EMBEDDING SUMS OF CANCELLATIVE MODES INTO FUNCTORIAL SUMS OF AFFINE SPACES

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Abstract. A mode is an idempotent and entropic algebra. We show that cancellative modes embed as subreducts into affine spaces. This result is then extended to certain general sums of cancellative modes. In this case we show that such sums embed as subreducts into functorial sums of affine spaces.

One of the main features of late 20th century algebra was the emergence of new "post-modern" algebraic structures, as opposed to the basic structures of "modern algbebra": groups, rings, fields and modules. On the one hand; the traditional structures are no longer adequate for addressing the many algebraic problems that arise in mathematics and other disciplines. On the other hand the new structures, for example ordered sets, monoids, monoid actions, quasigroups and loops, different types of lattices and more general ordered algebras (among them the BCK-algebras developed by Professor Iseki), semirings and semimodules, not only have a interesting theory, but also play an essential role in many applications of mathematics. These various structures are unified by techniques of universal algebra and category theory. The topic of the present paper is one of these new

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algebraic structures. The algebras are called modes, and originated as a common generalization of affine spaces, convex sets and semilattices. They are characterized by two basic properties: idempotence and entropicity, explained in Section 1. Other important examples are provided by subreducts of affine spaces, by normal bands, and by certain groupoids arising from combinatorics. See [14] for the basic definitions, properties and general theory as developed by 1985.

An important class of subreducts of affine spaces is formed by those that are cancellative. On the other hand general cancellative modes seem to play a special and important role within the theory of modes. (See e.g. [14].) So it is esssential to investigate the relationship between the classes of cancellative affine space subreducts and of general cancellative modes. This problem is addressed in Section 3, where we show that each cancellative mode embeds as a subreduct into an affine space. This theorem has several predecessors concerning the embeddability of cancellative entropic groupoids into commutative semigroups equipped with commuting endomorphisms [11], [20], [21], [22], [3], or into entropic quasigroups [17], [4, Theorem 5.3.1]. Note that in the case of an idempotent cancellative entropic groupoid, the corresponding entropic quasigroup is also idempotent, and hence is a mode. Since this mode has a Mal'cev operation, it is equivalent to an affine space [14].

The next step is to investigate certain "sums" of cancellative modes. The basic facts about modes constructed as so-called Lallement sums of their submodes are recalled in Section 4. Among such sums the so-called functorial sums are those yielding the best structural descriptions. In Section 5, we show that any Lallement sum of cancellative modes embeds as a subreduct into a functorial sum of affine spaces.

These two results belong to the broad realm of embeddability theorems, where one embeds a given algebra into another, usually one with a better known and richer structure. Such embeddings are usually a very efficient way of describing the structure of an algebra. The prototypes of such theorems yield the embedding of integral domains into fields and of commutative cancellative semigroups into commutative groups. On the other hand our second result uses some new and promising techniques for representing algebras as sums of subalgebras.

The notation and terminology of the paper is basically as in the book [14]. We refer the reader to this book and to the surveys [12] and [18] for undefined notions and results. The meanings of "term operation" and "derived operation" are the same. We use "reverse Polish" notation for words (terms) and (derived) operations, e.g. $x_1 \dots x_n \omega$ denotes a word (term) with variables $x_1, \dots, x_n$ or the corresponding derived operation in an algebra. We make an exception for certain binary operations that are denoted by traditional infix notation.

1. Modes, affine spaces and barycentric algebras.

Fix a type $\tau : \Omega \to \mathbb{N}$ of algebras. A τ-algebra (A, Ω) is said to be *entropic* if each basic operation ω in Ω is a homomorphism $\omega : (A^{\omega\tau}, \Omega) \to (A, \Omega)$. In other words, for each set

$\{\omega, \varphi\}$ of basic operations, say with $\omega\tau = n$ and $\varphi\tau = m$, the identity

$$(x_{11} \dots x_{1n}\omega) \dots (x_{m1} \dots x_{mn}\omega)\varphi = (x_{11} \dots x_{m1}\varphi) \dots (x_{1n} \dots x_{mn}\varphi)\omega \tag{1.1}$$

is satisfied. A τ-algebra (A, Ω) is said to be *idempotent* if each singleton subset of A is actually a subalgebra. In other words, for each basic operation ω, the identity

$$x \dots x\omega = x \tag{1.2}$$

is satisfied. A *mode* is an idempotent and entropic algebra. Modes are characterized by the following:

Proposition 1.1 [18]. *A τ-algebra (A, Ω) is a mode if and only if each polynomial operation of (A, Ω) is a homomorphism.* □

A τ-mode (A, Ω) is said to be *cancellative* if it satisfies the quasi-identity

$$(x_1 \dots x_{i-1} y x_{i+1} \dots x_n\omega = x_1 \dots x_{i-1} z x_{i+1} \dots x_n\omega) \to (y = z) \tag{1.3}$$

for each (n-ary) ω in Ω and each $i = 1, \dots, n$.

Initial examples of modes are provided by normal bands, idempotent semigroups satisfying the entropic identity

$$xy \cdot zt = xz \cdot yt.$$

Among them, there are semilattices and left and right zero bands. In this paper two other important examples will play a basic role.

Example 1.2 [affine spaces]. Let R be a commutative (unital) ring. Let R-$\underline{\underline{\underline{\mathrm{Mod}}}}$ be the variety of (unital, right) R-modules, construed as algebras $(E, +, 0, R)$ with a binary addition, nullary zero, and unary scalar multiplications. Given a module E, the corresponding *affine space* may be described algebraically as the set E equipped with all the idempotent linear or *affine* operations

$$E^n \to E, (a_1, \dots, a_n) \mapsto \sum_{i=1}^{n} a_i r_i \tag{1.4}$$

for each positive n and each element $(r_1, \dots, r_n)$ of R^n with $r_1 + \dots + r_n = 1$. In this sense, affine spaces are modes. Define a set $\underline{R}$ of binary operations

$$\underline{r} : E^2 \to E; (x, y) \mapsto x(1 - r) + yr \tag{1.5}$$

for r in R. Define the ternary Mal'cev operation

$$P : E^3 \to E; (x, y, z) \mapsto x - y + z. \tag{1.6}$$

As noted by Osterman and Schmidt [9], cf. [2], the affine operations (1.4) are precisely the derived operations of the algebra $(E, \underline{R}, P)$. Moreover, the algebra $(E, \underline{R}, P)$ has the affine group as its group of automorphisms, and may thus be identified with the affine geometry (cf. [8], [9]). Let $\underline{\underline{R}}$ denote the class of affine spaces over the commutative ring R. Then $\underline{\underline{R}}$ may be characterized [14, 255] as the variety of modes of type $(R \times \{2\}) \cup \{(P, 3)\}$ satisfying the Mal'cev identities

$$xxyP = y = yxxP \tag{1.7}$$

together with

$$(1.8)\qquad \begin{cases} (A1) & xyxP = yx\underline{2}\ ; \\ (A2) & xy\underline{p}\ xy\underline{q}\ \underline{r} = xy\underline{pqr}\ ; \\ (A3) & xy\underline{p}\ xy\underline{q}\ xy\underline{r}\ P = xy\ \underline{pqrP}\ ; \\ (A4) & xy\underline{0} = x = yx\underline{1} \end{cases}$$

for p, q, r in R. If 2 is invertible in R, then one may derive the Mal'cev operation (1.6) as

$$xyzP = y.xz\underline{2^{-1}}\ \underline{2}.$$

In this case, $\underline{\underline{R}}$ may be characterized as the variety of modes of type $R \times \{2\}$ satisfying (A2) and (A4) of (1.8) [9], [14, 256].

Example 1.3 [convex sets and barycentric algebras]. Let I° denote the open unit interval $]0,1[= \{x|0 < x < 1\}$ in a subfield F of the real field $\mathbb{R}$. Consider reducts $(E, \underline{I}^\circ)$ of affine spaces $(E, \underline{F})$ over F. The subalgebras of such reducts are precisely the convex sets. One may characterize the class $\underline{\underline{C}}$ of convex sets over F as the class of algebras $(A, \underline{I}^\circ)$ of type $I^\circ \times 2$ satisfying idempotence

$$xx\underline{p} = x, \tag{1.9}$$

skew commutativity

$$xy\underline{p} = yx\underline{(1-p)}, \tag{1.10}$$

skew associativity

$$xy\underline{p}\ z\underline{q} = x\ yz\underline{q/(p+q-pq)}\ \underline{p+q-pq}, \tag{1.11}$$

and *cancellativity*

$$(xy\underline{p} = xz\underline{p}) \to (y = z) \tag{1.12}$$

for p and q in I° [8], [14, 269]. Note that $\underline{\underline{C}}$ forms a quasivariety of modes. It does not form a variety. The smallest variety $\underline{\underline{B}}$ containing $\underline{\underline{C}}$ is called the variety of *barycentric algebras*. It is characterized as the class of $\underline{I}^\circ$-algebras satisfying the identities (1.9)–(1.11) [8], [14, 214]. Note that a semilattice $(H, \cdot)$ is also a barycentric algebra, with

$$xy\underline{p} = x \cdot y$$

for all x and y in H and p in I°. $\square$

2. Subreducts of affine spaces.

An Ω-*reduct* of an affine space $(A, \underline{R}, P)$ over a ring R is an algebra (A, Ω) with a subset Ω of derived operations of $(A, \underline{R}, P)$ as the set of basic operations. An Ω-*subreduct* of $(A, \underline{R}, P)$ is a subalgebra of an Ω-reduct (A, Ω). Ω-subreducts of affine spaces over R are characterized by certain separability conditions [15], and form a quasivariety [6].

For any variety $\underline{\underline{V}}$ of Ω-modes, its subquasivariety of Ω-subreducts of affine spaces may be described as the class of Ω-subreducts of affine spaces in the affiinization of $\underline{\underline{V}}$. The *affinization* of $\underline{\underline{V}}$ is the *tensor product* $\underline{\underline{\mathbb{Z}}} \otimes \underline{\underline{V}}$ of the varieties $\underline{\underline{\mathbb{Z}}}$ of integral affine spaces and $\underline{\underline{V}}$. This is the variety of modes $(A, \Omega \dot{\bigcup} P)$ with the reducts (A, Ω) in $\underline{\underline{V}}$ and the Mal'cev operation P. It is equivalent to the variety $\underline{\underline{R(V)}}$ of affine spaces over a certain ring $R(V)$ [15]. The Ω-reducts of $\underline{\underline{R(V)}}$-algebras are often very useful and informative models of $\underline{\underline{V}}$-algebras.

Example 2.1 [15]. Let $\underline{\underline{M\tau}}$ denote the variety of all modes of given plural type $\tau : \Omega \to (\mathbb{Z}^+, \leq)$ (cf. [14, p.32]). Recall that for a function $f : D \to (E, \leq)$ with ordered codomain, the *hypograph* is the set

$$\text{hyp } f = \{(d, e) \in D \times E | df \geq e\} \tag{2.1}$$

[18, (7.6)]. Consider the integral polynomial ring $\mathbb{Z}[\text{hyp } \tau]$ over the hypograph

$$\{(\omega, i) \in \Omega \times \mathbb{Z}^+ | 1 \leq i \leq \omega\tau\} \tag{2.2}$$

of the type τ. Then $R(M\tau)$ is the quotient

$$R(M\tau) = \mathbb{Z}[\text{hyp } \tau] / \langle 1 - \sum_{i=1}^{\omega\tau} (\omega, i) | \omega \in \Omega \rangle \tag{2.3}$$

of the polynomial ring $\mathbb{Z}[\text{hyp } \tau]$ by the ideal obtained by setting each sum $\sum_{i=1}^{\omega\tau} (\omega, i)$ to be 1. For ω in Ω, the corresponding operation in an affine space over $R(M\tau)$ is given by the formula

$$x_1 \dots x_{\omega\tau} \omega = \sum_{i=1}^{\omega\tau} x_i (\omega, i). \tag{2.4}$$

In this and similar formulas, one does not always distinguish between elements of $\mathbb{Z}[\text{hyp } \tau]$ and their corresponding images in $R(M\tau)$. □

Example 2.2 [15]. Consider again the variety $\underline{\underline{B}}$ of barycentric algebras over a subfield F of $\mathbb{R}$ as in Example 1.3. The affinization $\underline{\underline{R(B)}}$ of $\underline{\underline{B}}$ is the variety $\underline{\underline{F}}$ of affine spaces over F. The class of $\underline{I}^\circ$-subreducts of $\underline{\underline{R(B)}}$-algebras is precisely the class $\underline{\underline{C}}$ of convex subsets of affine F-spaces. The convex sets are characterized among barycentric algebras over F as those satisfying the cancellation laws (1.12). □

Note however that not all Ω-subreducts of affine spaces over a given commutative ring are necessarily cancellative.

3. Embedding cancellative modes into affine spaces.

Let (A, Ω) be a non-empty τ-mode, and let e be any element of A. For each ω in Ω, and for $1 \leq i \leq \omega\tau$, define the *i-th translation by e* as the mapping

$$e_i(\omega) : A \to A;\ x \mapsto e \dots exe \dots e\omega \tag{3.1}$$

with x as the i-th argument of ω. By entropicity and Proposition 1.1, one obtains:

Lemma 3.1. *Let (A, Ω) be a non-empty τ-mode and let e be any element of A. Then all the translations $e_i(\omega)$ are pairwise commuting endomorphisms of (A, Ω).* □

Now extend the type $\tau : \Omega \to \mathbb{N}$ of the algebra (A, Ω) by taking e as the element selected by a nullary operation, and by taking $e_i(\omega)$ for ω in Ω and $1 \leq i \leq \omega\tau$ as unary operations. In this way, one obtains the algebra (A, Ω') of type $\tau' : \Omega' \to \mathbb{N}$ with $\tau' = \tau \biguplus (\{e_i(\omega) | \omega \in \Omega, 1 \leq i \leq \omega\tau\} \times \{1\}) \biguplus \{(e, 0)\}$. Let $\underline{\underline{K}}$ be the class of algebras of type τ' on which each $e_i(\omega)$ is an injective endomorphism.

Proposition 3.2 [4, Proposition 1.1.1]. *For each $\underline{\underline{K}}$-algebra (A, Ω') there is an algebra $(\overline{A}, \Omega')$ containing a subalgebra isomorphic with (A, Ω') such that the operations $e_i(\omega)$ are automorphisms of $(\overline{A}, \Omega')$, and such that $(\overline{A}, \Omega')$ satisfies all the identities satisfied by (A, Ω').* □

The following is a corollary to Theorem 7 of [1].

Theorem 3.3. *Let (B, Ω) be a non-empty, plural τ-mode and let e be any element of B. If all the translations $e_i(\omega)$ are bijective, then the following hold:*

(a) *There is a commutative monoid $(B, +, e)$ with addition defined by*

$$x + y = xe_1(\omega)^{-1}ye_2(\omega)^{-1}e \dots e\omega \tag{3.2}$$

for any ω in Ω;

(b) *The translations $e_i(\omega)$ are pairwise commuting monoid and Ω-automorphisms;*

(c) *for each ω in Ω and $x_1, \dots, x_{\omega\tau}$ in B one has*

$$x_1 \dots x_{\omega\tau}\omega = x_1e_1(\omega) + \dots + x_{\omega\tau}e_{\omega\tau}(\omega). \quad \square \tag{3.3}$$

If the τ-mode (A, Ω) is cancellative, then cancellativity and Lemma 3.1 imply that all the translations $e_i(\omega)$ are injective endomorphisms. By Proposition 3.2, the algebra (A, Ω') is isomorphic to a subalgebra of an algebra $(\overline{A}, \Omega')$ on which the $e_i(\omega)$ are automophisms. In what follows we will identify (A, Ω) with this subalgebra of $(\overline{A}, \Omega')$. Then the assumptions of Theorem 3.3 are satisfied by $(\overline{A}, \Omega)$ with e in A, and one is led to the following result.

Corollary 3.4. *If (A, Ω) is a cancellative τ-mode, then there is a cancellative commutative monoid $(\overline{A}, +, e)$ on the set $\overline{A}$ and pairwise commuting automorphisms $e_i(\omega)$ for all ω in Ω and $i = 1, \dots, \omega\tau$ such that*

$$x_1 \dots x_{\omega\tau}\omega = x_1 e_1(\omega) + \dots + x_{\omega\tau} e_{\omega\tau}(\omega). \tag{3.4}$$

Moreover, under the operations (3.4), the algebra $(\overline{A}, \Omega)$ is a cancellative τ-mode.

Proof. The cancellativity of $(\overline{A}, +, e)$ remains to be verified. Now in the context of Proposition 3.2, the algebra $(\overline{A}, \Omega')$ is obtained as a union of subalgebras isomorphic with (A, Ω'). Since (A, Ω') is cancellative, it follows that $(\overline{A}, \Omega')$ is cancellative. The cancellativity of $(\overline{A}, \Omega)$ then implies cancellativity of the monoid $(\overline{A}, +, e)$ with addition given by (3.2) using automophisms $e_1(\omega)$ and $e_2(\omega)$. □

There is a very well-known standard construction, like the localization of a ring, that allows one to embed each cancellative commutative monoid C into an abelian group. One takes the direct power $C \times C$ of the set C, and defines the relation ρ on $C \times C$ by

$$((a, b), (c, d)) \in \rho :\Leftrightarrow a + d = b + c. \tag{3.5}$$

It is immediate that ρ is a congruence of the commutative monoid $(C \times C, +)$, and that $G = (C \times C, +)^\rho$ is an abelian group. The identity of G is the diagonal $(x, x)^\rho = \{(x, x) | x \in C\}$. The inverse of $(a, b)^\rho$ is $(b, a)^\rho$. The mapping

$$\Delta : C \to (C \times C)^\rho; x \mapsto (x + x, x)^\rho \tag{3.6}$$

embeds the monoid $(C, +, e)$ into the monoid $(G, +, 0)$ with $0 = (x, x)^\rho$. In what follows, we will identify elements x of C with $(x + x, x)^\rho$ and consider $(C, +, e)$ as a submonoid of the monoid $(G, +, 0)$. Note that the group G has the following universality property. Let H be any abelian group and let $f : (C, +, e) \to (H, +, e)$ be a monoid homomorphism. Then there is a unique group homomorphism $\overline{f} : (G, +, -, 0) \to (H, +, -, 0)$ such that $\Delta\overline{f} = f$, i.e. the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & G = (C \times C)^\rho \\ \| & & \downarrow \overline{f} \\ C & \xrightarrow{f} & H \end{array} . \tag{3.7}$$

The mapping $\overline{f}$ is defined by

$$\overline{f} : G \to H;\ (a, b)^\rho \mapsto af - bf. \tag{3.8}$$

Theorem 3.5. *Each cancellative mode (A, Ω) of a fixed type $\tau : \Omega \to \mathbb{Z}^+$ embeds as an Ω-subreduct into an affine space.*

Proof. Using Corollary 3.4, first embed (A, Ω) into the commutative monoid $(\overline{A}, +, e)$ equipped with Ω-operations (3.4). Then consider the abelian group $G = (\overline{A} \times \overline{A})^\rho$ from above as a $\mathbb{Z}$-module. We will extend the ring $\mathbb{Z}$ to a commutative ring R that will make G an R-module, and then show that the mode (A, Ω) embeds (as a subreduct) into the affine space $(G, \underline{R}, P)$.

The first aim is to define the ring R. Consider the translations $e_i(\omega)$ with $i = 1, \ldots, \omega\tau$ for all ω in Ω. The translations $e_i(\omega)$ are automorphisms of the monoid $(\overline{A}, +, e)$. The universality property for G allows one to extend these automorphisms to automorphisms of the group $(G, +, -, 0)$ as shown on the following diagram:

(3.9)
$$\begin{array}{ccc} \overline{A} & \xrightarrow{\ \Delta\ } & G \\ {\scriptstyle e_i(\omega)}\big\downarrow & & \big\downarrow{\scriptstyle \overline{e}_i(\omega) := \overline{e_i(\omega)\Delta}} \\ \overline{A} & \xrightarrow{\ \Delta\ } & G \end{array} .$$

The group homomorphisms $\overline{e}_i(\omega)$ are defined by

(3.10)
$$\begin{aligned} (a, b)^\rho \overline{e}_i(\omega) &= a e_i(\omega)\Delta - b e_i(\omega)\Delta \\ &= (a e_i(\omega) + a e_i(\omega), a e_i(\omega))^\rho \\ &+ (b e_i(\omega), b e_i(\omega) + b e_i(\omega))^\rho \\ &= (a e_i(\omega), e)^\rho + (e, b e_i(\omega))^\rho = (a e_i(\omega), b e_i(\omega))^\rho, \end{aligned}$$

and are obviously automorphisms, since the $e_i(\omega)$ are. Define the ring R to be the subring, generated by all the $\overline{e}_i(\omega)$, of the ring $\mathrm{End}(G, +)$ of group endomorphisms. Consider the R-module $(G, +, R)$ and the corresponding affine space $(G, \underline{R}, P)$. For each ω in Ω, define the operation ω on G as follows:

(3.11)
$$(a_1, b_1)^\rho \ldots (a_{\omega\tau}, b_{\omega\tau})^\rho \omega := (a_1, b_1)^\rho \overline{e}_1(\omega) + \cdots + (a_{\omega\tau}, b_{\omega\tau})^\rho \overline{e}_{\omega\tau}(\omega).$$

Now the elements $a_1, \ldots, a_{\omega\tau}$ in $\overline{A}$ are identified under Δ with $(a_1 + a_1, a_1)^\rho, \ldots,$ $(a_{\omega\tau} + a_{\omega\tau}, a_{\omega\tau})^\rho$. Hence

$$\begin{aligned} (a_1\Delta) \ldots (a_n\Delta)\omega &= (a_1 + a_1, a_1)^\rho \ldots (a_{\omega\tau} + a_{\omega\tau}, a_{\omega\tau})^\rho \omega \\ &= (a_1 + a_1, a_1)^\rho \overline{e}_1(\omega) + \cdots + (a_{\omega\tau} + a_{\omega\tau}, a_{\omega\tau})^\rho \overline{e}_{\omega\tau}(\omega) \\ &= (a_1 e_1(\omega) + a_1 e_1(\omega), a_1 e_1(\omega))^\rho + \cdots + (a_{\omega\tau} e_{\omega\tau}(\omega) + a_{\omega\tau} e_{\omega\tau}(\omega), a_{\omega\tau} e_{\omega\tau}(\omega))^\rho \\ &= a_1 e_1(\omega)\Delta + \cdots + a_{\omega\tau} e_{\omega\tau}(\omega)\Delta. \end{aligned}$$

It follows that for elements of $\overline{A}$, the definition (3.11) of ω on G coincides with that given on $\overline{A}$ by (3.3). Hence $(\overline{A}, \Omega)$ embeds as a subreduct of the affine space $(G, \underline{R}, P)$. □

The affine space of Theorem 3.5 was defined over a ring depending on the particular cancellative mode (A, Ω). However, one may also embed cancellative τ-modes into affine spaces over the ring $R(M\tau)$ of (2.3), a ring that is independent of the particular τ-mode being embedded.

Corollary 3.6. *Let $\tau : \Omega \to \mathbb{Z}^+$ be a plural type. Then each cancellative τ-mode (A, Ω) embeds as a Ω-subreduct of an affine space $(G, \underline{R(M\tau)}, P)$ over the ring $R(M\tau)$ of (2.3), the Ω-operations on G being defined by (2.4).*

Proof. For x in $\overline{A}$ and ω in Ω, the equation (3.4) and the idempotence of $(\overline{A}, \Omega)$ yield

$$x \sum_{i=1}^{\omega\tau} e_i(\omega) = x. \tag{3.12}$$

Consider an element $g = (a, b)^\rho$ of G. By (3.10) and (3.12), one has

$$\begin{aligned} g \sum_{i=1}^{\omega\tau} \overline{e_i}(\omega) &= (a, b)^\rho \sum_{i=1}^{\omega\tau} \overline{e_i}(\omega) \\ &= \left(a \sum_{i=1}^{\omega\tau} e_i(\omega), b \sum_{i=1}^{\omega\tau} e_i(\omega) \right)^\rho \\ &= (a, b)^\rho = g, \end{aligned}$$

so that $\sum_{i=1}^{\omega\tau} \overline{e_i}(\omega) = 1$ in R. It follows that the unique commutative ring homomorphism $\mathbb{Z}[\text{hyp}\tau] \to R$ defined by $(\omega, i) \mapsto \overline{e_i}(\omega)$ for each (ω, i) in hypτ induces a unique ring homomorphism

$$R(M\tau) \to R. \tag{3.13}$$

The composite $R(M\tau) \to R \to \text{End}\,(G, +)$ then makes G into an affine space over $R(M\tau)$, and the Ω-operations on $(G, \underline{R(M\tau)}, P)$ defined by (2.4) coincide with those on $(G, \underline{R}, P)$ defined by (3.11). $\square$

4. Sums of algebras.

Let $\underline{\underline{\tau}}$ be the variety of τ-algebras (A, Ω) of a fixed type $\tau : \Omega \to \mathbb{Z}^+$. Let (I, Ω) be a τ-algebra. The binary relation $\preceq$ defined on I by

$$\{(i, j) | \exists x_1 \dots x_n t \in X\Omega \text{ and } i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \in I \text{ with } j = i_1 \dots i_{k-1} i i_{k+1} \dots i_n t\},$$

where $X\Omega$ is the free τ-algebra over X, is a quasi-order. As a quasi-ordered set $(I, \preceq)$ is a (small) category, where the elements of I are its objects and there is a morphism $i \to j$ precisely when $i \preceq j$. The relation $\preceq$ is called the *algebraic quasi-order* of (I, Ω). The algebra (I, Ω) is called *naturally quasi-ordered*, if for each ω in Ω and each $i = 1, \dots, \omega\tau$, if $a_i \preceq b_i$ then $a_1 \dots a_{\omega\tau}\omega \preceq b_1 \dots b_{\omega\tau}\omega$.

Let $\underline{\underline{\mathbb{I}}}$ be the class of idempotent naturally quasi-ordered τ-algebras. Let (I, Ω) be an $\underline{\underline{\mathbb{I}}}$-algebra with the algebraic quasi-order $\preceq$. For each i in I, let a τ-algebra (A_i, Ω) and an extension (E_i, Ω) be given. For $i \preceq j$ in $(I, \preceq)$, let $\varphi_{i,j} : (A_i, \Omega) \to (E_j, \Omega)$ be an Ω-homomorphism such that:

(L1) for each (n-ary) ω in Ω and $i_1, \ldots, i_n$ in I with $i_1 \ldots i_n \omega = i$, one has

$$(A_{i_1}\varphi_{i_1,i}) \ldots (A_{i_n}\varphi_{i_n,i})\omega \subseteq A_i;$$

(L2) for each $i_1 \ldots i_n \omega = i \preceq j$ in $(I, \preceq)$ one has

$$a_{i_1}\varphi_{i_1,i} \ldots a_{i_n}\varphi_{i_n,i}\omega\varphi_{i,j} = a_{i_1}\varphi_{i_1,j} \ldots a_{i_n}\varphi_{i_n,j}\omega,$$

where $a_{i_k} \in A_{i_k}$ for $k = 1, \ldots n$;

(L3) $a_i\varphi_{i,i} = a$;

(L4) $E_j = \{a_i\varphi_{i,j} | i \preceq j\}$.

Moreover, (E_j, Ω) is an envelope of (A_j, Ω), i.e. equality is the only congruence θ on (E_j, Ω) having nat $\theta|_{A_j}$ injective. Then the disjoint union $A = \biguplus\ (A_i | i \in I)$ equipped with the operations

$$\omega : A_{i_1} \times \cdots \times A_{i_n} \to A_i; (a_{i_1}, \ldots, a_{i_n}) \mapsto a_{i_1}\varphi_{i_1,i} \ldots a_{i_n}\varphi_{i_n,i}\omega \tag{4.1}$$

for all ω in Ω, all $(i_1, \ldots, i_n) \in I^n$ and $i = i_1 \ldots i_n \omega$, is called the *Lallement sum of the algebras* (A_i, Ω) *over the algebra* (I, Ω) *by the mappings* $\varphi_{i,j}$ or briefly a *Lallement sum* of the A_i. (This concept was in fact developed in [13], [14], [16] and called a "Lallement sum" in honor of Lallement who introduced this type of construction in the case of semigroups [5].)

Theorem 4.1 [16]. *Let (A, Ω) be a τ-algebra having a homomorphism onto an $\underline{\underline{\mathbb{I}}}$-algebra, with corresponding fibres (A_i, Ω) for $i \in I$. Then (A, Ω) is a Lallement sum of (A_i, Ω) over (I, Ω).* □

Let $\underline{\underline{Q}}$ and $\underline{\underline{R}}$ be subquasivarieties of a quasivariety $\underline{\underline{S}}$ of τ-algebras, with $\underline{\underline{R}}$ contained in $\underline{\underline{\mathbb{I}}}$. Then the Mal'cev product $(\underline{\underline{Q}} \circ \underline{\underline{R}}) \cap \underline{\underline{S}}$ consists of those $\underline{\underline{S}}$-algebras (A, Ω) having a quotient in $\underline{\underline{R}}$ and corresponding congruence classes in $\underline{\underline{Q}}$. This Mal'cev product is known to be a quasivariety [7].

Theorem 4.2 [16]. *Each $((\underline{\underline{Q}} \circ \underline{\underline{R}}) \cap \underline{\underline{S}})$-algebra is a Lallement sum of $\underline{\underline{Q}}$-algebras over an $\underline{\underline{R}}$-algebra.* □

A Lallement sum (A, Ω) is called a *functorial* (or an "Agassiz") *sum* if it satisfies the *functoriality condition*

$$a_i\varphi_{i,j}\varphi_{j,k} = a_i\varphi_{i,k} \tag{f}$$

for each i in I and a_i in A_i and $i \preceq j \preceq k$. The condition (f) implies the conditions (L1), (L2) and (L4). Moreover, the envelopes E_i coincide with A_i. In the case where the algebra

(I,Ω) is an $(\Omega$-) semilattice, the quasi-order $\preceq$ coincides with the semilattice order $\preceq$, and the functorial sum is called a *Płonka sum.* (See e.g. [10].) In the case of a functorial sum, the conditions (L3) and (f) together mean that there is a functor $\Phi : I \to \underline{\underline{\tau}}$ acting on morphisms as follows:

$$\begin{array}{ccc} j & & jF = (A_j,\Omega) \\ \uparrow & \mapsto & \uparrow \varphi_{ij} \\ i & & iF = (A_i,\Omega) \end{array} . \tag{4.2}$$

We also say that (A,Ω) is the *functorial sum* of the functor Φ.

Now let us return to modes. Let $\underline{\underline{KM\tau}}$ be the class of cancellative τ-modes, and let $\underline{\underline{Q}}$ be a quasivariety of naturally quasi-ordered τ-modes.

Theorem 4.3 [16]. *Let (A,Ω) be a τ-mode in the quasi-variety $(\underline{\underline{KM\tau}} \circ \underline{\underline{Q}}) \cap \underline{\underline{M\tau}}$ of τ-modes with a projection π onto a $\underline{\underline{Q}}$-algebra (I,Ω) and corresponding $\underline{\underline{KM\tau}}$-fibres (A_i,Ω). Then (A,Ω) embeds into a functorial sum of cancellative envelopes (E_i,Ω) of (A_i,Ω) over (I,Ω) by a functor Φ.* $\square$

5. Embedding sums of cancellative modes into functorial sums of affine spaces.

Consider the quasivariety $\underline{\underline{KM\tau}}$ of cancellative τ-modes, and any quasivariety $\underline{\underline{Q}}$ of naturally quasi-ordered τ-modes. By Theorem 4.2, a $\underline{\underline{KM\tau}} \circ \underline{\underline{Q}}$-mode (B,Ω) is a Lallement sum of $\underline{\underline{KM\tau}}$-modes (B_i,Ω) over a $\underline{\underline{Q}}$-mode (I,Ω). By Theorem 4.3, the mode (B,Ω) embeds into a functorial sum of cancellative envelopes (E_i,Ω) of (B_i,Ω) over (I,Ω) by a functor Φ. Corollary 3.6 showed that a single cancellative τ-mode embedded into the Ω-reduct of an affine space over the ring $R = R(M\tau)$ of (2.4). The main result of this final section shows that (B,Ω) embeds into a functorial sum of Ω-reducts of affine spaces over R.

Theorem 5.1. *Each $\underline{\underline{KM\tau}} \circ \underline{\underline{Q}}$-mode (B,Ω) embeds into a functorial sum (A,Ω) of Ω-reducts (A_i,Ω) of affine R-spaces over a $\underline{\underline{Q}}$-algebra (I,Ω).*

Proof. Consider the functor $U : \underline{\underline{R}} \to \underline{\underline{M\tau}}$ that maps on affine R-space $(A,\underline{\underline{R}},P)$ to its Ω-reduct (A,Ω) with operations given by (2.4). This functor preserves underlying sets. By [19, Cor. IV 3.4.8], it thus has a left adjoint $F : \underline{\underline{M\tau}} \to \underline{\underline{R}}$. For each i in I, define the affine R-space A_i to be the image $A_i = E_iF$ of the cancellative envelope E_i under the functor F. Now by Corollary 3.6, there is an embedding $z_i : E_i \to G_iU$ of (E_i,Ω) into the Ω-reduct $G_iU = (G_i,\Omega)$ of an affine R-space $(G_i,\underline{\underline{R}},P)$. Consider the component $\eta_i : E_i \to A_iU$ at E_i of the unit η of the adjunction between $U : \underline{\underline{R}} \to \underline{\underline{M\tau}}$ and $F : \underline{\underline{M\tau}} \to \underline{\underline{R}}$. Since η_i is initial in the comma category (E_i,U) [19, Theorem III 3.1.4], there is an $\underline{\underline{R}}$-homomorphism $\theta : A_i \to G_i$ such that $\eta_i\theta^U = Z_i$. It follows that $\eta_i : E_i \to A_iU$ embeds E_i as an Ω-subreduct of the affine space $(A_i,\underline{R},P)$.

Now consider the composite functor $\Phi F : I \to \underline{\underline{R}}$, and the corresponding functorial sum $A = \dot{\bigcup}_{i\in I} A_i$. The functorial sum $E = \dot{\bigcup}_{i\in I} E_i$ of Φ embeds into A via the disjoint union $\eta = \dot{\bigcup}_{i\in I} \eta_i$. Then for each ω in Ω and elements e_{i_j} of E_{i_j} for $i \le i_j \le \omega\tau$, one has $e_{i_1} \dots e_{i_{\omega\tau}} \omega \eta_{i_1 \dots i_{\omega\tau}\omega}$

$$
\begin{aligned}
&= e_{i_1}\varphi_{i_1, i_1\dots i_{\omega\tau}\omega} \dots e_{i_{\omega\tau}}\varphi_{i_{\omega\tau}, i_1\dots i_{\omega\tau}\omega}\omega\eta_{i_1\dots i_{\omega\tau}\omega}\\
&= e_{i_1}\varphi_{i_1, i_1\dots i_{\omega\tau}\omega}\eta_{i_1\dots i_{\omega\tau}\omega} \dots e_{i_{\omega\tau}}\varphi_{i_{\omega\tau}, i_1\dots i_{\omega\tau}\omega}\eta_{i_1\dots i_{\omega\tau}\omega}\omega\\
&= e_{i_1}\eta_{i_1}\varphi^{FU}_{i_1, i_1\dots i_{\omega\tau}\omega} \dots e_{i_{\omega\tau}}\eta_{i_{\omega\tau}}\varphi^{FU}_{i_{\omega\tau}, i_1\dots i_{\omega\tau}\omega}\omega\\
&= e_{i_1}\eta_{i_1} \dots e_{i_{\omega\tau}}\eta_{i_{\omega\tau}}\omega,
\end{aligned}
$$

so that the embedding $\eta : E \to A$ is an Ω-homomorphism, as required. □

Example 5.2. Let $\tau : I^\circ \to \{2\}$ be the type of barycentric algebras. Let $\underline{\underline{Q}}$ be the variety of I°-semilattices. Then Theorem 5.1 generalizes the embedding of barycentric algebras into Płonka sums of real affine spaces [15, Ex. 5.1]. □

Example 5.3. Let $\tau = \{(2, 2)\}$ be the type of groupoids. Let $\underline{\underline{Q}}$ be the variety of semilattices. Then Theorem 5.1 generalizes the embedding of commutative binary modes into Płonka sums of affine spaces over the ring $\mathbb{Z}[X]/\langle 1 - 2X\rangle$ of dyadic rationals [15, Ex. 5.3]. □

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