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DIRECTIONAL ALGEBRAS

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Abstract. Directional algebras are generalizations of dimonoids, which may themselves be regarded as directional semigroups. Given a constant-free type, a directional type is obtained by pointing to each of the arguments of the original, undirected type. For each axiomatization of a variety of algebras of constant-free type, a corresponding directional variety is determined. Dimonoids and digroups are shown to arise from the general procedure. For quasigroups, various choices of equational bases lead to various varieties of directional quasigroups. Under one natural axiomatization, the variety of quasigroups is shown to be *directionally complete*, in the sense that the corresponding directional variety is again the variety of quasigroups. Another axiomatization yields $(4+2)$-*quasigroups*. Digroups are equivalent to a certain class of $(4+2)$-quasigroups.

1. Introduction

As part of a programme addressing questions from algebraic K-theory and noncommutative geometry, J.-L. Loday introduced dimonoids $(S, \lhd, \rhd)$, with dialgebras as their linear analogues, breaking a single associative multiplication up into two separate binary operations [6]. Commutators $[x, y] = x \lhd y - y \rhd x$ of dialgebras lead to Leibniz algebras, as “noncommutative” (more strictly: non-antisymmetric) analogues of Lie algebras. While Lie’s Third Theorem integrates Lie algebras to Lie groups, no structure has yet been identified to give a comparable integration of Leibniz algebras. Nevertheless, there are potential candidates such as digroups (compare [3]).

The intention of the present paper is to examine dimonoids, digroups, and related algebras from the standpoint of universal algebra. In the interests of

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simplicity, attention is focussed purely on the set-based, "combinatorial" setting, with no explicit mention of linear analogues. An initial discussion of dimonoids is presented in Section 2, with an alternative construction of free dimonoids in Section 3.

The general treatment begins in Section 4. A constant-free type $\tau\colon \Omega \to \mathbb{Z}^+$ has a corresponding directional type $\tau_\uparrow\colon \Omega_\uparrow \to \mathbb{Z}^+$, where a new operator (ω, i) is associated to each operator ω of type τ and each argument slot $1 \le i \le \omega\tau$ in ω. For example, a single binary operator μ yields two directional operators: infix $\lhd$ or postfix $(\mu, 1)$, and infix $\rhd$ or postfix $(\mu, 2)$. Each τ-algebra (A, Ω) has a *directional version*, a $\tau_\uparrow$-algebra $(A, \Omega_\uparrow)$ in which each directional operation (ω, i) coincides with the undirected operation ω. The directional versions of τ-algebras are described as *essentially undirected* $\tau_\uparrow$-algebras. Other models of $\tau_\uparrow$-algebras are provided by the *projection algebras* of Definition 4.3, where each directional operation (ω, i) just projects onto the i-th argument.

In order to prepare for the analysis of identities, the derived directional type $\tau'_\uparrow$ is introduced in Section 5. Each algebra $(A, \Omega_\uparrow)$ of directional type $\tau_\uparrow$ yields an algebra $\left(A, \Omega'_\uparrow\right)$ of derived directional type $\tau'_\uparrow$ (Lemma 5.4), the algebra *derived* from $(A, \Omega_\uparrow)$. Algebras of type $\tau'_\uparrow$ are known as *derived directional τ-algebras*.

Section 6 is the keystone of the paper, providing the machinery to translate from an equational presentation Σ of a variety $\mathbf{V}$ of τ-algebras into an equational presentation of a variety $\mathbf{V}'_\uparrow[\Sigma]$ of derived directional τ-algebras. The equational presentation Σ is formulated in terms of *projectively regular* identities — regular identities between words in operators from Ω and the set of projections (6.1). Some immediate examples formulate commutativity and idempotence as projectively regular identities. The case of idempotence demonstrates the sensitivity of the translation process to the specific choice of the presentation Σ. Theorem 7.1 shows how the process outlined in Section 6 extracts dimonoids from semigroups.

The next two sections examine the various directional versions of quasigroups, *diquasigroups*, that arise from various axiomatizations of the variety $\mathbf{Q}$ of quasigroups by projectively regular identities. Section 8 takes the rather comprehensive set Σ of (8.7). Theorem 8.3 then shows that the corresponding variety $\mathbf{Q}'_\uparrow[\Sigma]$ is the variety of algebras derived from *EU-quasigroups*, which are essentially undirected. In other words, when axiomatized by the set Σ of (8.7), the variety of quasigroups has the property of being *directionally complete*, reemerging unchanged from the translation process of Section 6.

Section 9 axiomatizes the variety of quasigroups by the sets Σ_4 of (9.2) and Σ_6 of (9.3). These sets of projectively regular identities yield directional algebras that consist of separate left and right quasigroup structures on the same set. The basic structures are 4-*diquasigroups* $(Q, \lhd, \rhd, \swarrow, \searrow)$, consisting of a left quasigroup $(Q, \lhd, \swarrow)$ and a right quasigroup $(Q, \rhd, \searrow)$, while $\mathbf{Q}'_{\uparrow}[\Sigma_4]$ is the variety of algebras derived from (4+2)-*diquasigroups* $(Q, \lhd, \rhd, \swarrow, \nearrow, \nwarrow, \searrow)$, namely 4-diquasigroups $(Q, \lhd, \rhd, \swarrow, \searrow)$ with additional magma structures $(Q, \nearrow)$ and $(Q, \nwarrow)$.

Section 10 examines digroups $(Q, \lhd, \rhd, {}^{-1}, 1)$ within the (constant-free) context of the paper, using a constant unary operation 1 to select the so-called bar unit. Proposition 10.2 show how digroups are obtained from a projectively regular axiomatization (10.1) of the variety of groups. A digroup $(Q, \lhd, \rhd, {}^{-1}, 1)$ yields a $(4 + 2)$-quasigroup $(Q, \lhd, \rhd, \swarrow, \nearrow, \nwarrow, \searrow)$ with

$$x \swarrow y = x \lhd y^{-1},\ x \nearrow y = x \rhd y^{-1},\ y \nwarrow x = y^{-1} \lhd x \ \text{ and } y \searrow x = y^{-1} \rhd x\,.$$

Indeed, it transpires that digroups are completely equivalent to a certain class of $(4 + 2)$-diquasigroups (Theorem 10.8).

For algebras with a bilinear multiplication, two other approaches to the passage from undirected to directed identities have appeared in the literature. Chapoton used an endofunctor on the category of operads obtained by tensoring with so-called permutation or perm algebras, essentially right or left normal bands [1, 10]. On the other hand, Kolesnikov provided a translation procedure based on a connection between dialgebras and conformal algebras ([4], compare Example 2.7). Coordination of these varying approaches remains a topic for future research.

2. Dimonoids

Definition 2.1. A *dimonoid* or *directional semigroup* $(S, \lhd, \rhd)$ is an algebra with two associative multiplications $\lhd, \rhd$, known respectively as the *left* and *right directional multiplications*, such that the *internal associativity*

$$(x \rhd y) \lhd z = x \rhd (y \lhd z)$$

and *bar side irrelevance* identities

$$(x \rhd y) \rhd z = (x \lhd y) \rhd z\,, \quad x \lhd (y \lhd z) = x \lhd (y \rhd z)$$

are satisfied. Note that in the products $x \lhd y$ and $y \rhd x$, the variable y is said to be on the *bar side* [6, p.11].

Example 2.2 (Projection dimonoids)**.** [6, Ex. 1.3(b)] Suppose that X is a set. Then the left directional multiplication $x \lhd y = x$ and right directional multiplication $x \rhd y = y$ constitute a dimonoid $(X, \lhd, \rhd)$. Each element x of X yields a subdimonoid $\{x\}$ of $(X, \lhd, \rhd)$.

A dimonoid $(S, \lhd, \rhd)$ is said to be *undirected* if the identity $x \lhd y = x \rhd y$ is satisfied. Thus undirected dimonoids are just *stammered* semigroups with the multiplication appearing twice as a fundamental operation (compare [8, p.60]). Conversely, the congruence υ generated by the set

$$\{(x \lhd y, x \rhd y) \mid x, y \in S\}$$

of pairs of elements of a dimonoid $(S, \lhd, \rhd)$ yields a projection to the *semigroup replica* or *undirected replica* S^υ of the dimonoid S. The congruence υ itself is known as the *undirected replica congruence*.

Example 2.3 (Square dimonoids)**.** [6, Ex. 1.3(c)] Let $(S, \cdot)$ be a semigroup, with multiplication denoted by $\cdot$ or juxtaposition. Defining

$$(s,t) \lhd (u,v) = (s, tuv) \quad \text{and} \quad (s,t) \rhd (u,v) = (stu, v)$$

on S^2 yields a dimonoid $\left(S^2, \lhd, \rhd\right)$, known as the *square dimonoid* of the semigroup S.

Example 2.4 (Action dimonoids)**.** Let G be a semigroup, and let X be a right G-set. Defining

$$(g,x) \lhd (h,y) = (gh, xh) \quad \text{and} \quad (g,x) \rhd (h,y) = (gh, y)$$

on $G \times X$ yields a dimonoid $(G \times X, \lhd, \rhd)$ known as an *action dimonoid* or the *dimonoid of the action* (X, G). Compare [6, Ex. 1.3(d)] for the left-handed group version.

Proposition 2.5. *Let G be a monoid, and let X be a right G-set. Then the undirected replica $(G \times X, \lhd, \rhd)^\upsilon$ of the action dimonoid $(G \times X, \lhd, \rhd)$ is the semigroup reduct $(G, \cdot)$ of the monoid G.*

Proof. It will be shown that the undirected replica congruence υ is the kernel congruence $\ker \pi$ of the projection $\pi \colon G \times X \to G; (g,x) \mapsto g$. Let g be an element of G. Let x and y be elements of the set X. Then

$$\big((g,x),(g,y)\big) = \big((g,x) \lhd (1,y), (g,x) \rhd (1,y)\big) \in \upsilon\,,$$

so $\ker \pi \subseteq \upsilon$. Conversely, for elements g, h of G and x, y of X, one has

$$\big((g,x) \lhd (h,y), (g,x) \rhd (h,y)\big) = \big((gh, xh), (gh, y)\big) \in \ker \pi\,,$$

so $\upsilon \subseteq \ker \pi$. □

Remark 2.6. Description of the undirected replica of the action dimonoid for a general semigroup action appears to be more complicated.

Example 2.7 (Associative conformal algebras)**.** [4, (3)] A *conformal algebra* is an abelian group C that is equipped with an endomorphism

$$C \to C; a \mapsto a' \tag{2.1}$$

and a bilinear operation or *multiplication*

$$C^2 \to C; (a, b) \mapsto a\,\underline{n}\,b \tag{2.2}$$

for each natural number n. The endomorphism (2.1) of the group C is a derivation $\big(a\,\underline{n}\,b\big)' = a'\,\underline{n}\,b + a\,\underline{n}\,b'$ for each multiplication $\underline{n}$, while successive multiplications are connected by the identity $a'\,\underline{n}\,b = (-na)\,\underline{n-1}\,b$. Finally, the products (2.2) are *local* in the sense that

$$\forall\ a, b \in C,\ \exists\ N \in \mathbb{N}\,.\ \forall\ n \geq N\,,\ a\,\underline{n}\,b = 0\,.$$

A conformal algebra C is said to be *associative* if the identity

$$a\,\underline{n}\ (b\,\underline{m}\,c) = \sum_{s\in\mathbb{N}} \binom{n}{s} (a\,\underline{n-s}\,b)\ \underline{m+s}\,c$$

is satisfied for all natural numbers m and n. Then in an associative conformal algebra C whose abelian group reduct satisfies the divisibility condition

$$\forall\ x \in C\,,\ \forall\ 0 \neq n \in \mathbb{Z}\,,\ \exists\ y \in C\,.\ ny = x\,,$$

the definitions $a \triangleright b = a\,\underline{0}\,b$ and

$$a \triangleleft b = \sum_{s\in\mathbb{N}} \frac{(-1)^s}{s!}\,(a\,\underline{s}\,b)^{[s]} \tag{2.3}$$

yield a dimonoid $(C, \triangleleft, \triangleright)$. Note that the index $^{[s]}$ at the end of (2.3) denotes an s-fold application of the derivation (2.1).

3. Free dimonoids

Various constructions of free dimonoids have been given in the literature, including the original version of Loday [6, Cor. 1.8] and a more elegant version due to Zhuchok [13, §2]. This section presents an alternative construction, which may be viewed as a combinatorial version of a construction of Loday in the linear case [6, Th. 2.5].

Let X be a set. Recall that X^*, the set of words in the alphabet X, is the free monoid on X. The identity element 1 is the "empty" word of length 0, while the set X is inserted into X^* as the set of words of length 1. The multiplication is just the concatenation of words or strings.

Theorem 3.1. *For a set X, define $X^D = X^* \times X \times X^*$ and*

$$\eta_X \colon X \to X^D ; x \mapsto (1, x, 1)\,.$$

Then with the left directional multiplication

$$(u^-, x, u^+) \lhd (v^-, y, v^+) = (u^-, x, u^+ v^- y v^+)$$

and right directional multiplication

$$(u^-, x, u^+) \rhd (v^-, y, v^+) = (u^- x u^+ v^-, y, v^+)\,,$$

the set X^D forms the free dimonoid $(X^D, \lhd, \rhd)$ over the set X.

Proof. It is straightforward to verify that $(X^D, \lhd, \rhd)$ is a dimonoid (compare Remark 3.2). Let S be a dimonoid, the codomain of a function $f \colon X \to S$. In view of the associativity and internal associativity properties of the dimonoid multiplications, a function $\overline{f} \colon X^D \to S$ is defined unambiguously by

$$\begin{aligned}
&(u_l^- \dots u_1^-, x, u_1^+ \dots u_r^+)\overline{f} \\
&\quad = \begin{cases} xf & \text{if } l = r = 0\,; \\ (u_l^- f \rhd \cdots \rhd u_1^- f) \rhd xf & \text{if } l > 0, r = 0\,; \\ xf \lhd (u_1^+ f \lhd \cdots \lhd u_r^+ f) & \text{if } l = 0, r > 0\,; \\ (u_l^- f \rhd \cdots \rhd u_1^- f) \rhd xf \lhd (u_1^+ f \lhd \cdots \lhd u_r^+ f) & \text{if } l > 0, r > 0 \end{cases}
\end{aligned}$$

for $u_1^-, \dots, u_l^-, x, u_1^+, \dots u_r^+$ in X. Then certainly $\eta_X \overline{f} = f$. It is routine to check that $\overline{f} \colon X^D \to S$ is a dimonoid homomorphism. The uniqueness of $\overline{f}$ as a homomorphic extension of f follows from the equations

$$\begin{aligned}
&(1, u_{l+1}^-, 1) \rhd (u_l^- \rhd \cdots \rhd u_1^-, x, u_1^+ \lhd \cdots \lhd u_r^+) \\
&\qquad = (u_{l+1}^- \rhd u_l^- \rhd \cdots \rhd u_1^-, x, u_1^+ \lhd \cdots \lhd u_r^+)
\end{aligned}$$

and

$$\begin{aligned}
&(u_l^- \rhd \dots \rhd u_1^-, x, u_1^+ \lhd \cdots \lhd u_r^+) \lhd (1, u_{r+1}^+, 1) \\
&\qquad = (u_l^- \rhd \cdots \rhd u_1^-, x, u_1^+ \lhd \cdots \lhd u_r^+ \lhd u_{r+1}^+)
\end{aligned}$$

proved by induction on the natural numbers l and r respectively. □

Remark 3.2. Note that the projection $\pi \colon X^D \to X; (u^-, x, u^+) \mapsto x$ is a homomorphism from the free dimonoid X^D over X to the projection dimonoid $(X, \lhd, \rhd)$ of Example 2.2. Then for each element x of X, the preimage $\pi^{-1}\{x\}$ forms a subdimonoid of X^D.

4. Directional types

If the codomain Y of a function $f\colon X \to Y$ forms a poset $(Y, \leq)$, then the *apograph* of f is the subset

$$\{(x, y) \in X \times Y \mid y \leq xf\}$$

of $X \times Y$.

Definition 4.1 (The directional type of a constant-free type)**.** Let $\tau\colon \Omega \to \mathbb{Z}^+$ be a constant-free type (thus with the well-ordered set $\mathbb{Z}^+$ of positive integers as its codomain). Consider the apograph $\Omega_\uparrow$ of $\tau\colon \Omega \to \mathbb{Z}^+$. Then the function

$$\tau_\uparrow\colon \Omega_\uparrow \to \mathbb{Z}^+; (\omega, i) \mapsto \omega\tau$$

is the *directional type* of the constant-free type τ. Elements (ω, i) of $\Omega_\uparrow$ are known as *directional operators*, and $\tau_\uparrow$-algebras are *directional* τ-algebras.

Example 4.2. For the magma type $\tau = \{(\mu, 2)\}$, the directional type $\tau_\uparrow$ is the dimonoid type $\tau_\uparrow\colon \lhd \mapsto 2, \rhd \mapsto 2$. Formally, $\lhd = (\mu, 1)$ and $\rhd = (\mu, 2)$ in $\Omega_\uparrow$.

Definition 4.3. Let $\tau\colon \Omega \to \mathbb{Z}^+$ be a constant-free type, with directional type $\tau_\uparrow\colon \Omega_\uparrow \to \mathbb{Z}^+; (\omega, i) \mapsto \omega\tau$. A $\tau_\uparrow$-algebra $(A, \Omega_\uparrow)$ is a *projection* $\tau_\uparrow$*-algebra* if

$$a_1 \dots a_{\omega\tau}(\omega, i) = a_i \tag{4.1}$$

for all $\omega \in \Omega$, $1 \leq i \leq \omega\tau$, and $a_1, \dots, a_{\omega\tau} \in A$.

If τ is the magma type of Example 4.2, then the projection $\tau_\uparrow$-algebras are the projection dimonoids of Example 2.2. Indeed, the construction of that example generalizes as follows.

Proposition 4.4. *Let $\tau\colon \Omega \to \mathbb{Z}^+$ be a constant-free type, with directional type $\tau_\uparrow\colon \Omega_\uparrow \to \mathbb{Z}^+; (\omega, i) \mapsto \omega\tau$. Let A be a set. Then the specifications* (4.1) *yield a projection $\tau_\uparrow$-algebra $(A, \Omega_\uparrow)$.*

Definition 4.5. Let $\tau\colon \Omega \to \mathbb{Z}^+$ be a constant-free type, with directional type $\tau_\uparrow\colon \Omega_\uparrow \to \mathbb{Z}^+; (\omega, i) \mapsto \omega\tau$.

(a) Let (A, Ω) be a τ-algebra. Then the $\tau_\uparrow$-algebra $(A, \Omega_\uparrow)$ with

$$a_1 \dots a_{\omega\tau}(\omega, i) = a_1 \dots a_{\omega\tau}\omega \tag{4.2}$$

for $\omega \in \Omega$, $1 \leq i \leq \omega\tau$, and $a_1, \dots, a_{\omega\tau} \in A$ is called the *directional version* of (A, Ω).

(b) A $\tau_\uparrow$-algebra $(A, \Omega_\uparrow)$ is said to be an *essentially undirected* $\tau_\uparrow$*-algebra* if it is the directional version of a τ-algebra (A, Ω).

Example 4.6. Stammered semigroups are essentially undirected $\tau_\uparrow$-algebras for the magma type τ of Example 4.2.

5. Derived directional types

Suppose that
$$\tau\colon \Omega \to \mathbb{Z}^+$$
is a constant-free type. The *derived directional type*
$$\tau'_\uparrow\colon \Omega'_\uparrow \to \mathbb{Z}^+$$
is defined recursively. The recursive basis has projections (π^n_i, i) with

$$\tau'_\uparrow\colon (\pi^n_i, i) \mapsto n \tag{5.1}$$

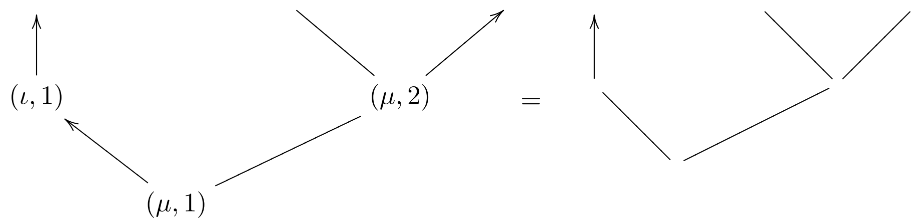
for $1 \le i \le n \in \mathbb{Z}$, including $(\iota, 1) = (\pi^1_1, 1)$ corresponding to a unary identity function ι, and
$$\tau'_\uparrow\colon (\omega, i) \mapsto \omega\tau$$
for directional operators $(\omega, i) \in \Omega_\uparrow$. The recursive step is
$$\tau'_\uparrow\colon \Big(u_1 \dots u_{\omega\tau}\omega, i_j + \sum_{i<j} u_i\tau'_\uparrow\Big) \mapsto \sum_{i=1}^{\omega\tau} u_i\tau'_\uparrow$$
with

$$\Big(u_1 \dots u_{\omega\tau}\omega, i_j + \sum_{i<j} u_i\tau'_\uparrow\Big) = (u_1, i_1)\dots(u_{\omega\tau}, i_{\omega\tau})(\omega, j) \tag{5.2}$$

for $(u_1, i_1), \dots, (u_{\omega\tau}, i_{\omega\tau}) \in \Omega'_\uparrow$ and $(\omega, j) \in \Omega_\uparrow$. The set $\Omega'_\uparrow$ that is recursively defined as the domain of the directional derived type is known as the *derived directional operator domain.*

Example 5.1. Consider the magma type $\tau = \{(\mu, 2)\}$ of Example 4.2 with directional, dimonoid type
$$\tau_\uparrow\colon\ \lhd \mapsto 2,\ \rhd \mapsto 2\,,$$
i.e. $\lhd = (\mu, 1)$ and $\rhd = (\mu, 2)$ in $\Omega_\uparrow$. The recursion basis comprises the projections
$$(\iota, 1) = (\pi^1_1, 1), \quad (\pi^2_1, 1),\ (\pi^2_2, 2), \quad (\pi^3_1, 1),\ (\pi^3_2, 2),\ (\pi^3_3, 3), \quad \dots$$
and the two directional operators $(\mu, 1)$ and $(\mu, 2)$. Then the first recursive step generates operators such as
$$(\iota, 1)(\mu, 2)(\mu, 1) = (x_1\iota x_2 x_3\mu\mu, 1)$$

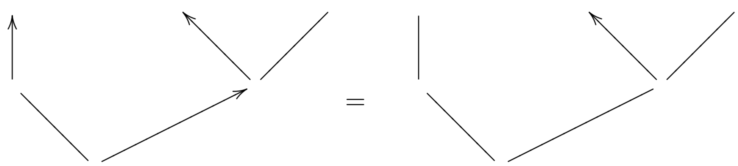
with $j = 1$, $i_1 = 1$, and $i_j + \sum_{i<j} u_i\tau'_{\uparrow} = 1 + 0 = 1$ in the notation of (5.2), represented graphically by the parsing trees



or

$$(\iota, 1)(\mu, 1)(\mu, 2) = (x_1\iota x_2 x_3\mu\mu, 2)$$

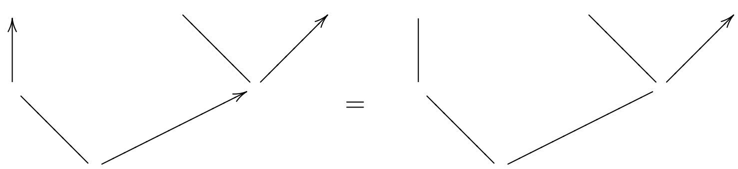
with $j = 2$, $i_2 = 1$, $(\iota, 1)\tau'_{\uparrow} = 1$, $i_j + \sum_{i<j} u_i\tau'_{\uparrow} = 1+1 = 2$, represented graphically (in slightly more abbreviated form) by



or

$$(\iota, 1)(\mu, 2)(\mu, 2) = (x_1\iota x_2 x_3\mu\mu, 3)$$

with $j = 2$, $i_2 = 2$, $(\iota, 1)\tau'_{\uparrow} = 1$, $i_j + \sum_{i<j} u_i\tau'_{\uparrow} = 2+1 = 3$, represented graphically by



Note how one follows the arrows up from the root of the parsing tree in order to locate the argument corresponding to the sum $i_j + \sum_{i<j} u_i\tau'_{\uparrow}$ in (5.2).

Definition 5.2. For a constant-free type $\tau\colon \Omega \to \mathbb{Z}^+$ with derived directional type $\tau'_{\uparrow}\colon \Omega'_{\uparrow} \to \mathbb{Z}^+$, a derived directional operator is said to *undirect* according to the recursive scheme with basis

$$(\pi_i^n, i) \mapsto \big(\pi_i^n\colon (x_1, \dots, x_n) \mapsto x_i\big) \tag{5.3}$$

for $1 \leq i \leq n \in \mathbb{Z}$ and

$$(\omega, i) \mapsto \omega$$

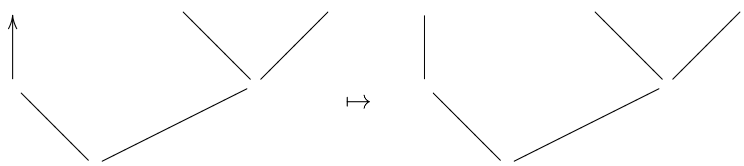
for $\omega \in \Omega$, and recursive step

$$\Big(u_1 \dots u_{\omega\tau}\omega, i_j + \sum_{i<j} u_i \tau'_{\uparrow}\Big) \mapsto u_1 \dots u_{\omega\tau}\omega$$

for $(u_1, i_1), \dots, (u_{\omega\tau}, i_{\omega\tau}) \in \Omega'_{\uparrow}$ and $(\omega, j) \in \Omega_{\uparrow}$.

Example 5.3. For the magma type $\tau = \{(\mu, 2)\}$ of Examples 4.2 and 5.1, the diagram

(5.4)



provides a graphical representation of the undirection from $(x_1 \iota x_2 x_3 \mu\mu, 1)$ to $x_1 \iota x_2 x_3 \mu\mu$.

Lemma 5.4. *Let $\tau \colon \Omega \to \mathbb{Z}^+$ be a constant-free type. Let $(A, \Omega_{\uparrow})$ be an algebra of directional type $\tau_{\uparrow}$. Then the recursive specification*

$$\begin{aligned} & a_1 \dots a_{\sum_{1 \leq i \leq \omega\tau} u_i \tau'_{\uparrow}} \Big(u_1 \dots u_{\omega\tau}\omega, i_j + \sum_{i<j} u_i \tau'_{\uparrow}\Big) \\ &\quad = \big(a_1 \dots a_{u_1 \tau'_{\uparrow}}(u_1, i_1)\big) \dots \big(a_{1+\sum_{1 \leq i < \omega\tau} u_i \tau'_{\uparrow}} \dots a_{\sum_{1 \leq i \leq \omega\tau} u_i \tau'_{\uparrow}}(u_{\omega\tau}, i_{\omega\tau})\big)(\omega, j) \end{aligned}$$

for $(u_1, i_1), \dots, (u_{\omega\tau}, i_{\omega\tau}) \in \Omega'_{\uparrow}$ and $(\omega, j) \in \Omega_{\uparrow}$, with the projections being interpreted according to (5.3)*, yields an algebra $\big(A, \Omega'_{\uparrow}\big)$ of derived directional type $\tau'_{\uparrow}$.*

Definition 5.5. In the context of Lemma 5.4, the algebra $\big(A, \Omega'_{\uparrow}\big)$ is called the algebra *derived* from $(A, \Omega_{\uparrow})$.

Example 5.6. If $(A, \Omega_{\uparrow})$ is a projection $\tau_{\uparrow}$-algebra, then one has

$$a_1 \dots a_{u\tau'_{\uparrow}}(u, i) = a_i$$

for all $(u, i) \in \Omega'_{\uparrow}$ and $a_1, \dots, a_{u\tau'_{\uparrow}} \in A$ in the algebra $\big(A, \Omega'_{\uparrow}\big)$ derived from $(A, \Omega_{\uparrow})$.

6. Identities of directional algebras

Let $\tau\colon \Omega \to \mathbb{Z}^+$ be a constant-free type. Let

$$\Pi = \{\pi_i^n\colon (x_1,\dots,x_n) \mapsto x_i \mid 1 \le i \le n \in \mathbb{Z}\} \tag{6.1}$$

be the full set of projections. Then an identity

$$x_1 \dots x_m u = y_1 \dots y_n v$$

between words in $\Omega \cup \Pi$ is said to be *projectively regular* if there is an equality

$$\{x_1,\dots,x_m\} = \{y_1,\dots,y_n\}$$

between the corresponding argument sets on each side of the identity.

Definition 6.1. Let $\tau\colon \Omega \to \mathbb{Z}^+$ be a constant-free type.

(a) An algebra $(A, \Omega'_\uparrow)$ of derived directional type $\tau'_\uparrow$ is said to be a *derived directional τ-algebra*.

(b) Let $\mathbf{V}$ or $\mathbf{V}[\Sigma]$ be a variety of τ-algebras presented by a specific set Σ of projectively regular identities, which may include redundant identities. In this context, the elements of Σ are known as the *basic identities*, while $\mathbf{V}$ is said to be the *undirected variety*. Then the *derived directional variety* $\mathbf{V}'_\uparrow$ or $\mathbf{V}'_\uparrow[\Sigma]$ is defined as the variety of derived directional τ-algebras defined by those identities

$$x_1 \dots x_{(u,i)\tau'_\uparrow}(u,i) = y_1 \dots y_{(v,i)\tau'_\uparrow}(v,i)\,, \tag{6.2}$$

with $(u,i),(v,i)$ in $\Omega'_\uparrow$, for which

$$x_1 \dots x_{(u,i)\tau'_\uparrow} u = y_1 \dots y_{(v,i)\tau'_\uparrow} v \tag{6.3}$$

is a basic identity from Σ for the undirected variety $\mathbf{V}[\Sigma]$.

(c) In (b), the identity (6.2) is said to *undirect* to the identity (6.3).

Example 6.2. Let $\tau = \{(\mu,2)\}$ be the magma type (Example 4.2). If the commutative law $xy\mu = yx\mu$ is a basic identity, as an instance of (6.3), then the corresponding derived directional identities (6.2) are $xy(\mu,1) = yx(\mu,1)$ and $xy(\mu,2) = yx(\mu,2)$, or $x \lhd y = y \lhd x$ and $x \rhd y = y \rhd x$ in infix notation. These are the commutativity identities for dimonoids as understood by Zhuchok [11, 12]. They are not satisfied by non-trivial projection dimonoids.

The following example gives an immediate illustration of how the derived directional variety $\mathbf{V}'_\uparrow$ may depend on the specific presentation of $\mathbf{V}$ by the set Σ of projectively regular basic identities in the context of Definition 6.1.

Example 6.3. Again, take $\tau = \{(\mu, 2)\}$ to be the magma type. Suppose that the idempotent law $xx\mu = x$ is interpreted as the basic identity $xx\mu = x\iota$ or $xx\mu = xx\pi_1^2$. Then the only corresponding derived directional identity is $xx(\mu, 1) = x(\iota, 1)$, or $x \triangleleft x = x$ in infix notation. On the other hand, a second basic interpretation of idempotence as $xx\mu = xx\pi_2^2$ has a unique corresponding derived directional identity $xx(\mu, 2) = xx(\pi_2^2, 2)$, or $x \triangleright x = x$ in infix notation. In this way, one recovers the two idempotent identities $x \triangleleft x = x = x \triangleright x$ for dimonoids as understood by Zhuchok [13, p.197].

Proposition 6.4. *For a given constant-free type $\tau\colon \Omega \to \mathbb{Z}^+$, suppose that $\mathbf{V}$ is a variety of τ-algebras defined by basic identities (6.3). Suppose that $\mathbf{V}'_\uparrow$ is the corresponding derived directional variety. Let (A, Ω) be an algebra in $\mathbf{V}$. Then the algebra derived from the directed version of (A, Ω) lies in $\mathbf{V}'_\uparrow$.*

Proof. Each derived identity (6.2) holds in the algebra that is derived from the directed version of (A, Ω), since the corresponding undirected basic identity (6.3) holds in the $\mathbf{V}$-algebra (A, Ω). □

7. Directional semigroups

As a "proof of concept," the following result shows how dimonoids emerge from semigroups as the corresponding derived directional variety.

Theorem 7.1. *Let $\tau = \{(\mu, 2)\}$ be the magma type. Let $\mathbf{Sgp}$ be the variety of semigroups, presented by the associative law. Then the derived directional variety $\mathbf{Sgp}'_\uparrow$ is the variety of algebras derived from dimonoids.*

Proof. The unique basic identity for $\mathbf{Sgp}$ is the associative identity

$$x_1x_2\mu x_3\iota\mu = x_1\iota x_2x_3\mu\mu\,, \tag{7.1}$$

written using the identical unary operation ι. The derived directed operators that undirect to $x_1x_2\mu x_3\iota\mu$ are

$$(x_1x_2\mu x_3\iota\mu, 1)\,,\ (x_1x_2\mu x_3\iota\mu, 2)\,,\ (x_1x_2\mu x_3\iota\mu, 3)\,,$$

while the derived directed operators that undirect to $x_1\iota x_2x_3\mu\mu$ are

$$(x_1\iota x_2x_3\mu\mu, 1)\,,\ (x_1\iota x_2x_3\mu\mu, 2)\,,\ (x_1\iota x_2x_3\mu\mu, 3)$$

(compare (5.4) in Example 5.3 for the first of these latter operators). Thus in the language of derived directed operators, the basic identities for $\mathbf{Sgp}'_\uparrow$ are the three identities

$$(x_1x_2\mu x_3\iota\mu, 1) = (x_1\iota x_2x_3\mu\mu, 1)\,, \tag{7.2}$$

$$(x_1x_2\mu x_3\iota\mu, 2) = (x_1\iota x_2x_3\mu\mu, 2)\,, \text{ and} \tag{7.3}$$

$$(x_1x_2\mu x_3\iota\mu, 3) = (x_1\iota x_2x_3\mu\mu, 3) \tag{7.4}$$

that undirect to the associative law (7.1). The case of (7.3) is the simplest to consider, since it has a unique interpretation

$$x_1x_2(\mu, 2)x_3(\mu, 1) = x_1x_2x_3(\mu, 1)(\mu, 2)$$

in the language of directional τ-algebras, becoming the internal associativity

$$(x_1 \rhd x_2) \lhd x_3 = x_1 \rhd (x_2 \lhd x_3)$$

in the language of dimonoids. Now while the left-hand side of (7.2) has a unique interpretation

$$x_1x_2(\mu, 1)x_3(\mu, 1)$$

in directional τ-algebras, the right-hand side has the two interpretations

$$x_1x_2x_3(\mu, 1)(\mu, 1) \quad \text{and} \quad x_1x_2x_3(\mu, 2)(\mu, 1)$$

there. The identity (7.2) thus yields the identities

$$x_1x_2(\mu, 1)x_3(\mu, 1) = x_1x_2x_3(\mu, 1)(\mu, 1)$$

and

$$x_1x_2(\mu, 1)x_3(\mu, 1) = x_1x_2x_3(\mu, 2)(\mu, 1)$$

in the language of directional τ-algebras, the associative identity

$$(x_1 \lhd x_2) \lhd x_3 = x_1 \lhd (x_2 \lhd x_3) \tag{7.5}$$

and the identity

$$(x_1 \lhd x_2) \lhd x_3 = x_1 \lhd (x_2 \rhd x_3) \tag{7.6}$$

in the language of dimonoids. While (7.6) was one of the original dimonoid identities given by Loday [6, §1.1 2], it is replaced by the bar side irrelevance $x_1 \lhd (x_2 \lhd x_3) = x_1 \lhd (x_2 \rhd x_3)$ in the presence of the associative law (7.5). The identity (7.4) is treated in dual fashion to (7.2). □

8. EU-Directional quasigroups

A *quasigroup* $(Q,\cdot,/,\backslash)$ is an algebra with three binary operations, the *multiplication* $\cdot$ and the *right* and *left divisions* $/,\backslash$, such that the identities

$$y\backslash(y\cdot x)=x=(x\cdot y)/y \quad \text{and} \tag{8.1}$$

$$y\cdot(y\backslash x)=x=(x/y)\cdot y \tag{8.2}$$

are satisfied. The respective quasigroup operations $\cdot,/,\backslash$ are written as μ,ρ,λ in postfix notation, with corresponding directional operations

$$(\mu,1)=\lhd\,,\ (\mu,2)=\rhd\,,\ (\rho,1)=\swarrow\,,\ (\rho,2)=\nearrow\,,\ (\lambda,1)=\nwarrow\,,\ (\lambda,2)=\searrow\ .$$

Definition 8.1. An *EU-diquasigroup* $(Q,\lhd,\rhd,\swarrow,\nearrow,\nwarrow,\searrow)$ is an algebra equipped with 6 binary operations, such that the identities

$$y\nwarrow(y\lhd x)=y\searrow(y\lhd x)=y\searrow(y\rhd x)=x\,, \tag{8.3}$$

$$(x\lhd y)\swarrow y=(x\rhd y)\swarrow y=(x\rhd y)\nearrow y=x\,, \tag{8.4}$$

$$y\lhd(y\nwarrow x)=y\rhd(y\nwarrow x)=y\rhd(y\searrow x)=x \quad \text{and} \tag{8.5}$$

$$(x\swarrow y)\lhd y=(x\nearrow y)\lhd y=(x\nearrow y)\rhd y=x \tag{8.6}$$

are satisfied.

The designation "EU" in Definition 8.1 arises as follows.

Proposition 8.2. *Each EU-diquasigroup is essentially undirected.*

Proof. Recall that for a magma $(Q,\circ)$, one has the *left multiplications*

$$L_\circ(q)\colon Q\to Q; x\mapsto q\circ x$$

and *right multiplications*

$$R_\circ(q)\colon Q\to Q; x\mapsto x\circ q$$

for elements q of Q. Let $(Q,\lhd,\rhd,\swarrow,\nearrow,\nwarrow,\searrow)$ be an EU-diquasigroup. Now the outside equalities in (8.3) and (8.5) show that each left multiplication $L_\nwarrow(q)$ is invertible, with inverse $L_\lhd(q)$. On the other hand, equality of the second and final terms in (8.5) shows that $L_\rhd(q)$ is also an inverse for $L_\nwarrow(q)$. Thus $L_\lhd(q)=L_\rhd(q)$ for all q in Q, so the two directed multiplications $(\mu,1)$ and $(\mu,2)$ coincide. Next, note how (8.3) shows that both $L_\nwarrow(q)$ and $L_\searrow(q)$ are inverses for $L_\lhd(q)$. It follows that the two directional left divisions $(\lambda,1)$ and $(\lambda,2)$ coincide. Finally, since the definition of EU-diquasigroups is self-dual between left and right, a dual argument shows that the two directional right divisions $(\rho,1)$ and $(\rho,2)$ coincide. □

Theorem 8.3. *Let $\tau = \{\mu, \rho, \lambda\} \times \{2\}$ be the quasigroup type. Let* $\mathbf{Q}$ *be the variety of quasigroups, presented by the following set Σ of projectively regular identities:*

$$\Sigma = \begin{cases} y\,yx\mu\lambda = xyy\pi_1^3 = yxy\pi_2^3 = yyx\pi_3^3\,, \\ xy\mu y\rho = xyy\pi_1^3 = yxy\pi_2^3 = yyx\pi_3^3\,, \\ y\,yx\lambda\mu = xyy\pi_1^3 = yxy\pi_2^3 = yyx\pi_3^3\,, \\ xy\rho y\mu = xyy\pi_1^3 = yxy\pi_2^3 = yyx\pi_3^3\,. \end{cases} \tag{8.7}$$

Then the derived directional variety $\mathbf{Q}'_{\uparrow}[\Sigma]$ is the variety of algebras derived from EU-diquasigroups.

Proof. It will suffice to consider the case of the quasigroup identities in the first line of (8.7), since the remaining identities in Σ all have the same general format. For $1 \le i \le 3$, the only derived directed operator that undirects to π_i^3 is (π_i^3, i). Thus the basic identities for $\mathbf{Q}'_{\uparrow}$ that undirect to the identities in the first line of (8.7) are

$$\begin{aligned} &y\,yx(\mu,1)(\lambda,1) = xyy(\pi_1^3,1)\,, \\ &y\,yx(\mu,1)(\lambda,2) = yxy(\pi_2^3,2)\,, \text{ and} \\ &y\,yx(\mu,2)(\lambda,2) = yyx(\pi_3^3,3)\,, \end{aligned}$$

or $y \nwarrow (y \lhd x) = x$, $y \searrow (y \lhd x) = x$, and $y \searrow (y \rhd x) = x$. These are the three EU-diquasigroup identities in (8.3) of Definition 8.1. The remaining nine are obtained in similar fashion. $\square$

A projectively regular presentation $\mathbf{V}[\Sigma]$ is said to be *directionally complete* if each algebra in $\mathbf{V}[\Sigma]'_{\uparrow}$ is essentially undirected. Theorem 9.5 shows that the projectively regular presentation $\mathbf{Q}[\Sigma]$ of the variety $\mathbf{Q}$ of quasigroups given by (8.7) is directionally complete.

9. n-Directional quasigroups

A quasigroup $(Q, \cdot, /, \backslash)$ satisfies the additional identities

$$y/(x\backslash y) = x = (y/x)\backslash y \tag{9.1}$$

[9, p.6]. A *right quasigroup* $(Q, \cdot, /)$ is an algebra with a binary multiplication and right division satisfying the right-hand identities in (8.1), (8.2). Dually, a *left quasigroup* $(Q, \cdot, \backslash)$ is an algebra equipped with a binary multiplication and left division satisfying the left-hand identities in (8.1), (8.2). The opposite of $(\lambda, 1)$ is written with infix notation as $x \nwarrow^{\mathsf{op}} y = y \nwarrow x$.

Definition 9.1. (a) A 4-*diquasigroup* $(Q, \lhd, \rhd, \swarrow, \searrow)$ is an algebra with 4 binary operations, such that the identities

$$y \searrow (y \rhd x) = x = (x \lhd y) \swarrow y \quad \text{and}$$
$$y \rhd (y \searrow x) = x = (x \swarrow y) \lhd y$$

are satisfied.

(b) A $(4+2)$-*diquasigroup* $(Q, \lhd, \rhd, \swarrow, \nearrow, \nwarrow, \searrow)$ is an algebra equipped with 6 binary operations, such that the identities

$$y \searrow (y \rhd x) = x = (x \lhd y) \swarrow y \quad \text{and}$$
$$y \rhd (y \searrow x) = x = (x \swarrow y) \lhd y$$

are satisfied. In other words, the reduct $(Q, \lhd, \rhd, \swarrow, \searrow)$ is a 4-quasigroup, while the reducts $(Q, \nearrow)$ and $(Q, \nwarrow)$ are arbitrary magmas.

(c) A 6-*diquasigroup* $(Q, \lhd, \rhd, \swarrow, \nearrow, \nwarrow, \searrow)$ is an algebra equipped with 6 binary operations, such that the identities

$$y \searrow (y \rhd x) = x = (x \lhd y) \swarrow y\,,$$
$$y \rhd (y \searrow x) = x = (x \swarrow y) \lhd y \quad \text{and}$$
$$y \nearrow (x \nwarrow y) = x = (y \nearrow x) \nwarrow y$$

are satisfied.

Proposition 9.2. (a) *The algebra* $(Q, \lhd, \rhd, \swarrow, \searrow)$ *is a* 4-*diquasigroup if and only if the set* Q *carries a right quasigroup structure* $(Q, \lhd, \swarrow)$ *and a left quasigroup structure* $(Q, \rhd, \searrow)$.

(b) *The algebra* $(Q, \lhd, \rhd, \swarrow, \nearrow, \nwarrow, \searrow)$ *is a* 6-*diquasigroup if and only if the set* Q *carries a right quasigroup structure* $(Q, \lhd, \swarrow)$ *and left quasigroup structures* $(Q, \rhd, \searrow)$, $(Q, \nwarrow^{\mathsf{op}}, \nearrow)$.

Example 9.3. The directed versions of quasigroups given in Definition 9.1 admit models on any set Q with

$$x \lhd y = x \swarrow y = x \nwarrow y = x \quad \text{and } x \rhd y = x \nearrow y = x \searrow y = y$$

(compare the projection dimonoids of Example 2.2).

Example 9.4. By Proposition 6.4, the directed versions of quasigroups given in Definition 9.1 admit models with

$$x \lhd y = x \rhd y = x \cdot y\,,\ x \swarrow y = x \nearrow y = x/y\,,\ \text{and } x \nwarrow y = x \searrow y = x\backslash y$$

on any quasigroup $(Q, \cdot, /, \backslash)$.

Theorem 9.5. *Let* $\tau = \{\mu, \rho, \lambda\} \times \{2\}$ *be the quasigroup type.*

(a) *Let* **Q** *be the variety of quasigroups defined with basic identities from the set*

$$\Sigma_4 = \begin{cases} y\,yx\mu\lambda = yyx\pi^3_3\,, \\ xy\mu y\rho = xyy\pi^3_1\,, \\ y\,yx\lambda\mu = yyx\pi^3_3\,, \\ xy\rho y\mu = xyy\pi^3_1\,. \end{cases} \tag{9.2}$$

The corresponding derived directional variety $\mathbf{Q}'_{\uparrow}[\Sigma_4]$ *coincides with the variety of algebras derived from* $(4+2)$*-diquasigroups.*

(b) *Let* **Q** *be the variety of quasigroups defined with basic identities from the following set:*

$$\Sigma_6 = \begin{cases} y\,yx\mu\lambda = yyx\pi^3_3\,, \\ xy\mu y\rho = xyy\pi^3_1\,, \\ y\,yx\lambda\mu = yyx\pi^3_3\,, \\ xy\rho y\mu = xyy\pi^3_1\,, \\ y\,yx\lambda\rho = yxy\pi^3_2\,, \\ yx\rho y\lambda = yxy\pi^3_2\,. \end{cases} \tag{9.3}$$

Then the corresponding derived directional variety $\mathbf{Q}'_{\uparrow}[\Sigma_6]$ *is the variety of algebras derived from* 6*-diquasigroups.*

Proof. It will suffice to consider the case of a single quasigroup identity, the top identity

$$y\,yx\mu\lambda = yyx\pi^3_3 \tag{9.4}$$

of (9.2) or (9.3), since the remaining five identities in (9.2) or (9.3) all have the same format. The only derived directed operator that undirects to π^3_3 is $(\pi^3_3, 3)$, so the only basic identity for $\mathbf{Q}'_{\uparrow}$ that undirects to (9.4) is

$$y\,yx(\mu,2)(\lambda,2) = yyx(\pi^3_3, 3)$$

or $y \searrow (y \rhd x) = x$. □

Proposition 9.6. *Let G and H be quasigroups. Let X be a set, with functions $R\colon G \to X!$ and $L\colon H \to X!$ to the group $X!$ of bijections of the set X. Define:*

$$\begin{aligned}(g_1, x_1, h_1) \lhd (g_2, x_2, h_2) &= (g_1 g_2, x_1 R(g_2), h_1 h_2)\,;\\ (g_1, x_1, h_1) \rhd (g_2, x_2, h_2) &= (g_1 g_2, x_2 L(h_1), h_1 h_2)\,;\\ (g_1, x_1, h_1) \swarrow (g_2, x_2, h_2) &= (g_1/g_2, x_1 R(g_2)^{-1}, h_1/h_2)\,;\\ (g_1, x_1, h_1) \searrow (g_2, x_2, h_2) &= (g_1\backslash g_2, x_2 L(h_1)^{-1}, h_1\backslash h_2)\end{aligned}$$

on $G \times X \times H$. Then $(G \times X \times H, \lhd, \rhd, \swarrow, \searrow)$ is a 4-diquasigroup.

Proof. Consider (g_i, x_i, h_i) in $G \times X \times H$, for $i = 1, 2$. Then

$$\begin{aligned}(g_2,&x_2, h_2) \searrow \big((g_2, x_2, h_2) \rhd (g_1, x_1, h_1)\big)\\ &= (g_2, x_2, h_2) \searrow (g_2 g_1, x_1 L(h_2), h_2 h_1)\\ &= (g_2\backslash(g_2 g_1), x_1 L(h_2) L(h_2)^{-1}, h_2\backslash(h_2 h_1)) = (g_1, x_1, h_1)\,,\end{aligned}$$

verifying the first of the identities in Definition 9.1(a). The remaining three identities are checked in similar fashion. □

Corollary 9.7. *Let G and H be quasigroups. Let (X, e) be a pointed set, with functions $R\colon G \to X!$ and $L\colon H \to X!$. Define:*

$$\begin{aligned}(g_1, x_1, h_1) \nearrow (g_2, x_2, h_2) &= (g_1/g_2, e, h_1/h_2)\,;\\ (g_1, x_1, h_1) \nwarrow (g_2, x_2, h_2) &= (g_1\backslash g_2, e, h_1\backslash h_2)\end{aligned}$$

on $G \times X \times H$. Then $(G \times X \times H, \lhd, \rhd, \swarrow, \nearrow, \nwarrow, \searrow)$ is a $(4+2)$-diquasigroup.

Corollary 9.8. *Suppose that G and H are groups, such that X is a right G-set and a left H-set. Then the reduct $(G \times X \times H, \lhd, \rhd)$ of $(G \times X \times H, \lhd, \rhd, \swarrow, \searrow)$ is a dimonoid if and only if the respective actions of G and H on X commute.*

Proof. The validity of the internal associativity law

$$\begin{aligned}(g_1, x_1,&h_1) \rhd \big((g_2, x_2, h_2) \lhd (g_3, x_3, h_3)\big)\\ &= \big((g_1, x_1, h_1) \rhd (g_2, x_2, h_2)\big) \lhd (g_3, x_3, h_3)\end{aligned}$$

for elements (g_i, x_i, h_i) of $G \times X \times H$ with $1 \le i \le 3$, namely

$$(g_1, x_1,h_1) \rhd (g_2 g_3, x_2 R(g_3), h_2 h_3) = (g_1 g_2, x_2 L(h_1), h_1 h_2) \lhd (g_3, x_3, h_3)\,,$$

reduces to the equality

$$(g_1 g_2 g_3, x_2 R(g_3) L(h_1), h_1 h_2 h_3) = (g_1 g_2 g_3, x_2 L(h_1) R(g_3), h_1 h_2 h_3)\,,$$

which holds if and only if the two actions commute. On the other hand, the associative laws for $\lhd$ and $\rhd$, as well as the two bar side irrelevance identities, always hold under the hypotheses of the corollary. □

10. Digroups

Various definitions of digroups have appeared in the literature [2, 3, 5, 7]. The following approach interprets these definitions with a constant-free type, specifying a *bar unit* 1 by a unary operation.

Definition 10.1. A *digroup* $(S, \lhd, \rhd, {}^{-1}, 1)$ is a dimonoid $(S, \lhd, \rhd)$ with a unary operation $x \mapsto x^{-1}$ and a constant unary operation $x \mapsto 1$ such that the *bar unit identities*

$$x \lhd 1 = x = 1 \rhd x$$

and *inversion identities*

$$x^{-1} \lhd x = 1 = x \rhd x^{-1}$$

are satisfied.

Digroups arise from groups using the machinery of Section 6 as follows. Compare Theorem 7.1, and the comments on idempotence of dimonoids in Example 6.3.

Proposition 10.2. *Consider the constant-free type $\tau = \{(\mu, 2), (\nu, 1), (\varepsilon, 1)\}$. Let $\mathbf{G}$ be the variety of groups defined with basic identities from the set*

$$\Sigma = \begin{cases} x_1 x_2 \mu x_3 \iota \mu = x_1 \iota x_2 x_3 \mu \mu\,, \\ x\iota\, x\varepsilon\mu = xx\pi_1^2\,, \\ x\varepsilon\, x\iota\mu = xx\pi_2^2\,, \\ x\nu\, x\iota\mu = x\varepsilon x\pi_1^2\,, \\ x\iota\, x\nu\mu = xx\varepsilon\pi_2^2\,. \end{cases} \tag{10.1}$$

Then the corresponding derived directional variety $\mathbf{G}'_{\uparrow}[\Sigma]$ coincides with the variety of algebras derived from digroups.

Proposition 10.3. *Let $(S, \lhd, \rhd, {}^{-1}, 1)$ be a digroup. Define*

$$x \swarrow y = x \lhd y^{-1} \quad \text{and} \quad y \searrow x = y^{-1} \rhd x$$

for $x, y \in S$. Then $(S, \lhd, \rhd, \swarrow, \searrow)$ is a 4-diquasigroup.

Proof. For elements x, y of S, one has

$$y \searrow (y \rhd x) = y^{-1} \rhd (y \rhd x) = (y^{-1} \rhd y) \rhd x = x\,,$$

the final equation holding by [3, Lemma 4.3(1)] (a known result in right groups). On the other hand, one also has

$$y \rhd (y \searrow x) = y \rhd (y^{-1} \rhd x) = (y \rhd y^{-1}) \rhd x = 1 \rhd x = x$$

directly from the defining axioms for digroups. The right-hand identities in Definition 9.1(a) are obtained dually. □

Corollary 10.4. *Let $(S, \lhd, \rhd, {}^{-1}, 1)$ be a digroup. Define*

$$x \nearrow y = x \rhd y^{-1} \quad \textit{and} \quad y \nwarrow x = y^{-1} \lhd x$$

for $x, y \in S$. Then $(S, \lhd, \rhd, \swarrow, \nearrow, \searrow, \nwarrow)$ is a $(4+2)$-diquasigroup.

Proposition 10.5. *Suppose that G and H are groups, such that X is a right G-set and a left H-set. Suppose that the actions of G and H on X commute. Let e be an element of X that is fixed by G and H. Then the dimonoid*

$$(G \times X \times H, \lhd, \rhd)$$

of Corollary 9.8, equipped with the bar unit

$$(1, e, 1)$$

and the inversion

$$(g, x, h)^{-1} = (g^{-1}, e, h^{-1})\,,$$

forms a digroup.

Proof. For the element (g, x, h) of $G \times X \times H$, one has

$$\begin{aligned} &(g, x, h) \lhd (1, e, 1) = (g1, xR(1), h1) = (g, x, h) \\ \text{and} \quad &(1, e, 1) \rhd (g, x, h) = (1g, xL(1), 1h) = (g, x, h)\,. \end{aligned}$$

Similarly,

$$\begin{aligned} &(g^{-1}, e, h^{-1}) \lhd (g, x, h) = (g^{-1}g, eR(g), h^{-1}h) = (1, e, 1) \\ \text{and} \quad &(g, x, h) \rhd (g^{-1}, e, h^{-1}) = (gg^{-1}, eL(h), hh^{-1}) = (1, e, 1)\,, \end{aligned}$$

completing the proof that $(G \times X \times H, \lhd, \rhd, {}^{-1}, 1)$ is a digroup. □

Example 10.6. Let Q be a set. Then the projection algebra $(Q, \lhd, \rhd, \swarrow, \searrow)$ is a 4-quasigroup with a dimonoid reduct $(Q, \lhd, \rhd)$. If Q has an element e, then the digroup $\{1\} \times Q \times \{1\}$ of Proposition 10.5 yields a 4-quasigroup $(\{1\} \times Q \times \{1\}, \lhd, \rhd, \swarrow, \searrow)$ as in Proposition 10.3 that is isomorphic to the projection algebra $(Q, \lhd, \rhd, \swarrow, \searrow)$.

Corollary 10.7. *With the directional division definitions of Proposition 10.3 and Corollary 10.4, the digroup* $(G \times X \times H, \lhd, \rhd, {}^{-1}, 1)$ *of Proposition 10.5 is equivalent to the* $(4 + 2)$*-diquasigroup structure* $(G \times X \times H, \lhd, \rhd, \swarrow, \nearrow, \nwarrow, \searrow)$ *of Proposition 9.6 and Corollary 9.7.*

Proof. For elements (g_i, x_i, h_i) of $G \times X \times H$, with $i = 1, 2$, one has

$$\begin{aligned}(g_1, x_1,h_1) \lhd (g_2, x_2, h_2)^{-1} &= (g_1, x_1, h_1) \lhd (g_2^{-1}, e, h_2^{-1})\\ &= (g_1/g_2, x_1R(g_2)^{-1}, h_1/h_2) = (g_1, x_1, h_1) \swarrow (g_2, x_2, h_2)\,,\end{aligned}$$

and a dual computation yields $(g_1, x_1, h_1) \searrow (g_2, x_2, h_2)$. Similarly,

$$\begin{aligned}(g_1, x_1,h_1) \rhd (g_2, x_2, h_2)^{-1} &= (g_1, x_1, h_1) \rhd (g_2^{-1}, e, h_2^{-1})\\ &= (g_1/g_2, eL(h_1), h_1/h_2) = (g_1/g_2, e, h_1/h_2)\\ &= (g_1, x_1, h_1) \nearrow (g_2, x_2, h_2)\,,\end{aligned}$$

and a dual computation yields $(g_1, x_1, h_1) \nwarrow (g_2, x_2, h_2)$. Conversely,

$$(g, x, h) \nearrow (g, x, h) = (gg^{-1}, e, gg^{-1}) = (1, e, 1)$$

and

$$(g, x, h)^{-1} = (g^{-1}, e, h^{-1}) = (1, e, 1) \swarrow (g, x, h)$$

for an element (g, x, h) of $G \times X \times H$. □

An important consequence of Corollary 10.7 may be summarized as follows.

Theorem 10.8. *Each digroup is equivalent to a* $(4 + 2)$*-diquasigroup structure.*

Proof. By a result of Kinyon [3, Th. 4.8], each digroup may be expressed in the form described in Proposition 10.5. (Note that the transitivity mentioned in [3, Ex. 4.2] is not required.) □

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