HIGHER HOMOTOPIES BETWEEN QUASIGROUPS

G.N. NOP[1] AND J.D.H. SMITH[2]

Abstract. The concept of homotopy has been a staple of quasigroup theory since Albert's work on isotopies in the early 1940s. In this paper, we introduce and study the dual concept of a higher homotopy.

This is a preprint version. Please cite the published version as: G. Nop and J.D.H. Smith, Higher homotopies between quasigroups, *J. of Algebra*, **526**, (2019), 309–332.

Contents

1. Introduction

The algebraic concept of an isotopy, as a triple of bijections that generalizes the notion of an isomorphism, was originally introduced by Albert for linear algebras [1]. He subsequently applied it to quasigroups [2], essentially in the form of Definition 2.1(b) below. Isotopy also arises from the geometric structures of 3-nets or 3-webs corresponding to, and coordinatized by, quasigroups: Two quasigroups are isotopic if and only if they coordinatize the same 3-net [4, §I.4], [14, Th. I.4.5].

2010 *Mathematics Subject Classification.* 20N05.

Key words and phrases. quasigroup, loop, homotopy, isotopy.

Relaxation of the bijectivity requirement placed on isotopies yields the algebraic concept of a homotopy. Thus a *homotopy*

$$(f, g, h) \colon (P, *) \to (Q, \circ)$$

from a quasigroup $(P, *)$ to a quasigroup $(Q, \circ)$ is a triple of functions $f, g, h \colon P \to Q$ such that $x^f \circ y^g = (x * y)^h$ for all x, y in P (as in Definition 2.1(a) below)[1]. Note that the 3-net of a quasigroup Q lives naturally on the space of homotopies from the singleton quasigroup to Q [11, Th. 1].

For each element p of a quasigroup $(P, *)$ with multiplication $x * y$, there are bijections $R(p)$ or

$$R_*(p) \colon P \to P; x \mapsto x * p$$

and $L(p)$ or

$$L_*(p) \colon P \to P; y \mapsto p * y$$

known respectively as *right* and *left multiplications* by p. Thus the quasigroup carries a *right division* operation $x/y = xR_*(y)^{-1}$ and a *left division* operation $x\backslash y = yL_*(x)^{-1}$, forming an algebraic structure $(P, *, /, \backslash)$ with three binary operations. In terms of these operations, one may formulate the *homotopy extension condition*

$$\forall\ x, y, z \in P\,,\ x^f \circ (x\backslash y)^g = (y/z)^f \circ z^g \tag{1.1}$$

that is necessary and sufficient for a pair $f, g \colon P \to Q$ of functions to extend to a homotopy

$$(f, g, h) \colon (P, *, /, \backslash) \to (Q, \circ, //, \backslash\backslash)$$

from a quasigroup $(P, *)$ to a quasigroup $(Q, \circ)$, as described in the first result of this paper (Theorem 2.4).

The homotopy extension condition (1.1) may be dualized to

$$\forall\ x, y, z \in P\,,\ x^f \backslash\backslash (x * y)^f = (y * z)^g // z^g \tag{1.2}$$

by switching the respective multiplications and divisions between the common domain P and codomain Q of the functions $f, g \colon P \to Q$. The pair (f, g) satisfies the *higher homotopy extension condition* if (1.2) holds. According to Definition 3.1(a), a quadruple $(f, g, h; m)$ of functions from a quasigroup $(P, *)$ to a quasigroup $(Q, \circ)$ is said to form a *higher homotopy* if the triples (f, m, f), (m, g, g) and (f, g, h) are homotopies from $(P, *)$ to $(Q, \circ)$. The core result of this paper, Theorem 3.5, then shows that a homotopy $(f, g, h) \colon (P, *) \to (Q, \circ)$

[1]The paper follows the algebraic convention of placing a function to the right of its argument, either on the line or as a superfix. Thus composites of functions are read in natural order from left to right. This convention minimizes the occurrence of brackets, which otherwise proliferate in the study of non-associative structures.

extends to a higher homotopy $(f, g, h; m)$ if and only if the pair (f, g) satisfies the condition (1.2).

A higher homotopy $(f, g, h; m)$ is a *higher isotopy* if its constituent functions biject. Higher isotopy is a stronger relationship between quasigroups than isotopy. For example, commutativity of the multiplication is preserved by higher isotopy — Proposition 4.2(c) — while it is not necessarily preserved by a mere isotopy. Again, Corollary 7.4 shows that higher isotopes of loops are loops, while ordinary isotopes of loops are not necessarily loops.

An isotopy is *principal* if its third component is an identity function — compare Definition 2.1(c). Each isotopy (f, g, h) factorizes as the composite of a principal isotopy $(fh^{-1}, gh^{-1}, 1)$ and an isomorphism h [2, Lemma 2]. According to Definition 4.4, a higher isotopy $(f, g, h; m)$ is *principal* if $h = fg = gf$. Theorem 4.5 shows that each higher isotopy factorizes as the composite of a principal higher isotopy and an isomorphism. The two notions of principality are independent of each other: Example 4.8 displays a principal higher isotopy $(f, g, h; m)$ where the isotopy (f, g, h) is not principal, while Example 4.9 displays a principal isotopy extending to a higher isotopy that is not principal.

Section 5, which is incidental to the quasigroup theory applications that appear elsewhere in the paper, gives an alternative view of the relationship between homotopies and higher homotopies, based on the commutation of squares in the diagram (5.3). In terms of the "vertical" functions $h_i^r \colon P^r \to Q^r$ (for $1 \leq r \leq 3$ and $1 \leq i \leq r$) that appear in the diagram, Theorem 5.2 characterizes homotopies by commutativity of the right hand square, where the "horizontal" functions have P^2 and Q^2 as domains. Dually, higher homotopies are characterized by the commutativity of the left hand squares, where the "horizontal" functions have P^2 and Q^2 as codomains.

Isotopies are fundamental in quasigroup theory, but not in group theory. This observation originates from Albert's computational result that a loop which is isotopic to a group is actually isomorphic to that group [2, Th. 2], [13, Th. 11.39]. Higher isotopy provides a conceptual framework for this fact: Theorem 6.4 shows that a principal isotopy $(f, g, 1_P)$, from a loop structure on a set P to a group structure on P, always extends to a higher isotopy $(f, g, 1_P; m)$ in which m is an isomorphism.

Section 7 shows that for large finite quasigroups, higher isotopies are rare. More precisely (Theorem 7.5), when a finite domain quasigroup is given, the proportion of isotopies extending to higher isotopies tends to zero as the order of the domain tends to infinity.

An element of a quasigroup is a *Moufang element* (Definition 8.1) if its left and right multiplications extend to an isotopy. Theorem 8.3 uses local associativity properties to characterize those Moufang elements for which this isotopy extends to a higher isotopy.

The final section of the paper introduces the category **Qhh** of higher homotopies between quasigroups, in relation to the category **Qtp** of quasigroup homotopies studied in some detail in [11]. The key tool is the faithful *lowering functor* $\Lambda_3\colon \mathbf{Qhh} \to \mathbf{Qtp}$, whose properties depend on Theorem 3.5. Thus Corollary 9.3 observes that the higher autotopy group of a quasigroup is a subgroup of its autotopy group. Theorem 9.4 provides a category-theoretical characterization of loops amongst general quasigroups: A quasigroup Q is a loop if and only if each **Qtp**-point of Q is the image, under the lowering functor Λ_3, of a **Qhh**-point of Q.

2. Homotopies

Definition 2.1. [13, Definitions 11.26, 11.31] Suppose that $(P,*)$ and $(Q,\circ)$ are quasigroups.

(a) A triple of functions $f\colon P \to Q$, $g\colon P \to Q$ and $h\colon P \to Q$ forms a (*quasigroup*) *homotopy*, denoted by

$$(f,g,h)\colon (P,*) \to (Q,\circ)\,, \tag{2.1}$$

if

$$x^f \circ y^g = (x*y)^h \tag{2.2}$$

for all x,y in P.

(b) The homotopy (2.1) is an *isotopy* if the functions f,g,h are bijections.

(c) An isotopy (2.1) is *principal* if the sets P and Q are equal, and h is the identity function $1_P\colon P \to P$ on the set P.

The following well-known observation will be useful later.

Lemma 2.2. *Suppose that*

$$(f_1,g_1,h_1)\colon (P,*) \to (Q,\circ) \quad \textit{and} \quad (f_2,g_2,h_2)\colon (P,*) \to (Q,\circ)$$

are homotopies. Then if two of the three equalities

$$f_1 = f_2\,, \quad g_1 = g_2\,, \quad h_1 = h_2$$

hold, so does the third.

Proof. Write the full quasigroup structures on P and Q as $(P, *, /, \backslash)$ and $(Q, \circ, /\!\!/, \backslash\!\!\backslash)$ respectively. First, suppose that $f_1 = f_2$ and $g_1 = g_2$. Since (f_i, g_i, h_i) are homotopies for $i = 1, 2$, the equation (2.2) implies

$$\begin{aligned} y^{h_1} = \big(y * (y\backslash y)\big)^{h_1} &= y^{f_1} \circ (y\backslash y)^{g_1} \\ &= y^{f_2} \circ (y\backslash y)^{g_2} = y^{f_2} \circ (y\backslash y)^{g_2} = y^{h_2} \end{aligned}$$

for all y in P, so $h_1 = h_2$. Now suppose that $g_1 = g_2$ and $h_1 = h_2$. Then equation (2.2) implies

$$\begin{aligned} x^{f_1} = \big(x^{f_1} \circ x^{g_1}\big) /\!\!/ x^{g_1} &= (x * x)^{h_1} /\!\!/ x^{g_1} \\ &= (x * x)^{h_2} /\!\!/ x^{g_2} = \big(x^{f_2} \circ x^{g_2}\big) /\!\!/ x^{g_2} = x^{f_2} \end{aligned}$$

for all x in P, so $f_1 = f_2$. The proof that $g_1 = g_2$ follows from $f_1 = f_2$ and $h_1 = h_2$ is similar. □

Now recall that an element e of a quasigroup $(P, *)$ is a *left unit* if $e * y = y$ for all y in P. Similarly, it is a *right unit* if $y * e = y$ for all y in P.

Lemma 2.3. *Consider a homotopy* (2.1)*.*

(a) *If the domain of the homotopy is a quasigroup* $(P, *, e)$ *with left unit element* e*, then* $h = gL_\circ(e^f)$*.*
(b) *If the domain of the homotopy is a quasigroup* $(P, *, e)$ *with right unit element* e*, then* $h = fR_\circ(e^g)$*.*

Proof. (a) Substitution into (2.2) yields $y^h = (e * y)^h = e^f \circ y^g = ygL_\circ(e^f)$ for all y in P.

(b) Substitution into (2.2) yields $x^h = (x * e)^h = x^f \circ e^g = xfR_\circ(e^g)$ for all x in P. □

Theorem 2.4. *Consider an ordered pair* (f, g) *of maps from a quasigroup* $(P, *, /, \backslash)$ *to a quasigroup* $(Q, \circ)$*.*

(a) *The pair* (f, g) *extends to a homotopy* (f, g, h) *if and only if*

$$\forall\ x, y, z \in P\,,\ \ x^f \circ (x\backslash y)^g = (y/z)^f \circ z^g\,. \tag{2.3}$$

(b) *If* (2.3) *holds, then the third map of the homotopy is unique, given by*

$$h\colon P \to Q; y \mapsto x^f \circ (x\backslash y)^g$$

for an arbitrary element x *of* P*.*

(c) *If* (2.3) *holds, then the third map of the homotopy is given by*

$$h\colon P \to Q; y \mapsto (y/z)^f \circ z^g$$

for an arbitrary element z *of* P*.*

Proof. If (f,g,h) is a homotopy, then the equation (2.2) implies

$$x^f\circ(x\backslash y)^g=\big(x*(x\backslash y)\big)^h=y^h=\big((y/z)*z\big)^h=(y/z)^f\circ z^g$$

for all x,y,z in P, so that (2.3) holds. Conversely, suppose that (2.3) holds. If P is empty, the claims of the theorem are trivial or vacuous. Otherwise, choose an element x_0 of P, and consider the function

$$h\colon P\to Q; y\mapsto x_0^f\circ(x_0\backslash y)^g\,.$$

By (2.3), one has

$$y^h=x_0^f\circ(x_0\backslash y)^g=(y/z)^f\circ z^g \tag{2.4}$$

for all y,z in P. Let x and z be elements of P. Substituting $y=x*z$ or $x=y/z$ in (2.4) yields $(x*z)^h=x^f\circ z^g$, so that (f,g,h) is a homotopy. This completes the proof for claim (a) of the theorem.

As a particular case of (2.4), one has

$$y^h=x_0^f\circ(x_0\backslash y)^g=(y/x_0)^f\circ x_0^g$$

for all y in P. By (2.3), one then has

$$y^h=(y/x_0)^f\circ x_0^g=x^f\circ(x\backslash y)^g \tag{2.5}$$

for all x,y in P. Claim (c) of the theorem then follows from (2.4), while claim (b) follows from (2.5). □

3. Higher homotopies

Definition 3.1. Let $(P,*)$ and $(Q,\circ)$ be quasigroups. Consider a quadruple

$$f\colon P\to Q\,,\quad g\colon P\to Q\,,\quad h\colon P\to Q\,,\quad m\colon P\to Q \tag{3.1}$$

of functions.

(a) The quadruple (3.1) forms a *higher homotopy*, denoted by

$$(f,g,h;m)\colon(P,*)\to(Q,\circ)\,, \tag{3.2}$$

if $(f,g,h)\colon(P,*)\to(Q,\circ)$ is a homotopy, and the equations

$$(x*y)^f=x^f\circ y^m\,,\qquad (y*z)^g=y^m\circ z^g \tag{3.3}$$

hold for all x,y,z in P. Here, the function $m\colon P\to Q$ is known as the *middle map* of the higher homotopy, while the homotopy (f,g,h) is known as the *homotopic part* of the higher homotopy.

(b) The higher homotopy (3.2) is a *higher isotopy* if the functions f,g,h,m are bijections. In this case, the homotopic part is called the *isotopic part*.

The "middle map" terminology used in Definition 3.1 is justified in Remark 5.3. A higher homotopy (3.2) may be visualized as a triple

$$(P,*) \;\begin{array}{c} \xrightarrow{(f,g,h)} \\ \xrightarrow{(f,m,f)} \\ \xrightarrow[(m,g,g)]{} \end{array}\; (Q,\circ) \tag{3.4}$$

of homotopies. The conditions (3.3) for a higher homotopy may also be formulated in terms of a single homotopy.

Lemma 3.2. *Consider quasigroups $(P,*)$ and $(Q,\circ)$. Let*

$$(f,g,h)\colon (P,*) \to (Q,\circ)$$

be a homotopy. Then for a map $m\colon P \to Q$, the quadruple $(f,g,h;m)$ is a higher homotopy if and only if

$$(f,m,f)\colon (P,*) \to (Q,\circ) \quad \text{and} \quad (m,g,g)\colon (P,*) \to (Q,\circ)$$

are homotopies, i.e., if and only if the triple

$$\big((f,m),(m,g),(f,g)\big)\colon (P,*)^2 \to (Q,\circ)^2$$

is a homotopy.

A natural question to ask is whether any given homotopy arises as the homotopic part of a higher homotopy. The following example provides positive and negative answers.

Example 3.3. Consider an abelian group $(A,+,0)$. Take $(P,*) = (A,-)$ and $(Q,\circ) = (A,+)$, with the isotopy

$$(1_A, -1_A, 1_A)\colon (P,*) \to (Q,\circ)\,. \tag{3.5}$$

Now suppose that the isotopy (3.5) may be extended to a higher isotopy $(f,g,h;m)$. Then the equations (3.3) become

$$x - y = x + y^m \quad \text{and} \quad -(y-z) = y^m + (-z)\,,$$

reducing to

$$y^m = -y \quad \text{and} \quad y^m = 2z - y$$

respectively. In other words, the isotopy (3.5) extends to a higher isotopy if and only the abelian group $(A,+,0)$ has exponent 2.

On the other hand, each homomorphism may be extended to a higher homotopy.

Lemma 3.4. *Let $h\colon (P,*) \to (Q,\circ)$ be a quasigroup homomorphism. Then the homotopy (h,h,h) extends to the higher homotopy $(h,h,h;h)$.*

Higher homotopies of the type $(h,h,h;h)$ appearing in Lemma 3.4 are described as *homomorphic*.

A complete formal determination of which homotopies extend to higher homotopies is provided by the following analogue of Theorem 2.4.

Theorem 3.5. *Suppose that* $(f,g,h)\colon (P,*) \to (Q,\circ,//,\backslash\backslash)$ *is a quasigroup homotopy.*

(a) *The homotopy* (f,g,h) *extends to a higher homotopy* $(f,g,h;m)$ *if and only if*

$$\forall\ x,y,z \in P\,,\ x^f\backslash\backslash(x*y)^f = (y*z)^g//z^g\,. \tag{3.6}$$

(b) *If* (3.6) *holds, then the middle map of the extension is unique, given by*

$$m\colon P \to Q; y \mapsto x^f\backslash\backslash(x*y)^f$$

for an arbitrary element x *of* P*.*

(c) *If* (3.6) *holds, then the middle map of the extension is given by*

$$m\colon P \to Q; y \mapsto (y*z)^g//z^g$$

for an arbitrary element z *of* P*.*

Proof. If $(f,g,h;m)$ is a higher homotopy, then the equations (3.3) imply

$$y^m = x^f\backslash\backslash(x*y)^f = (y*z)^g//z^g$$

for all x,y,z in P, so that (3.6) holds. Conversely, suppose that (3.6) holds. If P is empty, the claims of the theorem are trivial (or vacuous) by Lemma 3.4. Otherwise, choose an element x_0 of P, and consider the function

$$m\colon P \to Q; y \mapsto x_0^f\backslash\backslash(x_0*y)^f\,.$$

By (3.6), one has

$$y^m = x_0^f\backslash\backslash(x_0*y)^f = (y*z)^g//z^g \tag{3.7}$$

for all y,z in P, so the second equation of (3.3) holds for all y,z in P. In particular, one has

$$y^m = x_0^f\backslash\backslash(x_0*y)^f = (y*x_0)^g//x_0^g\,.$$

By (3.6), one then has

$$y^m = (y*x_0)^g//x_0^g = x^f\backslash\backslash(x*y)^f \tag{3.8}$$

for all x,y in P, so the first equation of (3.3) holds for all x,y in P. This completes the proof of (a). Claim (c) of the theorem then follows from (3.7), while claim (b) follows from (3.8). □

4. Principal higher isotopies

Consider the following consequence of (3.3) in Definition 3.1.

Lemma 4.1. *If the homotopic part of a higher homotopy is an isotopy, then the middle map bijects.*

In the context of Lemma 4.1, one may note the following.

Proposition 4.2. *Suppose that* $(f, g, h; m)\colon (P, *) \to (Q, \circ)$ *is a higher isotopy.*

(a) *There are isotopies* $\big(m, m, fh^{-1}g\big)$ *and* $\big(m, m, gh^{-1}f\big)$ *from* $(P, *)$ *to* $(Q, \circ)$.
(b) *In particular,* $fh^{-1}g = gh^{-1}f$.
(c) *If* $(P, *)$ *is commutative, then so is* $(Q, \circ)$. *In other words, commutativity is preserved by higher isotopy.*

Proof. (a) One obtains

$$\big(m, m, fh^{-1}g\big) = (f, m, f)(f^{-1}, g^{-1}, h^{-1})(m, g, g) \tag{4.1}$$

and

$$\big(m, m, gh^{-1}f\big) = (m, g, g)(f^{-1}, g^{-1}, h^{-1})(f, m, f) \tag{4.2}$$

as composites of three isotopies.

(b) Comparison of the isotopies (4.1) and (4.2) using Lemma 2.2 yields the given equation.

(c) Consider elements x' and y' of Q, with corresponding elements x and y of P such that $x^m = x'$ and $y^m = y'$. Then the isotopy (4.1) yields

$$y' \circ x' = x^m \circ y^m = (x * y)fh^{-1}g = (y * x)fh^{-1}g = y^m \circ x^m = y' \circ x'$$

as required. □

Corollary 4.3. *If* $m\colon (P, *) \to (Q, \circ)$ *is an isomorphism within the situation of Proposition 4.2, then* $m = fh^{-1}g$.

Proof. Since m is an isomorphism, (m, m, m) is an isotopy. The result then follows by Lemma 2.2, on comparing the homotopy (m, m, m) with the homotopy (4.1). □

Definition 4.4. A higher isotopy

$$(f, g, h; m)\colon (P, *) \to (P, \circ) \tag{4.3}$$

between quasigroup structures $(P, *)$ and $(P, \circ)$ on the same set P is said to be *principal* if $h = fg = gf$.

Theorem 4.5. *Suppose that* $(f,g,h;m)\colon (P,*) \to (Q,\circ)$ *is a higher isotopy. Then there is a quasigroup structure* $(P,\circ)$ *such that* $(f,g,h;m)$ *factorizes as the product of a principal higher isotopy*

$$(a,b,c;d)\colon (P,*) \to (P,\circ) \tag{4.4}$$

and a homomorphic higher isotopy $(\theta,\theta,\theta;\theta)\colon (P,\circ) \to (Q,\circ)$.

Proof. Consider the bijection $\theta = fh^{-1}g = gh^{-1}f$ from P to Q given by Proposition 4.2. Define the quasigroup structure $(P,\circ)$ on P by declaring $\theta\colon (P,\circ) \to (Q,\circ)$ to be an isomorphism. The existence of the isotopies

$$(f,g,h)(\theta^{-1},\theta^{-1},\theta^{-1}) = (hg^{-1}, hf^{-1}, hf^{-1}\cdot hg^{-1})\,,$$
$$(f,m,f)(\theta^{-1},\theta^{-1},\theta^{-1}) = (hg^{-1}, mf^{-1}\cdot hg^{-1}, hg^{-1})\,, \text{ and}$$
$$(m,g,g)(\theta^{-1},\theta^{-1},\theta^{-1}) = (mg^{-1}\cdot hf^{-1}, hf^{-1}, hf^{-1})$$

from $(P,*)$ to $(P,\circ)$ then yields a higher isotopy (4.4) with $a = hg^{-1}$, $b = hf^{-1}$, $c = ba = ab$, and $d = m\theta^{-1}$. Thus this higher isotopy is principal, and $(f,g,h;m) = (a,b,c;d)(\theta,\theta,\theta;\theta)$. □

An application of Corollary 4.3 yields the following.

Corollary 4.6. *Suppose that* $m\colon (P,*) \to (Q,\circ)$ *is an isomorphism within the situation of Theorem 4.5. Then* $m = \theta$, *and the principal higher isotopy* (4.4) *reduces to* $(a,b,c;1_P)$.

Example 4.7. Consider the quasigroup structures on the set $P = Q = \{1,2,3,4,5,6\}$ with the following multiplication tables:

$(P,*)$	1	2	3	4	5	6
1	1	3	2	4	5	6
2	3	1	4	2	6	5
3	2	4	5	6	3	1
4	4	2	6	5	1	3
5	5	6	3	1	2	4
6	6	5	1	3	4	2

and

$(Q,\circ)$	1	2	3	4	5	6
1	4	2	3	1	5	6
2	2	4	1	3	6	5
3	3	1	5	6	2	4
4	1	3	6	5	4	2
5	5	6	2	4	3	1
6	6	5	4	2	1	3

.

Since $(P,*)$ has an idempotent element (namely 1), while $(Q,\circ)$ does not, the two quasigroups are not isomorphic. However,

$$(f,g,h;m)\colon (P,*) \to (Q,\circ)\,,$$

with $f = g = (1\ 2)(3\ 4)(5\ 6)$, $h = (1\ 4)(2\ 3)$, and $m = (1)$, is a higher isotopy. The factorization of $(f,g,h;m)$ given by Theorem 4.5 contains the principal higher isotopy

$$\big((1\ 3)(2\ 4)(5\ 6), (1\ 3)(2\ 4)(5\ 6), (1); (1\ 4)(2\ 3)\big)$$

and the homomorphism $(1\ 4)(2\ 3)\colon (P,\circ) \to (Q,\circ)$.

This section concludes with two examples that contrast principal isotopies with principal higher isotopies. The first example exhibits a principal higher isotopy whose isotopic part is not principal.

Example 4.8. Consider the quasigroup $(\mathbb{Z}/_3,+)$ of integers modulo 3 under addition. Consider the higher isotopy

$$(f,g,h;m)\colon (\mathbb{Z}/_3,+) \to (\mathbb{Z}/_3,+) \tag{4.5}$$

with $f = g = R_+(1)$, $h = R_+(2)$, and $m = 1_{\mathbb{Z}/_3}$. Since $h = fg$, the higher isotopy (4.5) is principal. On the other hand, the isotopy (f,g,h) is not principal.

The second example exhibits a principal isotopy which extends to a higher isotopy that is not principal.

Example 4.9. Consider the quasigroups $(\mathbb{Z}/_3,+)$ and $(\mathbb{Z}/_3,x+y+1)$. Consider the higher isotopy

$$(f,g,h;m)\colon (\mathbb{Z}/_3,+) \to (\mathbb{Z}/_3,x+y+1) \tag{4.6}$$

with $f = g = R_+(1)$ and $h = m = 1_{\mathbb{Z}/_3}$. Note that the isotopy (f,g,h) is principal. On the other hand, since $fg \neq h$, the higher isotopy (4.6) is not principal.

5. Homotopy/higher homotopy duality

Consider an ordered pair (f,g) of maps from a quasigroup $(P,*,/,\backslash)$ to a quasigroup $(Q,\circ,//,\backslash\backslash)$. The condition (2.3) of Theorem 2.4 for extension to a homotopy is

$$\forall\ x,y,z \in P\,,\ x^f \circ (x\backslash y)^g = (y/z)^f \circ z^g\,.$$

On the other hand, the condition (3.6) of Theorem 3.5 for extension to a higher homotopy is

$$\forall\ x,y,z \in P\,,\ x^f\backslash\backslash(x*y)^f = (y*z)^g//z^g\,.$$

The juxtaposition of these two extension conditions reveals a certain duality between homotopies and higher homotopies. For example, while the homotopy extension condition (2.3) involves division on the domain and multiplication on the codomain of the maps f,g, the higher homotopy extension condition (3.6) involves division on their codomain and multiplication on their domain. This section (which is not essential for the subsequent content of the paper) serves to suggest the origin of the duality.

The main task is to provide an alternative, diagrammatic description of homotopies and higher homotopies from $(P, *, /, \backslash)$ to $(Q, \circ, //, \backslash\backslash)$. The multiplication on P is written as

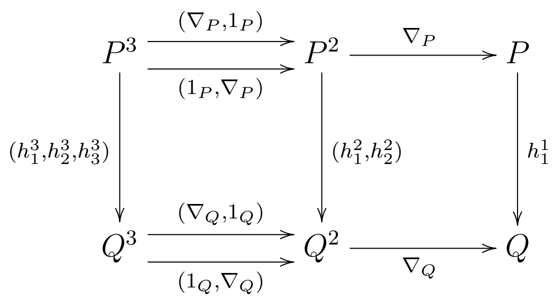
$$\nabla_P \colon P^2 \to P; (x, y) \mapsto x * y \tag{5.1}$$

with the cartesian square P^2 as the domain. The multiplication on Q is written as

$$\nabla_Q \colon Q^2 \to Q; (x, y) \mapsto x \circ y \tag{5.2}$$

in similar fashion. Now consider the diagram

(5.3)



written with similar notation. For example, we have

$$(\nabla_P, 1_P) \colon P^3 \to P^2; (x, y, z) \mapsto (x * y, z)$$

with $1_P \colon P \to P; z \mapsto z$ as the identity map on P. The diagram (5.3) is said to *commute* if the equations

$$(\nabla_P, 1_P)(h_1^2, h_2^2) = (h_1^3, h_2^3, h_3^3)(\nabla_Q, 1_Q)\,, \tag{5.4}$$

$$(1_P, \nabla_P)(h_1^2, h_2^2) = (h_1^3, h_2^3, h_3^3)(1_Q, \nabla_Q)\,, \tag{5.5}$$

$$\nabla_P h_1^1 = (h_1^2, h_2^2)\nabla_Q \tag{5.6}$$

hold. Here, the condition (5.4) is described as the *commuting of the top square on the left hand side* of (5.3). The condition (5.5) is described as the *commuting of the bottom square on the left hand side* of (5.3). Finally, the condition (5.6) is described as the *commuting of the square on the right hand side* of (5.3).

Remark 5.1. Note that other commutation relationships are possible within the diagram (5.3), beyond (5.4)–(5.6). For example,

$$(\nabla_Q, 1_Q)\nabla_Q = (1_Q, \nabla_Q)\nabla_Q$$

corresponds to associativity for the multiplication (5.2), a situation which will be considered in the following section.

Theorem 5.2. *Consider quasigroups P and Q, equipped with respective multiplications* (5.1) *and* (5.2).

(a) *The triple* $(h^2_1, h^2_2, h^1_1)\colon P \to Q$ *is a homotopy if and only if the square on the right hand side of* (5.3) *commutes.*

(b) *Suppose that* $(h^2_1, h^2_2, h^1_1)\colon P \to Q$ *is a homotopy. Consider a map* $h^3_2\colon P \to Q$*. Then* $(h^2_1, h^2_2, h^1_1; h^3_2)\colon P \to Q$ *is a higher homotopy if and only if there are maps* $h^3_1, h^3_3\colon P \to Q$ *such that the two squares on the left hand side of* (5.3) *commute.*

(c) *Consider a quadruple* $(h^2_1, h^2_2, h^1_1; h^3_2)\colon P \to Q$ *of maps from* P *to* Q*. Then* $(h^2_1, h^2_2, h^1_1; h^3_2)$ *is a higher homotopy if and only if there are maps* $h^3_1, h^3_3\colon P \to Q$ *such that the diagram* (5.3) *commutes.*

Proof. (a): The commuting (5.6) of the right hand side of (5.3) is equivalent to the condition $\forall\ x, y \in P\,,\ (x * y)^{h^1_1} = x^{h^2_1} \circ y^{h^2_2}$, which is just condition (2.2) for (h^2_1, h^2_2, h^1_1) to be a homotopy.

(b): First, suppose that there are maps $h^3_1, h^3_3\colon P \to Q$ such that the two squares on the left hand side of (5.3) commute. Let (x, y, z) be an element of P^3. The commuting of the top square means that

$$\big((x * y)^{h^2_1}, z^{h^2_2}\big) = (x^{h^3_1} \circ y^{h^3_2}, z^{h^3_3})\,,$$

so that $h^3_3 = h^2_2$ and

$$(x * y)^{h^2_1} = x^{h^3_1} \circ y^{h^3_2}\,. \tag{5.7}$$

The commuting of the bottom square means that

$$\big(x^{h^2_1}, (y * z)^{h^2_2}\big) = (x^{h^3_1}, y^{h^3_2} \circ z^{h^3_3})\,,$$

so that $h^3_1 = h^2_1$ and

$$(y * z)^{h^2_2} = y^{h^3_2} \circ z^{h^3_3}\,. \tag{5.8}$$

Substituting $h^3_1 = h^2_1$ back into (5.7) yields

$$(x * y)^{h^2_1} = x^{h^2_1} \circ y^{h^3_2}\,. \tag{5.9}$$

Substituting $h^3_3 = h^2_2$ back into (5.8) yields

$$(y * z)^{h^2_2} = y^{h^3_2} \circ z^{h^2_2}\,. \tag{5.10}$$

Together, (5.9) and (5.10) yield the equations (3.3) which are needed to conclude that $(h^2_1, h^2_2, h^1_1; h^3_2)\colon P \to Q$ is a higher homotopy.

Conversely, let $(h^2_1, h^2_2, h^1_1; h^3_2)\colon P \to Q$ be a higher homotopy. Define $h^3_1 = h^2_1$ and $h^3_3 = h^2_2$. Then the equations (3.3) may be rewritten as

$$(x * y)^{h^2_1} = x^{h^3_1} \circ y^{h^3_2}\,, \qquad (y * z)^{h^2_2} = y^{h^3_2} \circ z^{h^3_3}\,.$$

Thus

$$\big((x * y)^{h^2_1}, z^{h^2_2}\big) = (x^{h^3_1} \circ y^{h^3_2}, z^{h^3_3}) \text{ and } \big(x^{h^2_1}, (y * z)^{h^2_2}\big) = (x^{h^3_1}, y^{h^3_2} \circ z^{h^3_3}),$$

meaning that the two squares on the left hand side of (5.3) commute.

(c): Follows from (a) and (b). □

Remark 5.3. In Theorem 5.2(b),(c), the middle map of the higher homotopy is h^3_2. This is the reason for the terminology that was used in Definition 3.1(a): Looking at the left hand side of the diagram (5.3), the map h^3_2 is the middle entry of the triple (h^3_1, h^3_2, h^3_3).

Now, based on the diagram (5.3), the homotopy/higher homotopy duality may be explained. By Theorem 5.2(a), the homotopy condition corresponds to the commuting of the right hand square, involving maps with P^2 and Q^2 as domains. On the other hand, by Theorem 5.2(b), the higher homotopy conditions correspond to the commuting of the left hand squares, involving maps with P^2 and Q^2 as codomains.

6. Unit elements in domains

In this section, we consider higher homotopies whose domain is a loop $(P, *)$ with identity element e, or a quasigroup with a left or right unit element. The following result is an analogue of Lemma 2.3.

Lemma 6.1. *Consider a higher homotopy* (3.2)*.*

(a) *If the domain of the higher homotopy is a quasigroup* $(P, *, e)$ *with left unit element* e*, then* $f = mL_\circ(e^f)$*.*
(b) *If the domain of the higher homotopy is a quasigroup* $(P, *, e)$ *with right unit element* e*, then* $g = mR_\circ(e^g)$*.*

Proof. (a): Substitution into (3.3) yields $y^f = (e * y)^f = e^f \circ y^m = ymL_\circ(e^f)$ for y in P.

(b): Substitution into (3.3) yields $y^g = (y * e)^g = y^m \circ e^g = ymR_\circ(e^g)$ for y in P. □

Proposition 6.2. *Consider a higher homotopy* (3.2) *from a quasigroup* $(P, *)$*, with a left or right unit element* e*, to a group* $(Q, \circ)$*. Then the middle map* $m\colon (P, *) \to (Q, \circ)$ *is a homomorphism.*

Proof. Suppose that e is a left unit element. Then for x, y in P, one has

$$e^f \circ (x * y)^m = (x * y)^f = x^f \circ y^m = (e^f \circ x^m) \circ y^m = e^f \circ (x^m \circ y^m) \tag{6.1}$$

by successive application of Lemma 6.1(a), the first equation in (3.3), Lemma 6.1(a), and the associative law in the group $(Q, \circ)$. Cancelation of the group element e^f from (6.1) then yields the desired result. The case where e is a right unit is similar, making use of Lemma 6.1(b). □

Corollary 6.3. *Consider a higher isotopy* (3.2) *from a loop* $(P, *, e)$ *to a group* $(Q, \circ)$*. Then* $m\colon (P, *) \to (Q, \circ)$ *is an isomorphism.*

If there is an isotopy from a loop to a group, there is an isomorphism between them [2, Th. 2], [13, Th. 11.39]. Corollary 6.3 shows that if there is a higher isotopy $(f, g, h; m)$ from a loop to a group, then the middle map m of the higher isotopy is one such isomorphism from the loop to the group. Then Corollary 4.3 shows that $fh^{-1}g$ is that isomorphism. Indeed, the following result shows that the hypothesis of Corollary 6.3, imposing the existence of the higher isotopy *a priori*, is not needed.

Theorem 6.4. *A principal isotopy*

$$(f, g, 1_P)\colon (P, *, e) \to (P, \cdot, 1) \tag{6.2}$$

from a loop $(P, *, e)$ *to a group* $(P, \cdot, 1)$ *extends to a higher isotopy* $(f, g, 1_P; m)$*, with middle map*

$$m\colon (P, *, e) \to (P, \cdot, 1); x \mapsto (e^f)^{-1}x(e^g)^{-1}$$

as an isomorphism of groups.

Proof. Lemma 2.3 yields $x^f = x(e^g)^{-1}$ and $y^g = (e^f)^{-1}y$ for x, y in Q. Now $x(e^g)^{-1}(e^f)^{-1}y = x^f y^g = x * y$, so

$$x^f y^m = x(e^g)^{-1}(e^f)^{-1}y(e^g)^{-1} = (x * y)(e^g)^{-1} = (x * y)^f\,.$$

Similarly,

$$x^m y^g = (e^f)^{-1}x(e^g)^{-1}(e^f)^{-1}y = (e^f)^{-1}(x * y) = (x * y)^g\,.$$

Thus (f, m, f) and (m, g, g) are isotopies: The isotopy $(f, g, 1_P)$ extends to a higher isotopy $(f, g, 1_P; m)$. Corollary 6.3 confirms that m is an isomorphism. □

Corollary 6.5. *Consider the context of Theorem 6.4.*

(a) *The middle map is* $m = fg = gf$*.*
(b) *The higher isotopy* $(f, g, 1_P; m)$ *is principal iff* $m = 1_P$*.*

Proof. (a) For an element x of P, one has

$$x^{fg} = [x(e^g)^{-1}]^g = (e^f)^{-1}x(e^g)^{-1} = x^m\,.$$

Then $fg = gf$ follows by associativity in the group $(P, \cdot, 1)$, or on more general grounds by Proposition 4.2(b).

(b) By Definition 4.4, $(f, g, 1_P; m)$ is principal if and only if $1_P = fg$. But $m = fg$ by (a). □

7. Counting higher homotopies

The main result of this section, Theorem 7.5 below, will show that for a given domain quasigroup $(P,*)$, the proportion of isotopies extending to higher isotopies tends to zero as the order of P tends to infinity. In preparation for the proof of the theorem, it is necessary to examine the quadratic structure of quasigroups. For previous discussions of this structure, see [3, 4, 7, 8, 12].

Definition 7.1. Let $(P,\cdot)$ be a quasigroup.

(a) An element s of P is a *square* in $(P,\cdot)$ if $s = r\cdot r$ for some element r of P.
(b) The set of squares in $(P,\cdot)$ is written as $\mathrm{Sq}(P,\cdot)$.
(c) For a square s in $(P,\cdot)$, an element r of P is a *square root* of s in $(P,\cdot)$ if $s = r\cdot r$.
(d) If P has finite positive order n, then the *quadratic partition* of $(P,\cdot)$ is the integer partition

$$\sum_{s\in\mathrm{Sq}(P,\cdot)} \left|\{r\in P \mid s = r\cdot r\}\right|$$

of n.

Note that the squares of a quasigroup $(P,\cdot)$ are precisely the elements appearing down the main diagonal of the body of the multiplication table of $(P,\cdot)$.

Lemma 7.2. *Let e be an element of a quasigroup $(P,*,/,\backslash)$.*

(a) *Each element of P is a square root of e in $(P,/)$ if and only if e is a left unit for $(P,*)$.*
(b) *Each element of P is a square root of e in $(P,\backslash)$ if and only if e is a right unit for $(P,*)$.*

Proof. For (a), note that $e*y = y$ for an element y of P if and only if $e = y/y$. The proof of (b) is dual. □

Proposition 7.3. *Suppose that*

$$(f,g,h;m)\colon (P,*,/,\backslash)\to(P,\circ,//,\backslash\backslash)$$

is a higher isotopy.

(a) *The permutation m of P takes squares in $(P,/)$ to squares in $(P,//)$.*
(b) *Let s be an element of P. Then the permutation g of P takes square roots of s in $(P,/)$ to square roots of s^m in $(P,//)$.*
(c) *If P has finite positive order, then the quadratic partition of $(P,/)$ coincides with the quadratic partition of $(P,//)$.*

(d) *The permutation m of P takes squares in $(P, \backslash)$ to squares in $(P, \backslash\backslash)$.*
(e) *Let s be an element of P. Then the permutation f of P takes square roots of s in $(P, \backslash)$ to square roots of s^m in $(P, \backslash\backslash)$.*
(f) *If P has finite positive order, then the quadratic partition of $(P, \backslash)$ coincides with the quadratic partition of $(P, \backslash\backslash)$.*

Proof. For (a), (b), and (c), suppose that the equation $s = r/r$ holds for elements s and r of P, so that $s * r = r$. Application of the isotopy (m, g, g) then shows that $r^g = (s * r)^g = s^m \circ r^g$, so that $s^m = r^g // r^g$. The proof of (d), (e), and (f) is dual, involving the isotopy (f, m, f). $\square$

Corollary 7.4. *Suppose that the domain of a higher isotopy has a left (or, respectively, right) unit. Then so does its codomain. In particular, higher isotopes of loops are loops.*

Proof. Apply Lemma 7.2 and Proposition 7.3. $\square$

Theorem 7.5. *For each positive integer n, suppose that $(P, *, /, \backslash)$ is a quasigroup of order n. Let I_n be the number of isotopies from $(P, *)$ to any quasigroup structure on P. Let H_n be the number of higher isotopies from $(P, *)$ to any quasigroup structure on P. Then* $\lim_{n\to\infty} H_n/I_n = 0$.

Proof. The number I_n counts isotopies of the form

$$(f, g, h)\colon (P, *, /, \backslash) \to (P, \circ, //, \backslash\backslash)\,, \tag{7.1}$$

while the number H_n counts higher isotopies of the form

$$(f, g, h; m)\colon (P, *, /, \backslash) \to (P, \circ, //, \backslash\backslash)\,. \tag{7.2}$$

In (7.1), the triple (f, g, h) constituting the isotopy may be chosen arbitrarily from the direct cube $P! \times P! \times P!$ of the permutation group $P!$ on P, the codomain quasigroup structure being given by

$$x \circ y = \left(xf^{-1} * yg^{-1}\right)^h$$

for x, y in P. Thus $I_n = (n!)^3$. Determination of H_n breaks into two cases.

Case I: There is a left unit e in $(P, *)$.

Here, consider any one of the $n \cdot (n!)^2$ possible arbitrary choices of an element (e^f, g, m) of $P \times P! \times P!$. Now for a higher isotopy (7.2), the triple (m, g, g) is an isotopy. Thus

$$x \circ y = \left(xm^{-1} * yg^{-1}\right)^g$$

for all x, y in P, and the codomain quasigroup structure $(P, \circ, //, \backslash\backslash)$ of (7.2) is determined. Then h is determined as $gL_\circ(e^f)$ by Lemma 2.3(a), while f is determined as $mL_\circ(e^f)$ by Lemma 6.1(a). In summary, $H_n = n \cdot (n!)^2$, and $H_n/I_n = 1/(n-1)! \to 0$ for $n \to \infty$.

Case II: There is no left unit in $(P, *)$.

Here, Lemma 7.2(a) shows that the quadratic partition of $(P, /)$ has more than one summand. Let k be the largest such summand, so that $0 < k < n$. Consider an element s of P with $k = \big|\{r \in P \mid s = r/r\}\big|$. There are at most $(n!)^2$ choices for f and m, thence determining the codomain quasigroup structure $(P, \circ, //, \backslash\backslash)$ of (7.2) by

$$x \circ y = \left(xf^{-1} * ym^{-1}\right)^f$$

for all x, y in P. By Proposition 7.3(b), the permutation g of P is constrained to take each of the k square roots of s in $(P, /)$ to one of the k square roots of s in $(P, //)$. Thus there are at most $k! \cdot (n-k)!$ choices for g. With the respective choices for f, m and g made, the remaining component h of (7.2) is fixed (by Theorem 2.4(b), for example). Thus $H_n \leq k! \cdot (n-k)! \cdot (n!)^2 \leq (n-1)! \cdot (n!)^2$, and $H_n/I_n = \leq 1/n \to 0$ for $n \to \infty$. □

8. Moufang elements

A loop $(Q, \cdot, /, \backslash, e)$ with identity element e is described as having the *inverse property* if there is an *inversion* function $J \colon Q \to Q; x \mapsto x^{-1}$ such that

$$x/y = xy^{-1} \quad \text{and} \quad y\backslash x = y^{-1}x$$

for all x, y in Q [4, §VII.1]. The following definition was originally given in the context of inverse property loops [4, §VII.2].

Definition 8.1. An element u of a quasigroup $(Q, \cdot, /, \backslash)$ is a *Moufang element* if

$$(u \cdot x) \cdot (y \cdot u) = \big(u \cdot (xy)\big) \cdot u \tag{8.1}$$

for all x, y in Q.

The condition (8.1) may be rephrased as the statement that

$$\big(L(u), R(u), L(u)R(u)\big) \colon (Q, \cdot, /, \backslash) \to (Q, \cdot, /, \backslash) \tag{8.2}$$

is an isotopy. We determine when (8.2) extends to a higher isotopy. For the first two parts of the following definition, see [6, §2.2].

Definition 8.2. Let $(Q, \cdot)$ be a quasigroup.

(a) An element u of Q is *left associative* if $(u \cdot x) \cdot y = u \cdot (x \cdot y)$ for all x, y in Q.
(b) An element u of Q is *right associative* if $(x \cdot y) \cdot u = x \cdot (y \cdot u)$ for all x, y in Q.

(c) An element u of Q is *laterally associative* if it is both left and right associative.

It is apparent from (8.1) that laterally associative elements of a quasigroup are Moufang elements. Note that if u is a Moufang element of a loop with identity element e, setting $x = e$ in (8.1) shows that $L(u)R(u) = R(u)L(u)$. For further discussion of Moufang elements in loops, see [5, 9].

Theorem 8.3. *Let u be a Moufang element of a quasigroup $(Q, \cdot, /, \backslash)$.*

(a) *If the isotopy* (8.2) *extends to a higher isotopy, then $L(u)R(u) = R(u)L(u)$.*

(b) *The isotopy* (8.2) *extends to a higher isotopy if and only if u is laterally associative in Q.*

Proof. (a) Under the given condition, Proposition 4.2 shows that

$$L(u)R(u)^{-1}L(u)^{-1}R(u) = R(u)R(u)^{-1}L(u)^{-1}L(u) = 1\,,$$

so that $R(u)^{-1}L(u)^{-1} = L(u)^{-1}R(u)^{-1}$ and thus $L(u)R(u) = R(u)L(u)$.

(b) When applied to the isotopy (8.2), the extension condition (3.6) of Theorem 3.5 takes the form

$$(u \cdot x)\backslash\big(u \cdot (x \cdot y)\big) = \big((y \cdot z) \cdot u\big)/(z \cdot u)\,. \tag{8.3}$$

If u is laterally associative, then each side of (8.3) reduces to y. Now suppose that (8.3) holds for all x, y, z in Q. Setting $z = e$ in (8.3) yields $(u \cdot x)\backslash\big(u \cdot (x \cdot y)\big) = (y \cdot u)/u = y$. Left multiplication of both sides by $u \cdot x$ then shows that u is left associative. Setting $x = e$ in (8.3) yields $\big((y \cdot z) \cdot u\big)/(z \cdot u) = u\backslash(u \cdot y) = y$. Right multiplication of both sides by $z \cdot u$ then shows that u is right associative. ☐

Corollary 8.4. *Suppose that u is a laterally associative element of the quasigroup $(Q, \cdot, /, \backslash)$. Then the middle map of the higher isotopy extending the isotopy* (8.2) *is the identity 1_Q on Q.*

Proof. The image of an element y of Q under the middle map is given as

$$\Big(\big(y \cdot (u/u)\big) \cdot u\Big)\Big/\big((u/u) \cdot u\big) = \Big(y \cdot \big((u/u) \cdot u\big)\Big)\Big/u = (y \cdot u)/u = y$$

by Theorem 3.5(c). ☐

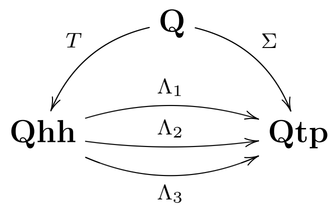
9. The category of higher homotopies

The class of quasigroups may be taken as the class of objects of three categories:

- The category $\mathbf{Q}$ of homomorphisms between quasigroups;

- The category **Qtp** of homotopies between quasigroups;
- The category **Qhh** of higher homotopies between quasigroups.

For the first two, see [10, §1.2]. There is a commutative diagram



of functors which preserve objects, and have the following morphism parts:

- $\Sigma\colon \mathbf{Q}_1 \to \mathbf{Qtp}_1; f \mapsto (f, f, f)$ [10, (1.4)];
- $T\colon \mathbf{Q}_1 \to \mathbf{Qhh}_1; f \mapsto (f, f, f; f)$ — compare Lemma 3.4;
- $\Lambda_1\colon \mathbf{Qhh}_1 \to \mathbf{Qtp}_1; (f, g, h; m) \mapsto (f, m, f)$;
- $\Lambda_2\colon \mathbf{Qhh}_1 \to \mathbf{Qtp}_1; (f, g, h; m) \mapsto (m, g, g)$;
- $\Lambda_3\colon \mathbf{Qhh}_1 \to \mathbf{Qtp}_1; (f, g, h; m) \mapsto (f, g, h)$.

The functor T is described as the *trivial functor*, while the functors $\Lambda_1, \Lambda_2, \Lambda_3$, arising from (3.4), are called the *lowering functors*. The first two are just recorded here for reference, while the third plays an important role in what follows.

Proposition 9.1. *The lowering functor Λ_3 is faithful. In other words, for each pair P, Q of quasigroups, the restriction*

$$\Lambda_3\colon \mathbf{Qhh}(P, Q) \to \mathbf{Qtp}(P, Q)$$

of the morphism part of Λ_3 is injective.

Proof. Suppose that $(f, g, h; m)$ and $(f, g, h; m')$ are higher homotopies from P to Q. Thus the homotopy (f, g, h) from P to Q satisfies the condition (3.6) of Theorem 3.5(a). By the remaining parts of that theorem, it then follows that $m = m'$. □

Note that the higher isotopies are the invertible morphisms in the category **Qhh**, just as the isotopies are the invertible morphisms in the category **Qtp**.

Definition 9.2. Let Q be a quasigroup.

(a) The *autotopy group* of Q is the group $\mathbf{Qtp}(Q, Q)^*$ of units of the monoid $\mathbf{Qtp}(Q, Q)$ of homotopies from Q to Q.
(b) The *higher autotopy group* of Q is the group $\mathbf{Qhh}(Q, Q)^*$ of units of the monoid $\mathbf{Qhh}(Q, Q)$ of higher homotopies from Q to Q.

Corollary 9.3. *Let Q be a quasigroup. Then the lowering functor Λ_3 embeds the higher autotopy group of Q as a subgroup of the autotopy group of Q.*

Recall that a *terminal object* of a category $\mathbf{C}$ is an object $\top$ such that $\mathbf{C}(X, \top)$ is a singleton for each object X of $\mathbf{C}$. A $\mathbf{C}$-*point* of an object X of $\mathbf{C}$ is an element of $\mathbf{C}(\top, X)$.

A singleton quasigroup $\top = \{e\}$ forms a terminal object for each of the quasigroup categories featured in this section. If $(Q, *)$ is a quasigroup, then each $\mathbf{Qtp}$-point $(f, g, h) \colon \top \to Q$ of Q corresponds to a triple

$$(u, v, u * v) = (e^f, e^g, e^h) \tag{9.1}$$

of elements of Q. The geometry formed by these points is the web or net of the quasigroup Q [11, Th. 1]. Note that the set of points of the empty quasigroup is empty in each of the three quasigroup categories under consideration.

Theorem 9.4. *Let $(Q, *, /, \backslash)$ be a non-empty quasigroup.*

(a) *The $\mathbf{Qtp}$-point (9.1) of the quasigroup Q is the image under the lowering functor Λ_3 of a $\mathbf{Qhh}$-point of Q if and only if $u\backslash u = v/v$.*

(b) *The quasigroup Q is a loop if and only if each $\mathbf{Qtp}$-point of the quasigroup Q is the image under the lowering functor Λ_3 of a $\mathbf{Qhh}$-point of Q.*

Proof. (a) By Theorem 3.5(a), the homotopy $(f, g, h) \colon \top \to Q$ extends to a higher homotopy if and only if the condition (3.6) is satisfied. Here, the condition reads as $e^f\backslash e^f = e^g/e^g$, i.e., as $u\backslash u = v/v$.

(b) By (a), each $\mathbf{Qtp}$-point is (the image under Λ_3 of) a $\mathbf{Qhh}$-point if and only if $u\backslash u = v/v$ for all elements u, v of Q, i.e., if and only if the identity [10, (1.13)] holds. The validity of this identity in a non-empty quasigroup Q is equivalent to Q forming a loop. □

References

[1] A.A. Albert, Non-associative algebras I. Fundamental concepts and isotopy. *Ann. of Math.* **43** (1942), 685–707.

[2] A.A. Albert, Quasigroups I. Fundamental concepts and isotopy. *Trans. Amer. Math. Soc.* **54** (1943), 507–519.

[3] V.D. Belousov. The core of a Bol loop. (Russian.) *Studies in General Algebra (Sem.) (Russian) Akad. Nauk Moldav. SSR, Kishinev* (1965), 53–66.

[4] R.H. Bruck, *A Survey of Binary Systems*, Springer, Berlin, 1958.

[5] I.A. Florja, Loops with one-sided invertibility (Russian). Bul. Akad. Štiince RSS Moldoven **1965** (1965), 68–79.

[6] G.N. Garrison, Quasi-groups. *Ann. of Math.* **41** (1940), 474–487.
[7] G. Glauberman. On loops of odd order. *J. Algebra* **1** (1964), 374–396.
[8] G. Glauberman. On loops of odd order. II. *J. Algebra* **8** (1968), 393–414.
[9] J.D. Phillips, The Moufang laws, global and local. J. Algebra Appl. **8** (2009), 477–492.
[10] J.D.H. Smith, *An Introduction to Quasigroups and Their Representations*, CRC Press, Boca Raton, FL, 2006.
[11] J.D.H. Smith, Quasigroup homotopies, semisymmetrization, and reversible automata. *Internat. J. Algebra Comput.* **18** (2008), 1203–1221.
[12] J.D.H. Smith, On 2-Engel groups and Bruck loops. *J. Group Theory* **16** (2013), 87–106.
[13] J.D.H. Smith, *Introduction to Abstract Algebra* (2nd. ed.), CRC Press, Boca Raton, FL, 2016.
[14] J.D.H. Smith and A.B. Romanowska, *Post-Modern Algebra*, Wiley, New York, NY, 1999.

Department of Mathematics, Iowa State University, Ames, Iowa, 50011, USA
Email address: [1]gnnop@iastate.edu
Email address: [2]jdhsmith@iastate.edu