow[ur, hook, shift left] &
\end{tikzcd}
\tag{3.15}
$$

gives a set-theoretical representation of the respective order relations (3.7) and (3.8) on the concept lattice. For compactness, subsets are written as concatenations of their elements.

Corollary 3.14. [9, Th. 11.9] *If $(P,\leq)$ is a (bounded) lattice, then* (3.13) *is a (bounded) lattice homomorphism.*

Example 3.15. Consider the three-element chain $\mathbf{3}=\{0<\frac{1}{2}<1\}$, which corresponds to the following poset context:

$$
\begin{array}{c|ccc}
\leq & 0 & \frac{1}{2} & 1 \\
\hline
0 & \times & \times & \times \\
\frac{1}{2} & & \times & \times \\
1 & & & \times
\end{array}\,.
\tag{3.16}
$$

Its MacNeille completion is abstractly isomorphic to $\mathbf{3}$, by means of the bounded lattice isomorphism (3.13). Then

$$
\begin{tikzcd}
(0\tfrac{1}{2}1|1) \arrow[d, hook, shift left] \\
(0\tfrac{1}{2}|\tfrac{1}{2}1) \arrow[u, hook, shift left] \arrow[d, hook, shift left] \\
(0|0\tfrac{1}{2}1) \arrow[u, hook, shift left]
\end{tikzcd}
\tag{3.17}
$$

gives a set-theoretical representation of the respective order relations (3.7) and (3.8), where subsets are again written as concatenations of their elements.

Remark 3.16. The lattice set representations (3.15) and (3.17) will reappear later as lattice fibers in a quasilattice set representation.

3.5. **Bonds.**

Definition 3.17. [3, §5.1] Consider a *domain context* $(\Omega_d, \Pi_d, \alpha_d)$ and a *codomain context* $(\Omega_c, \Pi_c, \alpha_c)$. Then a *bond* from the domain to the codomain is a relation $\beta_{dc} \subseteq \Omega_d \times \Pi_c$ such that:

(a) For each domain object x, the set

$$x^{\beta_{dc}} = \{q \in \Pi_c \mid x \ \beta_{dc} \ q\} \tag{3.18}$$

is an intent in the codomain, and dually

(b) For each codomain property q, the set

$$^{\beta_{dc}}q = \{x \in \Omega_d \mid x \ \beta_{dc} \ q\} \tag{3.19}$$

is an extent in the domain.

Example 3.18. For any domain context $(\Omega_d, \Pi_d, \alpha_d)$ and any codomain context $(\Omega_c, \Pi_c, \alpha_c)$, the relation $\Omega_d \times \Pi_c$ forms a bond from $(\Omega_d, \Pi_d, \alpha_d)$ to $(\Omega_c, \Pi_c, \alpha_c)$, since $\Pi_c = \varnothing^s$ in $(\Omega_c, \Pi_c, \alpha_c)$ and $\Omega_d = \varnothing^r$ in $(\Omega_d, \Pi_d, \alpha_d)$.

Example 3.19. For any context (Ω, Π, α), the polarity α is a bond from (Ω, Π, α) to (Ω, Π, α). Indeed, comparing (3.18) with (3.1) and (3.19) with (3.2) shows that

$$x^{\alpha} = \{x\}^r \quad \text{and} \quad {}^{\alpha}p = \{p\}^s \tag{3.20}$$

for $x \in \Omega$ and $p \in \Pi$.

The notation of (3.18) and (3.19) is extended to subsets X of Ω_d and P of Π_c, so that

$$X^{\beta_{dc}} = \{p \in \Pi_c \mid \forall \ x \in X \ , \ x \ \beta_{dc} \ p\} = \bigcap_{x \in X} x^{\beta_{dc}} \tag{3.21}$$

and

$$^{\beta_{dc}}P = \{x \in \Omega_d \mid \forall \ p \in P \ , \ x \ \beta_{dc} \ p\} = \bigcap_{p \in P} {}^{\beta_{dc}}p \tag{3.22}$$

[3, Defn. 17]. Thus in the situation of Example 3.19, one has

$$X^r = X^{\alpha} = \{p \in \Pi \mid \forall \ x \in X \ , \ x \ \alpha \ p\}$$

and

$$P^s = {}^{\alpha}P = \{x \in \Omega_d \mid \forall \ p \in P \ , \ x \ \alpha \ p\}$$

for $X \subseteq \Omega$ and $P \subseteq \Pi$.

Lemma 3.20. *Let β_{dc} be a bond from $(\Omega_d, \Pi_d, \alpha_d)$ to $(\Omega_c, \Pi_c, \alpha_c)$.*

(a) *For each subset X of Ω_d, the set $X^{\beta_{dc}}$ is the intent of*

$$\sum_{x\in X}\big((x^{\beta_{dc}})^s\big|x^{\beta_{dc}}\big)$$

in $\mathcal{L}(\Omega_c,\Pi_c,\alpha_c)$.

(b) *For each subset P of Π_d, the set ${}^{\beta_{dc}}P$ is the extent of*

$$\prod_{p\in P}\big({}^{\beta_{dc}}p\big|({}^{\beta_{dc}}p)^r\big)$$

in $\mathcal{L}(\Omega_c,\Pi_c,\alpha_c)$.

Proof. (a) Use Definition 3.17(a), (3.21), and (3.12).

(b) Use Definition 3.17(b), (3.22), and (3.11). □

4. Categorical aspects of classical concept analysis

4.1. The category of bonds.

Proposition 4.1. [3, §5.1] *Let β_{dc} be a bond from $(\Omega_d,\Pi_d,\alpha_d)$ to $(\Omega_c,\Pi_c,\alpha_c)$. Let β_{cb} be a bond from $(\Omega_c,\Pi_c,\alpha_c)$ to $(\Omega_b,\Pi_b,\alpha_b)$.*

(a) *The relation*

$$\begin{aligned}\beta_{dc}\beta_{cb} &= \bigcup_{(A|B)\in\mathcal{L}(\Omega_c,\Pi_c,\alpha_c)} {}^{\beta_{dc}}B\times A^{\beta_{cb}}\\ &= \{(x,n)\in\Omega_d\times\Pi_b \mid (x^{\beta_{dc}})^s\subseteq {}^{\beta_{cb}}n\}\end{aligned}\tag{4.1}$$

is a bond from $(\Omega_d,\Pi_d,\alpha_d)$ to $(\Omega_b,\Pi_b,\alpha_b)$.

(b) *Suppose that β_{db} is a bond from $(\Omega_d,\Pi_d,\alpha_d)$ to $(\Omega_b,\Pi_b,\alpha_b)$. Then $\beta_{db}\subseteq\beta_{dc}\beta_{cb}$ if and only if*

$$x^{\beta_{db}}\subseteq\Big(\big(x^{\beta_{dc}}\big)^s\Big)^{\beta_{cb}}\tag{4.2}$$

for all $x\in\Omega_d$, or

$${}^{\beta_{db}}n\subseteq {}^{\beta_{dc}}\Big(\big({}^{\beta_{cb}}n\big)^r\Big)\tag{4.3}$$

for all $n\in\Pi_b$.

Remark 4.2. The condition (4.1) on a pair (x,n) from $\Omega_d\times\Pi_b$ to be in $\beta_{dc}\beta_{cb}$ may be summarized by the following diagram:

$$\begin{tikzcd}
x \arrow[dr, "\beta_{dc}"] \arrow[drr, bend left] & & \\
y \arrow[r, bend left, "\alpha_c"] \arrow[rr, bend right] \arrow[dr, "\beta_{cb}"'] & m \arrow[r, dotted, no head] & m' \\
& n &
\end{tikzcd}$$

Here, the object y from Ω_c is α_c-related to all of the elements $m,\dots,m'$ that are β_{dc}-related to x. Thus y represents a typical element of $(x^{\beta_{dc}})^s$. The diagram then expresses the defining requirement (4.1) for such an element to lie in ${}^{\beta_{cb}}n$; namely, that it should be β_{cb}-related to n. This diagrammatic representation of the bond product may be contrasted with the representations used in [3, §5.1].

Proposition 4.3. *Let β_{dc} be a bond from $(\Omega_d,\Pi_d,\alpha_d)$ to $(\Omega_c,\Pi_c,\alpha_c)$. Then $\alpha_d\beta_{dc}=\beta_{dc}$ and $\beta_{dc}\alpha_c=\beta_{dc}$.*

Proof. Suppose that an element (x,q) of $\Omega_d\times\Pi_c$ lies in $\alpha_d\beta_{dc}$, so that

$$\{x\}^{rs}=(x^{\alpha_d})^s\subseteq {}^{\beta_{dc}}q=\{y\in\Omega_d\mid y\ \beta_{dc}\ q\}\,.$$

Then since $\{x\}\subseteq\{x\}^{rs}$, it follows that $x\ \beta_{dc}\ q$. Hence $\alpha_d\beta_{dc}\subseteq\beta_{dc}$. Conversely, consider an arbitrary element q of Π_c. Then

$${}^{\beta_{dc}}q\subseteq\left({}^{\beta_{dc}}q\right)^{rs}={}^{\alpha_d}\!\left(\left({}^{\beta_{dc}}q\right)^r\right),$$

so $\beta_{dc}\subseteq\alpha_d\beta_{dc}$ by (4.3).

Dually, suppose that an element (x,q) of $\Omega_d\times\Pi_c$ lies in $\beta_{dc}\alpha_c$, so that

$$(x^{\beta_{dc}})^s\subseteq {}^{\alpha_c}q=\{q\}^s\,. \tag{4.4}$$

Since β_{dc} is a bond, the set $x^{\beta_{dc}}$ is closed. Applying the order-reversing map r to (4.4) then yields

$$x^{\beta_{dc}}=\left(x^{\beta_{dc}}\right)^{sr}\supseteq\{q\}^{sr}\supseteq\{q\}$$

or $x\ \beta_{dc}\ q$. Hence $\beta_{dc}\alpha_c\subseteq\beta_{dc}$. Conversely, consider an arbitrary element x of Ω_d. Then

$$x^{\beta_{dc}}\subseteq\left(x^{\beta_{dc}}\right)^{sr}=\left(\left(x^{\beta_{dc}}\right)^s\right)^{\alpha_c},$$

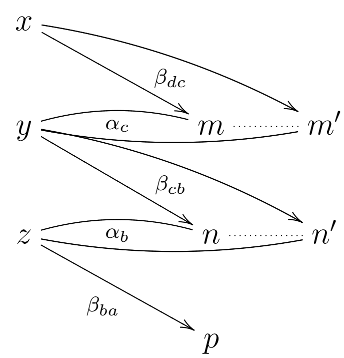
so $\beta_{dc}\subseteq\beta_{dc}\alpha_c$ by (4.2). $\square$

Proposition 4.4. *Let β_{dc} be a bond from $(\Omega_d,\Pi_d,\alpha_d)$ to $(\Omega_c,\Pi_c,\alpha_c)$. Let β_{cb} be a bond from $(\Omega_c,\Pi_c,\alpha_c)$ to $(\Omega_b,\Pi_b,\alpha_b)$. Let β_{ba} be a bond from $(\Omega_b,\Pi_b,\alpha_b)$ to $(\Omega_a,\Pi_a,\alpha_a)$. Then*

$$\left(\beta_{dc}\beta_{dc}\right)\beta_{dc}=\beta_{dc}\left(\beta_{dc}\beta_{dc}\right),$$

so the product (4.1) of bonds is associative.

Proof. Using the conventions of Remark 4.2, consider the following diagram:



The (indirect) connections from x to the elements $n, \ldots, n'$ represent that x is $\beta_{dc}\beta_{cb}$-related to the elements $n, \ldots, n'$ of $x^{\beta_{dc}\beta_{cb}}$. Thus the diagram represents an element (x, p) of $\big(\beta_{dc}\beta_{dc}\big)\beta_{dc}$. On the other hand, the indirect connection from y to p represents that y is $\beta_{cb}\beta_{ba}$-related to the element p of Π_a, so the diagram as a whole represents that (x, p) is also an element of $\beta_{dc}\big(\beta_{dc}\beta_{dc}\big)$. In other words, an element (x, p) of $\Omega_d \times \Pi_a$ lies in $\big(\beta_{dc}\beta_{dc}\big)\beta_{dc}$ if and only if it lies in $\beta_{dc}\big(\beta_{dc}\beta_{dc}\big)$. $\square$

Definition 4.5. The *category* **Bond** *of bonds* has the class of contexts as its object class, and the class of bonds as its morphism class. The product of morphisms is given by (4.1). The identity morphism at a context (Ω, Π, α) is the bond α.

4.2. **The category of bond pairs.**

Definition 4.6. Let $(\Omega_d, \Pi_d, \alpha_d)$ be a domain context. Let $(\Omega_c, \Pi_c, \alpha_c)$ be a codomain context. Then a *bond pair* (β_{dc}, β_{cd}) from $(\Omega_d, \Pi_d, \alpha_d)$ to $(\Omega_c, \Pi_c, \alpha_c)$ consists of a bond β_{dc} from $(\Omega_d, \Pi_d, \alpha_d)$ to $(\Omega_c, \Pi_c, \alpha_c)$ and a bond β_{cd} from $(\Omega_c, \Pi_c, \alpha_c)$ to $(\Omega_d, \Pi_d, \alpha_d)$.

Lemma 4.7. (a) *If (Ω, Π, α) is a context, then (α, α) is a bond pair from (Ω, Π, α) to (Ω, Π, α).*

(b) *If (β_{dc}, β_{cd}) is a bond pair from $(\Omega_d, \Pi_d, \alpha_d)$ to $(\Omega_c, \Pi_c, \alpha_c)$, then (β_{cd}, β_{dc}) is a bond pair from $(\Omega_c, \Pi_c, \alpha_c)$ to $(\Omega_d, \Pi_d, \alpha_d)$.*

Lemma 4.8. *Suppose that (β_{dc}, β_{cd}) is a bond pair from $(\Omega_d, \Pi_d, \alpha_d)$ to $(\Omega_c, \Pi_c, \alpha_c)$, and that (β_{cb}, β_{bc}) is a bond pair from $(\Omega_c, \Pi_c, \alpha_c)$ to $(\Omega_b, \Pi_b, \alpha_b)$. Then*

$$(\beta_{dc}, \beta_{cd})(\beta_{cb}, \beta_{bc}) := (\beta_{dc}\beta_{cb}, \beta_{bc}\beta_{cd}) \tag{4.5}$$

is a bond pair from $(\Omega_d, \Pi_d, \alpha_d)$ to $(\Omega_b, \Pi_b, \alpha_b)$.

Lemma 4.9. *Suppose that* (β_{dc}, β_{cd}) *is a bond pair from* $(\Omega_d, \Pi_d, \alpha_d)$ *to* $(\Omega_c, \Pi_c, \alpha_c)$. *Then the equations* $(\alpha_d, \alpha_d)(\beta_{dc}, \beta_{cd}) = (\beta_{dc}, \beta_{cd})$ *and* $(\beta_{dc}, \beta_{cd})(\alpha_c, \alpha_c) = (\beta_{dc}, \beta_{cd})$ *hold.*

Proof. Apply Proposition 4.3. Thus

$$(\alpha_d, \alpha_d)(\beta_{dc}, \beta_{cd}) = (\alpha_d \beta_{dc}, \beta_{cd} \alpha_d) = (\beta_{dc}, \beta_{cd}) \,,$$

and similarly for the other equation. □

Definition 4.10. The *category* **BdPr** *of bond pairs* comprises the class of contexts as its object class. The class of bond pairs is its morphism class. The product of morphisms is given by (4.5). Then the identity morphism at a context (Ω, Π, α) is the bond pair (α, α).

4.3. **Concept lattice homomorphisms and bond pairs.** In this paragraph, we consider the correspondence connecting certain bond pairs with complete lattice homomorphisms between concept lattices. To understand these results better, it is convenient to consider the category **CtLt** of concept lattices and complete lattice homomorphisms between them. Note that there is an inclusion functor

$$\mathcal{A}\colon \mathbf{CtLt} \hookrightarrow \mathbf{CLat} \tag{4.6}$$

known as *abstraction*.

Definition 4.11. Consider a given bond pair (β_{dc}, β_{cd}) from a domain context $(\Omega_d, \Pi_d, \alpha_d)$ to a codomain context $(\Omega_c, \Pi_c, \alpha_c)$. The bond pair is a *bonding* if the condition

$$\forall\ (A|B) \in \mathcal{L}(\Omega_d, \Pi_d, \alpha_d)\,,\ \left(A^{\beta_{dc}}\right)^r = {}^{\beta_{cd}}B \ \text{ and } \ \left({}^{\beta_{cd}}B\right)^s = A^{\beta_{dc}} \tag{4.7}$$

is satisfied.

Proposition 4.12. [3, §7.2] *For a domain context* $(\Omega_d, \Pi_d, \alpha_d)$ *and a codomain context* $(\Omega_c, \Pi_c, \alpha_c)$, *suppose that*

$$\varphi\colon \mathcal{L}(\Omega_d, \Pi_d, \alpha_d) \to \mathcal{L}(\Omega_c, \Pi_c, \alpha_c)$$

is a complete lattice homomorphism. Define

$$\beta_{dc} = \{(x, q) \in \Omega_d \times \Pi_c \mid x^\varepsilon \varphi \leq q^\eta\} \tag{4.8}$$

and

$$\beta_{cd} = \{(y, p) \in \Omega_c \times \Pi_d \mid y^\varepsilon \leq p^\eta \varphi\}\,. \tag{4.9}$$

Then (β_{dc}, β_{cd}) *is a bonding from* $(\Omega_c, \Pi_c, \alpha_c)$ *to* $(\Omega_d, \Pi_d, \alpha_d)$.

Proposition 4.13. [3, §7.2] *For a domain context* $(\Omega_d, \Pi_d, \alpha_d)$ *and a codomain context* $(\Omega_c, \Pi_c, \alpha_c)$*, suppose that there is a bonding* (β_{dc}, β_{cd}) *from the domain to the codomain. Then*

$$\mathcal{L}(\Omega_d, \Pi_d, \alpha_d) \to \mathcal{L}(\Omega_c, \Pi_c, \alpha_c) \colon (A|B) \mapsto \left({}^{\beta_{cd}}B \middle| A^{\beta_{dc}}\right)$$

defines a complete lattice homomorphism.

The construction of Proposition 4.12 gives a functor **CtLt** $\to$ **BdPr**. (Functoriality follows along the lines of [3, Hilfs./Prop. 113].) Then the image of this functor enables one to consider a category **Bdg** of bondings as a subcategory of **BdPr**. Thus Propositions 4.12 and 4.13 provide an isomorphism

$$\mathcal{L} \colon \mathbf{Bdg} \to \mathbf{CtLt} \tag{4.10}$$

from **Bdg** to **CtLt**.

Example 4.14. Suppose that $\varphi \colon \mathbf{3} \to \mathbf{2} \times \mathbf{2}$ is the complete lattice homomorphism from the three-element chain to the direct square of the two-element chain taking the intermediate element $\frac{1}{2}$ of $\mathbf{3}$ to the upper bound of $\mathbf{2} \times \mathbf{2}$. Interpreting the domain and codomain of φ as the respective concept lattices presented in Examples 3.13 and 3.15, we have $\varphi \colon (0\tfrac{1}{2}1|1), (0\tfrac{1}{2}|\tfrac{1}{2}1) \mapsto (ab|\emptyset)$, $(0|0\tfrac{1}{2}1) \mapsto (\emptyset|ab)$ as the action of the complete lattice homomorphism. The preimage bonding (β_{dc}, β_{cd}) under (4.10) is given by $\beta_{dc} = \{(0, a), (0, b)\}$ in record of the fact that the image of 0^ε is below a^η and b^η, according to (4.8), and $\beta_{cd} = \{(a, \frac{1}{2}), (b, \frac{1}{2}), (a, 1), (b, 1)\}$ recording that a^ε and b^ε are below the image of 1^η, according to (4.9). The bonding may be displayed as

(4.11)

	0	$\frac{1}{2}$	1	a	b
0	×	×	×	×	×
$\frac{1}{2}$		×	×		
1			×		
a		×	×	×	
b		×	×		×

in conjunction with the contexts from Examples 3.13 and 3.15.

5. Concept quasilattices

We are now in a position to extend the ideas of concept analysis to complex systems with a number of distinct levels that are indexed by a semilattice. The semilattice may be representing a hierarchical arrangement of levels in a complex system. As an alternative, in the case in which the underlying semilattice is a chain, it may be interpreted as presenting a series of time points. A quasilattice with a semilattice replica of this type may be describing the history of a context, with its concomitant concept lattice, that is evolving over the time series. These quasilattices may be considered as dynamic versions of concept lattices. In general, regardless of whether the underlying semilattice is a chain or not, it turns out to be convenient to use a temporal terminology ("earlier," "later," etc.), as was done in (2.5). In fact, as complex structures evolve, the higher levels in their hierarchy tend to arise later in the evolution, emerging on the basis of lower, earlier levels.

5.1. **Complex polarities.** We begin by extending the ideas of §3.1 to complex systems with a hierarchy of distinct levels, indexed by a semilattice.

Definition 5.1. A *complex polarity* or *quasicontext* is a contravariant functor $P\colon H \to \mathbf{Bdg}$ from a semilattice to the category of bondings.

Example 5.2. The polarities and bonding displayed in (4.11) may be interpreted as a complex polarity $P\colon H \to \mathbf{Bdg}$ from the time interval semilattice H of (2.5):

- The image of the object `early` is the poset context (3.14) from Example 3.13;
- The image of the object `late` is the poset context (3.16) from Example 3.15;
- The image of the morphism `early` $\to$ `late` is the bonding (4.11).

Note that polarities or contexts in the traditional sense are complex polarities or quasicontexts for which the indexing semilattice H is a singleton.

5.2. **The concept quasilattice of a complex polarity.** As recorded in Theorem 3.6, concept lattices are complete. The quasilattices that we will be using as the analogues of concept lattices for complex systems will have the property of local completeness, as introduced in §2.9. Recall the isomorphism functor $\mathcal{L}$ of (4.10) and the abstraction functor $\mathcal{A}$ of (4.6).

Definition 5.3. Suppose that $P\colon H \to \mathbf{Bdg}$ is a complex polarity. Then the *concept quasilattice* of P is the locally complete quasilattice that is given, according to Proposition 2.20, by the Płonka sum of the representation

$$H \xrightarrow{P} \mathbf{Bdg} \xrightarrow{\mathcal{L}} \mathbf{CtLt} \xrightarrow{\mathcal{A}} \mathbf{CLat}$$

of the semilattice H in the category $\mathbf{CLat}$ of complete lattices.

Consider the complex polarity that was presented in Example 5.2. The concept quasilattice of this complex polarity is given abstractly by the representation $R\colon \{\texttt{early} \to \texttt{late}\} \to \mathbf{Lat}$ that was exhibited in Example 2.11. The representation may actually be taken to be a contravariant functor $R\colon \{\texttt{early} \to \texttt{late}\} \to \mathbf{CLat}$, in view of the fact that $\varphi\colon \mathbf{3} \to \mathbf{2} \times \mathbf{2}$ is indeed a complete lattice homomorphism. The respective meet semilattice and join semilattice structures of the abstract quasilattice are given by the Hasse diagrams (2.6) and (2.7). These Hasse diagrams then appear within the set representation of the concept quasilattice that is displayed in Fig. 1. Here, the separate meet-semilattice and join-semilattice order relationships between concepts correspond to containments of extents and intents, exactly as in the classical concept lattice specifications (3.7) and (3.8).

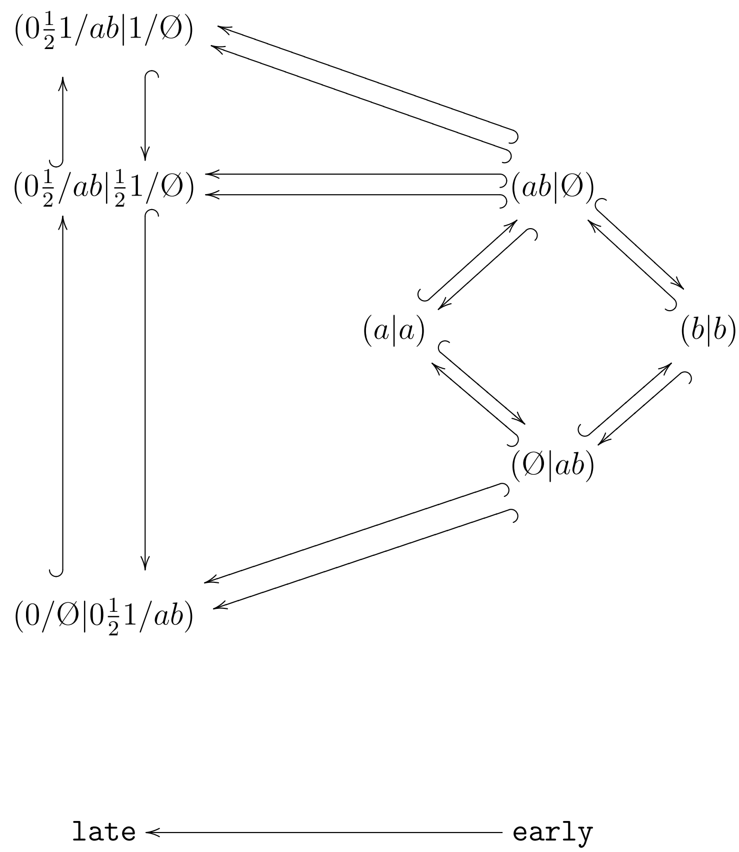


FIGURE 1. Set representation of a concept quasilattice as a record of history.

The containments are opposed within the lattice fibres, as is usual for concept lattices. On the other hand, the containments actually coincide along the edges in the Hasse diagrams that give the actions of the lattice homomorphism displayed by barred arrows in (2.4). In order to implement these containments, the extents and intents of the domain concepts are augmented by the respective extents and intents of their images in the codomain concept lattice. In the figure, the augmented domain concepts are separated by a slash, suggesting that the later concepts appear on top of the corresponding earlier concepts in so-called *layers of history*.

In the time-series interpretation of quasicontexts, these layers may be understood as analogues of geological strata, or the successive layers of an archaeological site. In the complex system interpretation of quasicontexts, the upper layers represent advanced or high-level phenomena that have emerged from more primitive, low-level phenomena.

Looking at Fig. 1, it may be seen that the singleton extents and intents a, b are lost over the time period, corresponding to the fact that they are not part of the image of the complete lattice homomorphism in the structural representation of the concept quasilattice. On the other hand, the early full and empty intents and extents ab and $\varnothing$ are preserved over the time period, and reappear as lower layers of history in the late period. In particular, the early concept $(ab|\varnothing)$ evolves into two distinct late concepts, $(0\frac{1}{2}/ab|\frac{1}{2}1/\varnothing)$ and $(0\frac{1}{2}1/ab|1/\varnothing)$.

5.3. **Quasilattices as concept quasilattices.** The following result extends Theorem 3.8 to locally complete quasilattices. In particular, Theorem 5.4(b) provides a set representation of locally complete quasilattices, as illustrated in Example 5.5 below.

To obtain a set representation for a quasilattice which is fiber finite but not locally complete (such as the quasilattices exhibited in §§ 2.7.2–2.7.4), one may first embed the fiber finite quasilattice into a locally complete lattice using Theorem 2.25, and then apply Theorem 5.4(b).

Theorem 5.4. *Let Q be a locally complete quasilattice, constructed as the Płonka sum of the contravariant functor $R\colon H \to \mathbf{CLat}$. For each element h of H, consider the concept lattice $hP := \mathcal{L}(hR, hR, \leq)$. If $k \to h$ in H, consider the bonding $(k \to h)P$ from hP to kP that is determined by the complete lattice homomorphism*

$$(k \to h)R\colon \mathcal{L}(hR, hR, \leq) \to \mathcal{L}(kR, kR, \leq)\,.$$

(a) *The assignment $P\colon H \to \mathbf{Bdg}$ is a complex polarity.*
(b) *The quasilattice Q is isomorphic to the concept quasilattice of the complex polarity P.*

Example 5.5. Consider the locally complete quasilattice of §2.7.1. The concept lattices of its lattice fibers are $(\{\infty\}|\{\infty\})$ and

$$\begin{tikzcd} (\{0\}|\{0,1\}) \arrow[r, hook, shift left] & (\{0,1\}|\{1\}) \arrow[l, hook', shift left] \end{tikzcd} \ ,$$

with $0^\varepsilon = 0^\eta = (\{0\}|\{0,1\})$ and $1^\varepsilon = 1^\eta = (\{0,1\}|\{1\})$. Then the concept lattice version of (2.8) is

$$\begin{tikzcd} & (\{0,1\}|\{1\}) \arrow[dl, dotted, mapsto] \arrow[dd, hook, shift left] \\ (\{\infty\}|\{\infty\}) & \\ & (\{0\}|\{0,1\}) \arrow[ul, dotted, mapsto] \arrow[uu, hook, shift left] \end{tikzcd} \tag{5.1}$$

The complex polarity, including the bonding

$$(\{(0,\infty),(1,\infty)\},\{(\infty,0),(\infty,1)\})$$

that is specified from the Płonka complete lattice homomorphism (2.8) by Proposition 4.12, is

(5.2)

	0	1	∞
0	$\times$	$\times$	$\times$
1		$\times$	$\times$
∞	$\times$	$\times$	$\times$

We then obtain

$$
\begin{tikzcd}
& (\{0,1\}\cup\{\infty\}|\{1\}\cup\{\infty\}) \arrow[dd, hook, shift left=3] \\
(\{\infty\}|\{\infty\}) \arrow[ur, hook, shift left] \arrow[ur, hook, shift right] \arrow[dr, hook, shift left] \arrow[dr, hook, shift right] & \\
& (\{0\}\cup\{\infty\}|\{0,1\}\cup\{\infty\}) \arrow[uu, hook]
\end{tikzcd}
\tag{5.3}
$$

as a set representation for the concept quasilattice of the quasicontext (5.2). The union operator appearing in the extents and intents on the right-hand side of the diagram may be replaced by slashes denoting layers of history, as in Figure 1.

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