FROM AFFINE TO PROJECTIVE GEOMETRY VIA CONVEXITY

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1. **INTRODUCTION** This paper sets out to study affine geometry, projective geometry, and the relationship between them. The study is undertaken from an algebraic standpoint subject to requirements of invariance and directness. "Invariance" means following Klein's Erlanger Programm for geometry [Kl,p.7] [Kn,p.463]:

> Suppose given a geometry and a group of transformations of the geometry; one should investigate features belonging to the geometry with regard to properties that are invariant under the transformations of the group.

The usual coordinatization of an affine geometry by a module over a commutative ring is not invariant under the affine group in this sense, since the zero element of the module is disturbed by affine translations. The algebraic descriptions of affine geometries to be used here are required to have affine groups as groups of automorphisms. The advantage of satisfying this requirement is that one may then identify the algebraic structure with the affine geometry, rather than having to maintain a duality between a description (the coordinatizing module) and the thing described (the affine geometry).

"Directness" is a requirement motivated by recent developments in applied mathematics [RS,p.103], [Sl,pp.388-9]. It means avoiding "secondary constructs" - features of a mathematical model introduced merely for pure mathematical convenience, and not corresponding to any phenomenon being modelled. In the present context, secondary constructs might take the form of ideal elements or points at infinity. Thus the aim of the paper is to describe affine geometry, projective geometry, and the passage between them purely algebraically, in an invariant way avoiding the setting up of auxiliary constructions.

2. **THE ALGEBRAIC FRAMEWORK** The algebraic approach to affine geometry here is that of [OS], [RS,2.5]. Let K be a field, and E a vector space over K. For each element k of K, define a binary operation

$$\underline{k}: E \times E \to E;\ (x,t) \mapsto xy\underline{k} := x(1-k) + yk \tag{2.1}$$

on E, so that (E,K) becomes an algebra with the set K of binary operations. "Algebra" is meant in the sense of universal algebra: see [Co] or [RS,Chapter 1] for explanations of universal algebraic notions. On E, define the parallelogram-completion operation

$$P:E \times E \times E \to E;\ (x,y,z) \mapsto x - y + z. \tag{2.2}$$

Then the algebra (E,K,P) with the ternary operation P and set K of binary operations has as its derived operations (those obtained from successive compositions of the basic operations P and $\underline{k}$ for k in K) precisely the affine combinations $x_1k_1 +\ldots+ x_rk_r$ (with $k_1 +\ldots+ k_r = 1$) of elements $x_1,\ldots,x_r$ of E. It follows that the algebra (E,K,P) has the affine group as its group of automorphisms, and may thus be identified as the affine geometry.

The corresponding projective geometry consists of the set $L(E)$ of linear or vector subspaces of the vector space E, ordered by inclusion. This may be described algebraically as $(L(E),+)$, where for subspaces U and V of E, $U + V$ is the sum of U and V. The incidence or inclusion structure is recovered from $(L(E),+)$ via $U \leqslant V$ iff $U + V = V$. Algebraically, $(L(E),+)$ is a semilattice - the binary operation $+$ is commutative, associative, and idempotent.

Both the algebras $(L(E),+)$ describing the projective geometries and the algebras (E,K,P) describing the affine geometries have two special properties. They are idempotent, in the sense that each singleton is a subalgebra, and they are entropic, i.e. each operation, as a mapping from a direct power of the algebra into the algebra, is actually a homomorphism. Algebras with these two properties are called modes. They are studied in detail in [RS]. Given a mode (A,Ω), as a set A with a set Ω of operations $\omega:A^{\omega\tau} \to A$ on it, one may form the set $(A,\Omega)S$ or AS of non-empty subalgebras of (A,Ω). This set AS carries an Ω-algebra structure under the complex products

$$\omega:AS^{\omega\tau} \to AS;\ (X_1,\ldots,X_{\omega\tau}) \mapsto \{x_1\ldots x_{\omega\tau}\omega \mid x_i \in X_i\},$$

and it turns out that the algebra (AS,Ω) is again a mode, preserving many of the algebraic properties of (A,Ω) [RS,146]. The key idea of the current paper is to examine the algebras $(E,K,P)S$ arising in this way from an affine space (E,K,P). The method is to obtain varieties nicely containing such $(E,K,P)S$, and to describe the structure of the $(E,K,P)S$ within the varieties - see (3.7), (4.9), (5.13). It is the structure of the $(E,K,P)S$ which yields the direct, invariant passage from the affine to the projective geometry.

The internal structure of the $(E,K,P)S$ is described using a construction known as the Płonka sum [Pł], [RS,236]. Let Ω be a non-empty domain of operations with an arity mapping $\tau:\Omega \to \{n \in \mathbb{N} \mid n > 1\}$. A semilattice $(H,+)$ may be considered as an Ω-algebra (H,Ω), a so-called Ω-semilattice, on defining $h_1\ldots h_{\omega\tau}\omega = h_1 +\ldots+ h_{\omega\tau}$ for h_i in H. The semilattice operation $+$ is then recovered as $h + k = hk\ldots k\omega$ for any ω from Ω. The semilattice $(H,+)$ may also be considered as a (small) category (H) with set H of objects, and with a unique morphism $h \to k$ precisely when $h + k = k$, i.e. $h \leqslant k$. The notion of Płonka sum depends on viewing the semilattice $(H,+)$ in this way both as a category and as an Ω-algebra. Let (Ω) denote the (concrete) category of Ω-algebras and homomorphisms between them. Let $F:(H) \to (\Omega)$ be a functor. Then the Płonka sum of the Ω-algebras (hF,Ω) (for h in H) over the semilattice $(H,+)$ by the functor F is the disjoint union $HF = \dot{\bigcup}_{h \in H} hF$ of the underlying sets hF (h in H), equipped with the Ω-algebra structure given for an h-ary operation ω in Ω and $h_1,\ldots,h_n,k = h_1 +\ldots+ h_n$ in H by

$$\omega:h_1F \times\ldots\times h_nF \to kF; \tag{2.3}$$

$$(x_1,\ldots,x_n) \mapsto x_1(h_1 \to k)F\ldots x_n(h_n \to k)F\omega.$$

The projection of the Płonka sum HF is the homomorphism $\pi_F:(HF,\Omega) \to (H,\Omega)$ with restrictions $\pi_F:hF \to \{h\}$. The subalgebras $(hF,\Omega) = (\pi_F^{-1}h,\Omega)$ of (HF,Ω) are referred to as the Płonka fibres.

The union $\{P\} \cup K$ forms a set of operations on $(E,K,P)S$, namely the set of corresponding complex products. For each of the three cases for K to be studied - namely characteristic 0, odd characteristic, and characteristic 2 - considerations of convexity suggest taking a certain subset Ω_K of $\{P\} \cup K$. The set Ω_K may be viewed as an algebraic analogue of the open unit interval in the field of rationals, and will actually be this interval in the characteristic zero case. The algebra $((E,K,P)S,\Omega_K)$ will then be shown to be a Płonka sum over the Ω_K-semilattice $(L(E),\Omega_K)$. The main result of this paper, the direct invariant passage from an affine to a projective geometry, follows:

THEOREM 2.4. Let K be a field, and let (E,K,P) be an affine space over K. Then the projective geometry $(L(E),+)$ is the largest Ω_K-semilattice quotient of the algebra $((E,K,P)S,\Omega_K)$ of affine subspaces of (E,K,P).

The three subsequent sections give the proof of the various cases of Theorem 2.4 in turn: Section 3 deals with K of characteristic 0, Section 4 with K of odd characteristic, and Section 5 with K of characteristic 2. In

the course of describing the algebra $((E,K,P)S,\Omega_K)$ for the characteristic 2 case in Section 5, attention is drawn to some general results of Płonka theory. A method of determining a basis for the identities of the regularisation of a strongly irregular variety from a basis for the irregular variety is given. This method preserves finiteness of the basis. It is also observed that the automorphism group of a free algebra on a set in a regular variety is the permutation group of the set.

3. **CHARACTERISTIC ZERO** This section considers the case that the field K is of characteristic zero. Since 2 has the inverse 1/2 in K, the parallelogram operation P of (2.2) may be written as

(3.1) $$xyzP = yxz\ \underline{1/2}\ \underline{2},$$

the latter term being $y(1-2) + (x(1-\frac{1}{2}) + z\frac{1}{2})2 = -y + x + z$. This means that the algebraic structure (E,K,P) describing an affine geometry over K may be replaced by the structure (E,K). The class $\underline{\underline{K}}$ consisting of these algebras (E,K) and the empty space (ϕ,K) forms a variety in the sense of universal algebra - the class of all algebras satisfying a given set of identities.

THEOREM 3.2 [OS,Satz 7] [RS,256]. *The class $\underline{\underline{K}}$ of affine K-spaces is the variety of modes (E,K) of type $\tau:K \to \{2\}$ satisfying the identities:*

(a) $xy\underline{0} = x$;

(b) $xy\underline{1} = y$;

(c) $xy\underline{p}\ xy\underline{q}\underline{r} = xy(pq\underline{r})$.

Given such an algebra (E,K), one may consider its reduct (E,I^o) obtained by admitting only those binary operations $\underline{k}$ as in (2.1) for which k lies in the open unit interval $I^o = \{x \in \mathbf{Q} \mid 0 < x < 1\}$ of the rationals $\mathbf{Q}$, the prime field of K. In the notation of Section 2, $I^o = \Omega_K$. The subalgebras (X,I^o) of (E,I^o) are precisely the $\mathbf{Q}$-convex subsets X of E. The class of all $\mathbf{Q}$-convex sets,i.e. the class of all such (X,I^o) for all such (E,I^o), does not itself form a variety, but the smallest variety containing the class, the variety of so-called *rational barycentric algebras*, is described as follows, where p' denotes 1 - p.

THEOREM 3.3 [RS,214]. *The variety of rational barycentric algebras is the class of algebras of type $\tau:I^o \to \{2\}$ satisfying the identities:*

(a) $x\underline{x}\underline{p} = x$

(b) $xy\underline{p} = yx\underline{p'}$

(c) $xy\underline{p}z\underline{q} = xyz(\underline{q/(p'q')'})(\underline{p'q'})'$.

(The reference [RS,214] actually treated the case of real barycentric algebras, with operations from the open unit interval in the reals, but the statement and proof for the rational case are completely analogous.) The identity (a) is just idempotence; (b) is a skew form of commutativity, while (c) is a skew form of associativity. Examples of rational barycentric algebras that are not convex sets are furnished by I^o-semilattices. For these, all the operations $\underline{p}$ with p in I^o coincide, so (b) becomes genuine commutativity and (c) becomes genuine associativity.

The choice of the set I^o of binary operations here is made for two reasons. Firstly, it leads to the readily available variety of rational barycentric algebras. Secondly, I^o avoids the elements 0 and 1 of K. The significance of this resides in the following general proposition.

PROPOSITION 3.4. *Let K be a field other than GF(2), and let k be an element of K distinct from 0 and 1. Let E be a vector space over K, and (L(E),+) the corresponding projective geometry.*

(i) *There is a homomorphism*

$$\pi:((E,K,P)S,k) \to (L(E),+);\ x + U \mapsto U.$$

(ii) *For the functor $F:L(E) \to (\{k\})$ with $UF = \pi^{-1}(U)$ and*

$$(U \to V)F;\ \pi^{-1}(U) \to \pi^{-1}(V);\ x + U \mapsto x + V,$$

the algebra $((E,K,P)S,k)$ is the Płonka sum over $(L(E),+)$ by the functor F.

Proof. For x,y in E and U,V in L(E), $(x+U)(y+V)\underline{k} = \{(x+u)(y+v)\underline{k} \mid u \in U, v \in V\} = xy\underline{k} + UV\underline{k} = xy\underline{k} + U(1-k) + Vk = xy\underline{k} + (U+V)$. This shows that π is a homomorphism. Further, $xy\underline{k} + (U+V) = (x + (U+V))(y + (U+V))\underline{k} = ((x+U)(U \to U+V)F)((y+V)(V \to U+V)F)\underline{k}$, so this latter term is equal to $(x+U)(y+V)\underline{k}$, showing that $((E,K,P)S,k)$ is a Płonka sum as claimed.

In the case that K has characteristic zero, this proposition leads to the following theorem describing the structure of the I^o-algebra of affine subspaces of an affine K-space.

THEOREM 3.5. *For an affine K-space (E,K) in $\underline{\underline{K}}$, the I^o-algebra $((E,K)S,I^o)$ of affine subspaces of (E,K) is a Płonka sum of $\mathbf{Q}$-convex sets over the projective geometry (L(E),+) by the functor $F:(L(E)) \to (I^o)$ with $UF = \{x + U \mid x \in E\}$ and $(U \to V)F:UF \to VF;\ x + U \mapsto x + V$.*

Proof. That $((E,K)S,I^o)$ is the Płonka sum over $(L(E),+)$ by the functor F follows directly from Proposition 3.4. The Płonka fibres are the I^o-algebras $\{x + U | x \in E\}$ for fixed subspaces U, with $(x+U)(y+U)p = xyp + U$. Since these are the reducts $(E/U,I^o)$ of the affine geometries $(E/U,K)$ coming from the quotient vector spaces E/U, they are $\mathbf{Q}$-convex sets.

COROLLARY 3.6. The I^o-algebra $((E,K)S,I^o)$ of affine subspaces of an affine K-space (E,K) is a rational barycentric algebra.

Proof. The Płonka fibres $(E/U,I^o)$, being $\mathbf{Q}$-convex sets, satisfy the identities (a)-(c) of Theorem 3.3. These identities are regular: in any of the identities, the same set of variables appears on each side of the identity. By a result of Płonka [Pł,Theorem I] [RS,238], it follows that the Płonka sum $((E,K)S,I^o)$ of $\mathbf{Q}$-convex sets also satisfies the identities, and is thus a rational barycentric algebra.

Theorem 3.5 and Corollary 3.6 may be summarized as follows:

(3.7) { The rational barycentric algebra of affine subspaces of affine K-space is a Płonka sum of $\mathbf{Q}$-convex sets over a projective geometry.

By Theorem 3.5, the projective geometry $(L(E),+)$ is an I^o-semilattice quotient of $((E,K)S,I^o)$ by the projection π_F. To complete the proof of Theorem 2.4 for the characteristic zero case, it must be shown that the projective geometry is the largest semilattice quotient of $((E,K)S,I^o)$. Since there is such a largest quotient, the so-called I^o-semilattice replica of $((E,K)S,I^o)$ [Ma,11.3] [RS,1.5], and since projection onto this replica factorises the projection of $((E,K)S,I^o)$ onto any semilattice quotient, it suffices to show that the Płonka fibres UF of Theorem 3.5 have no non-trivial I^o-semilattice quotient. Now these fibres are the I^o-reducts $(E/U,I^o)$ of affine K-spaces $(E/U,K)$, so the proof of Theorem 2.4 in the characteristic zero case is concluded by the following result.

THEOREM 3.8. For an affine K-space (E,K), the reduct (E,I^o) has no non-trivial I^o-semilattice quotient.

Proof. If (E,I^o) has a non-trivial semilattice quotient, it has the two-element semilattice $\{0,1\}$ with $0 < 1$ as quotient, say by a surjective homomorphism $f:(E,I^o) \to (\{0,1\},I^o)$. Then E is the disjoint union of non-empty subsets $f^{-1}(0)$ and $f^{-1}(1)$. Take x' in $f^{-1}(0)$ and y' in $f^{-1}(1)$. Consider the $\mathbf{Q}$-affine span of x' and y' in (E,K). This is a rational affine line $(\mathbf{Q},\mathbf{Q})$. The homomorphism $f:(E,I^o) \to (\{0,1\},I^o)$ restricts to a surjective homomorphism $g:(Q,I^o) \to (\{0,1\},I^o)$, decomposing the rational affine line as a disjoint union of non-empty fibres $g^{-1}(0)$ and $g^{-1}(1)$. These fibres are subalgebras of (Q,I^o), and so are convex subsets of $\mathbf{Q}$. Without loss of generality, assume that elements of $g^{-1}(0)$ are less than elements of $g^{-1}(1)$ in the order $(Q,<)$. Then there are elements x,y of $g^{-1}(0)$ and z,t of $g^{-1}(1)$ such that $x < y < z < t$. Take $p = (y-x)/(t-x)$ and $q = (z-x)/(t-x)$, so that $xtp = y$ and $xtq = z$. Then $0 = yg = xtpg = xgtgp$ and $1 = zg = xtqg = xgtgq$. But $xgtgp = xgtgq$ in an I^o-semilattice, a contradiction. It follows that (E,I^o) has no non-trivial I^o-semilattice quotient.

4. **ODD CHARACTERISTIC** In this section, the case that K has odd characteristic is considered. As for the characteristic zero case, 2 is invertible here, so the parallelogram operation P may be written in terms of the binary operations $1/2$ and 2 using (3.1). It follows that the algebraic structures (E,K,P) describing affine K-spaces may be replaced by the structures (E,K). Then the set $(E,K)S$ of non-empty subalgebras of (E,K) is the set of affine K-subspaces of (E,K).

Let J denote the prime subfield of K. This subset J of K plays a role analogous to that of the unit interval I in the rationals. The reduct (E,J) of (E,K) has as its subalgebras the affine J-subspaces of (E,K). These may be viewed as analogues of the convex subsets of a rational affine space. Consider the binary operation $1/2$. Under this operation, $(E,1/2)$ is a commutative binary mode, a reduct of (E,J).

PROPOSITION 4.1. The binary derived operations of the algebra $(E,1/2)$ are the operations p with p from J.

Proof. The binary derived operations of the algebra (E,J) are all of the form p for p in J. Let X be the subset of J consisting of those p for which p is a binary derived operation of $(E,1/2)$. Certainly X contains $1/2$. Now for p,q in K,

(4.2) $$xxypq = xy(pq).$$

Thus X is a subsemigroup of the multiplicative group of $J - \{0\}$. In particular, 1 and 2 lie in X, and $2X$ is a subset of X. Since

(4.3) $$xy\ 2p\ xy\ 2q\ 1/2 = xy(p+q)$$

for p,q in K, it follows that X is a subring of J. But X contains 1, and so is all of J.

In view of Proposition 4.1, Ω_K for K of odd characteristic will be taken to be the single binary operation $1/2$, often written as a multiplication $\cdot$ or juxtaposition. Since $(E,\cdot)$ is a commutative binary mode, it follows [RS,146] that the set of affine subspaces of (E,K) forms a commutative binary mode $((E,K)S,\cdot)$. The choice of Ω_K here is again made for two reasons similar to those involved in the choice of Ω_K for K of characteristic zero: firstly, Ω_K avoids 0 and 1, and secondly there is a readily available theory of commutative binary modes, due primarily to Ježek and Kepka [JK], [RS,Chapter 4].

The theory of commutative binary modes is based on the observation that the free commutative binary mode on the two-element set $\{0,1\}$ may be realised as the unit interval D_1 in the set $\mathbf{D} = \{m2^{-n} | m,n \in \mathbf{Z}\}$ of dyadic rationals under the operation $1/2$ [RS,424]. For an odd natural number m, let $l(m)$ denote the least integer greater than $\log_2 m$, e.g. $l(3) = 2$. Then $m\cdot 2^{-l(m)}$ and $m\cdot 2^{-l(m)-1}$, as elements of $(D_1,1/2)$, represent words $w_m(0,1)$ and $w'_m(0,1)$ in $\{0,1\}$ respectively. For example, $3/4 = 110\ 1/2\ 1/2$ and $3/8 = 0110\ 1/2\ 1/2\ 1/2$, so $w_3(0,1) = 110\ 1/2\ 1/2$ and $w'_3(0,1) = 0110\ 1/2\ 1/2\ 1/2$. Let $\underline{m}$ denote the variety of commutative binary modes satisfying the identity $x = w_m(x,y)$, and $\tilde{\underline{m}}$ the variety of those satisfying the identity $w_m(x,y) = w'_m(x,y)$. There is then the following classification theorem [JK,Theorem 4.9] [RS,454].

THEOREM 4.4. Apart from the variety of all commutative binary modes, the varieties of commutative binary modes are the varieties $\underline{m}$ and $\tilde{\underline{m}}$ for odd natural numbers m. For each such m, $\tilde{\underline{m}}$ is the variety of algebras satisfying the regular identities of $\underline{m}$.

Recall that a binary algebra $(A,\cdot)$ is said to be a quasigroup if there are derived binary operations $/$ (called right division) and $\backslash$ (called left division) on A such that the identities

$$(4.5)\qquad \begin{cases} (x\cdot y)/y = x, & (x/y)\cdot y = x, \\ y\backslash(y\cdot x) = x, & y\cdot(y\backslash x) = x \end{cases}$$

are satisfied. The commutative binary modes $(E,\cdot) = (E,1/2)$ coming from affine K-spaces (E,K) may then be described as follows.

PROPOSITION 4.6. Let u be the multiplicative order of 2 in the field K. Then the reduct $(E,1/2)$ of an affine K-space (E,K) is a quasigroup in the variety $\underline{m}$ for $m = 2^u - 1$.

Proof For $m=2^u - 1$, $l(m) = u$. Then $m\cdot 2^{-l(m)} = 1 - (1/2^u) = 10(1/2^u) = 1\ldots1\ 0\ 1/2\ldots1/2$ with u applications of $1/2$, the latter equality coming from (4.2). Thus $w_m(x,y) = y\ldots yx\ 1/2\ldots1/2$. In $(E,1/2)$, $w_m(x,y) = y\ldots yx\ 1/2\ldots1/2 = yx(1/2^u) = yx\ 1 = x$, the second equality coming from (4.2). Thus the commutative binary mode $(E,1/2)$ lies in the variety $\underline{m}$.

Consider the binary operations λ and ρ on E with $yx\lambda = yx(1/2^{u-1})$ and $xy\rho = yx\lambda$. Using (4.2) and the commutativity of $1/2$, the word $w_m(x,y) = y\ldots yx\ 1/2\ldots1/2$ in $(E,1/2)$ may be written variously as $w_m(x,y) = y(yx1/2)\lambda = y(yx\lambda)1/2 = (xy1/2)y\rho = (xy\rho)y1/2$. The identity $w_m(x,y) = x$ in $(E,1/2)$ then gives the quasigroup identities (4.5). Since λ and ρ are derived operations of $(E,1/2)$, it follows that $(E,1/2)$ is a quasigroup.

Propositions 3.4 and 4.6 may then be combined to give the following structural description of the commutative binary mode $((E,K)S,\cdot)$ of affine subspaces of the affine K-space (E,K).

THEOREM 4.7. Let K be a field of odd characteristic p. Let m be the least integer multiple of p of the form $2^u - 1$ for a natural number u. Then for an affine K-space (E,K), the commutative binary mode $((E,K)S,\cdot)$ is a Płonka sum of quasigroups in the variety $\underline{m}$ over the projective geometry $(L(E),+)$ by the functor $F:(L(E)) \to (\{\cdot\})$ with $UF = \{x + U | x \in E\}$ and $(U \to V)F:UF \to VF; x + U \mapsto x + V$.

Proof. That $((E,K)S,\cdot)$ is a Płonka sum over $(L(E),+)$ by the functor F follows directly from Proposition 3.4 with $k = 1/2$. The Płonka fibres are the algebras $(\{x + U | x \in E\},1/2)$ for fixed subspaces U, with $(x+U)(y+U)1/2 = xy1/2 + U$. Since these are the reducts $(E/U,1/2)$ of the affine geometries $(E/U,K)$ coming from the quotient vector subspaces E/U, they are quasigroups in the variety $\underline{m}$ by Proposition 4.6.

COROLLARY 4.8. The commutative binary mode $((E,K)S,\cdot)$ lies in the variety $\tilde{\underline{m}}$.

Proof. By Theorem 4.7 and Płonka's result [Pł,Theorem I] [RS,238], the algebra $((E,K)S,\cdot)$ satisfies the regular identities of $\underline{m}$ - the regular identities satisfied by each of the Płonka fibres of $((E,K)S,\cdot)$. Theorem 4.4 then shows that $((E,K)S,\cdot)$ lies in the variety $\tilde{\underline{m}}$.

In analogy with (3.7), Theorem 4.7 and Corollary 4.8 may be summarized as:

(4.9) $$\begin{cases} \text{The } \tilde{\underline{m}}\text{-algebra of affine subspaces of an affine } K\text{-space} \\ \text{is a Płonka sum of } \underline{m}\text{-quasigroups over a projective geometry.} \end{cases}$$

Just as for the characteristic zero case, the proof of Theorem 2.4 for the case that K has odd characteristic is completed by showing that the Płonka fibres UF of Theorem 4.7 have no non-trivial semilattice quotient. Now these Płonka fibres, lying in $\underline{m}$, satisfy the identity $w_m(x,y) = x$. In a semilattice quotient $(H,+)$ of such a fibre, this identity becomes $x + y = x$ or $y \leqslant x$. Thus for two elements h,k of H, one has $h \leqslant k$ and $k \leqslant h$, whence $h = k$ and the triviality of H.

5. **CHARACTERISTIC TWO** This section considers the case that the field K has characteristic 2. Since $2 = 0$ is no longer invertible, the ternary parallelogram operation P can no longer be made redundant by (3.1), and the full algebra structure (E,K,P) is needed to give the affine geometry. As in Section 4, J will denote the prime subfield $GF(2)$ of K. Note that every subset of E is a subalgebra of the reduct (E,J) of (E,K,P), since the binary operations J are just the projections $xy\underline{0} = x$ and $xy\underline{1} = y$. Let Ω_K in this case denote the singleton $\{P\}$ consisting of the ternary parallelogram operation (2.2). Thus the "convex subsets" of E will be taken to be the subalgebras of (E,P), the J-affine subspaces of (E,K,P). By [OS], [RS,255], the class of all J-affine spaces, together with the empty set, is the variety of all <u>minority modes</u> (A,P), algebras with a ternary operation P satisfying the entropic law

(5.1) $$x_{11}x_{12}x_{13}Px_{21}x_{22}x_{23}Px_{31}x_{32}x_{33}PP = x_{11}x_{21}x_{31}Px_{12}x_{22}x_{32}Px_{13}x_{23}x_{33}PP$$

and the identities

(5.2) $$yxyP = x, \quad xyyP = x, \quad yyxP = x.$$

The name comes from the observation that the value of the operation P in the identities (5.2) reduces to that one of its arguments, if any, that is in the minority. Note that idempotence is a consequence of (5.2), so minority modes really are modes.

By [RS,146], the set of affine subspaces of the affine K-space forms a ternary mode $((E,K,P)S,P)$. The structure of this algebra is given by the

following theorem.

THEOREM 5.3. <u>Let</u> K <u>be a field of characteristic</u> 2. <u>Then for an affine</u> K<u>-space</u> (E,K,P), <u>the ternary mode</u> $((E,K,P)S,P)$ <u>of affine subspaces is a Płonka sum of minority modes over the projective geometry</u> $(L(E),+)$ <u>by the functor</u> $F:(L(E)) \to (\{P\})$ <u>with</u> $UF = \{x + U \mid x \in E\}$ <u>and</u> $(U \to V)F:UF \to VF:x + U \mapsto x + V$.

<u>Proof</u>. Since K is of characteristic 2, the operation P on E as in (2.2) becomes $xyzP = x + y + z$. For vector subspaces U,V,W, and $X = U + V + W$ of E, and for corresponding affine subspaces $x + U$, $y + V$, and $z + W$, one has $(x+U)(y+V)(z+W)P = x + U + y + V + z + W = xyzP + X = (x+X)(y+X)(z+X)P$ $= x(U \to X)Fy(V \to X)Fz(W \to X)FP$. Thus $((E,K,P)S,P)$ is a Płonka sum as claimed. The Płonka fibres $UF = (E/U,P)$, as J-affine spaces, are minority modes.

In the characteristic zero case, Corollary 3.6 to the Structure Theorem 3.5 for the algebra of affine subspaces specified this algebra as lying in the variety of rational barycentric algebras, with identities given by Theorem 3.3. In the case that K had odd characteristic, Corollary 4.8 to the Structure Theorem 4.7 for the algebra of affine subspaces specified this algebra as lying in the variety $\tilde{\underline{m}}$ of commutative binary modes satisfying the identity $w_m(x,y) = w'_m(x,y)$. It is thus of interest in the current case to find a variety nicely containing the algebra of affine subspaces, so that this algebra is described well as lying in the variety. By Theorem 5.3 and the result of Płonka quoted earlier [Pł,Theorem I] [RS,238], the algebra $((E,K,P)S,P)$ may be described as satisfying each regular identity satisfied by its Płonka fibres, i.e. each regular identity satisfied by each minority mode. Unfortunately, there are infinitely many such identities involving the single operation P, so that this description seriously lacks conciseness. The problem is to find a finite set of identities, a so-called <u>finite basis</u>, of which the set of all regular identities satisfied by all minority modes is the consequence.

A little universal algebra, essentially implicit in the work of Płonka, serves to solve the problem. A variety $\underline{V}$ of algebras (A,Ω) is called <u>strongly irregular</u> if there is a binary derived operation $*$ such that $\underline{V}$-algebras may be characterised as the Ω-algebras satisfying some set of regular identities and the single irregular identity $x * y = x$. For example, the variety $\underline{m}$ of commutative binary modes is strongly irregular. Taking the binary derived operation $x * y = w_m(x,y)$, the variety $\underline{m}$ is specified by the regular commutative, idempotent, and entropic identities, together with the single irregular identity $w_m(x,y) = x$, i.e. $x * y = x$. In the present context, the variety of minority modes is strongly irregular. Define

(5.4) $$x * y = yxyP.$$

Then the variety of minority modes is the variety of algebras (A,P) satisfying the regular identities of idempotence and entropicity (5.1), together with the three identities (5.2). The first of these is just $x * y = x$. When this obtains, the second and third of them, which appear to be irregular, may in fact be rewritten as the regular identities

$$xyyP = x * y \quad \text{and} \quad yyxP = x * y,$$

i.e. as

(5.5) $$xyyP = yxyP \quad \text{and} \quad yyxP = yxyP.$$

In other words, minority modes are the ternary algebras (A,P) satisfying idempotence, entropicity (5.1), (5.5), and the irregular identity $x * y = x$ with $*$ as in (5.4).

A variety $\underline{V}$ of algebras (A,Ω) is called irregular if there is an irregular identity satisfied by each $\underline{V}$-algebra. The regularised variety or regularisation $\tilde{\underline{V}}$ of such a variety $\underline{V}$ is the variety of algebras satisfying all the regular identities satisfied by all $\underline{V}$-algebras. The current task is to specify the regularisation of a strongly irregular variety. Now a binary operation $*$ on an algebra (A,Ω) is said to be a partition operation on A if (A,Ω) satisfies the following identities: $(A,*)$ is a left normal band, i.e.

(5.6) $$\begin{cases} x * x = x, \\ (x * y) * z = x *(y * z), \text{ and} \\ x * y * z = x * z * y; \end{cases}$$

$*$ distributes from the right over ω in Ω, i.e.

(5.7) $$x_1 \ldots x_n \omega * y = (x_1 * y) \ldots (x_n * y)\omega;$$

and $*$ breaks ω from the left, i.e.

(5.8) $$y * (x_1 \ldots x_n \omega) = y * x_1 * \ldots * x_n.$$

(Note that no bracketing is necessary in the right hand side of (5.8) once (5.6) holds.) The significance of partition operations comes from the following result of Płonka.

PROPOSITION 5.9. [Pł] [RS,237]. An algebra (A,Ω) is a Płonka sum iff there is a partition operation $*$ on A. If these conditions obtain, the identity $x * y = x$ is satisfied by each fibre.

Using this result, the following characterisation of the regularisation of a strongly irregular variety may be given.

THEOREM 5.10. Let $\underline{V}$ be a strongly irregular variety, specified by $x * y = x$ and a set R of regular identities. Then the regularisation $\tilde{\underline{V}}$ is specified by R and the identities (5.6), (5.7), (5.8).

Proof. Let $\underline{W}$ be the variety of algebras, of the same type as $\underline{V}$, satisfying R and (5.6), (5.7), (5.8). Let (A,Ω) be an algebra in $\underline{W}$. By (5.6), (5.7), 5.8), the derived binary operation $*$ is a partition operation on (A,Ω). Proposition 5.9 then shows that (A,Ω) is a Płonka sum of algebras satisfying the identity $x * y = x$. Since the Płonka fibres, as subalgebras of (A,Ω), also satisfy the identities in R, it follows that the fibres lie in $\underline{V}$. Consequently [Pł] [RS238] (A,Ω) satisfies the regular identities of $\underline{V}$, and so is in the regularisation $\tilde{\underline{V}}$. This shows that $\tilde{\underline{V}}$ contains $\underline{W}$.

Conversely, consider a $\underline{V}$-algebra (B,Ω). Since $x * y = x$ on B, the identities (5.6), (5.7), (5.8) are all satisfied by (B,Ω). As a $\underline{V}$-algebra, (B,Ω) also satisfies the identities R. Thus (B,Ω) lies in $\underline{W}$, and $\underline{W}$ contains $\underline{V}$. But since the identities specifying $\underline{W}$ are all regular, $\underline{W}$ also contains $\tilde{\underline{V}}$. The equality of $\tilde{\underline{V}}$ with $\underline{W}$, and the theorem, follow.

COROLLARY 5.11. If a strongly irregular variety $\underline{V}$ has a finite basis for its identities, then so does its regularisation $\tilde{\underline{V}}$.

Define an algebra (A,P) with a single ternary operation P to be a regularised minority mode if it satisfies the identities of idempotence, entropicity (5.1), (5.5), the associative law $zyxyPzP = zyzPxzyzPP$, the left normal law $zyxyPzP = yzxzPyP$, and the left breaking law $xyzPtxyzPP = zyxtxPyPzP$. Writing the derived operation $*$ as in (5.4), the idempotence, associative, and left normal laws show that (5.6) hold for regularised minority modes. The distributive law (5.7) follows from the idempotence and entropicity, while (5.8) follows from the left breaking law. Theorem 5.10 then shows that the regularisation of the variety of minority modes may thus be concisely described as the finitely based variety of regularised minority modes.

Regularised minority modes appear to have some interesting properties worthy of further investigation. There is a result of Płonka stating that the free algebra over a set X in the regularisation of a strongly irregular variety is a Płonka sum over the join semilattice of non-empty subsets of X, by the free algebra functor for the strongly irregular variety [Po] [RS,273]. As a consequence of this, it turns out that the free regularised minority mode on $n + 1$ elements has the cardinality of n-dimensional projective space over GF(3). As the following

general result implies, the two structures have different automorphism groups, but it would nevertheless be useful to set up some correspondence between them in order to facilitate manipulation of the identities for regularised minority modes.

THEOREM 5.12. Let $\underline{W}$ be a variety specified by regular identities. For a set X, let XW denote the free $\underline{W}$-algebra on X. Then the automorphism group of XW is isomorphic to the group of permutations of X.

Proof. Since the identities of $\underline{W}$ are regular, an element x of X can only lie in the subalgebra of XW generated by a subset Y of XW if x actually appears as an element of Y. If f is an automorphism of XW, this forces Xf to be equal to X. Conversely, knowledge of the restriction of f to the generating set X determines f uniquely. Thus restriction to X provides an isomorphism from the automorphism group of XW to the permutation group of X.

Returning to the algebra of affine subspaces of an affine K-space, it is now possible to formulate the following corollary to Theorem 5.3.

COROLLARY 5.13. The ternary algebra ((E,K,P)S,P) of affine subspaces of an affine K-space (E,K,P) is a regularised minority mode.

Proof. By Theorem 5.3, ((E,K,P)S,P) is a Płonka sum of minority modes. Thus [Pł,Theorem I] [RS,238] it satisfies the regular identities satisfied by minority modes. Theorem 5.9 then shows that it is a regularised minority mode.

Theorem 5.3 and Corollary 5.12 summarize as:

(5.13) { The regularised minority mode of affine subspaces of an affine K-space is a Płonka sum of minority modes over a projective geometry.

To complete the proof of Theorem 2.4 for the case that K has characteristic 2, note that the Płonka fibres UF of Theorem 5.3 satisfy the irregular identity $x * y = x$. An argument identical to that given in the odd characteristic case then shows that these fibres have no non-trivial semilattice quotient, so the projective geometry (L(E),+) is the largest such quotient of ((E,K,P)S,P).

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