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Duality for semilattice representations

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Abstract

The paper presents general machinery for extending a duality between complete, cocomplete categories to a duality between corresponding categories of semilattice representations (i.e. sheaves over Alexandrov spaces). This enables known dualities to be regularized. Among the applications, regularized Lindenbaum–Tarski duality shows that the weak extension of Boolean logic (i.e. the semantics of PASCAL-like programming languages) is the logic for semilattice-indexed systems of sets. Another application enlarges Pontryagin duality by regularizing it to obtain duality for commutative inverse Clifford monoids.

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1. Introduction

Duality theory is generally viewed as more of an art than a science. The few broad techniques that are available, such as the enrichment of the structure of a schizophrenic object [14, Section VI.4.4], tend to be of uncertain efficacy. Thus, development of a duality for a particular class of objects is usually the result of an ad hoc procedure, and may often become the source of considerable interest.

The purpose of the current paper is to propose one general technique for obtaining dualities. The starting point is a duality $D : \mathfrak{A} \rightleftarrows \mathfrak{X} : E$ (2.1) between two complete and cocomplete concrete categories. Conventionally, these are described as the category \mathfrak{A} of “algebras” and the category \mathfrak{X} of “representation spaces”. In the example of Priestley duality [23, 24], \mathfrak{A} is the category of distributive lattices and \mathfrak{X} is the category of compact Hausdorff zero-dimensional partially ordered spaces. The basis for the technique is the duality $C : \underline{\mathfrak{S}}\mathfrak{L} \rightleftarrows \mathfrak{B} : F$ (2.2) between the category $\underline{\mathfrak{S}}\mathfrak{L}$ of

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semilattices (commutative, idempotent semigroups) and the category \mathfrak{B} of compact Hausdorff zero-dimensional bounded semilattices. Up to juggling with constants, this is the duality $\underline{\text{Sl}}_0 \rightleftarrows \underline{\mathcal{Z}}$ between the category $\underline{\text{Sl}}_0$ of commutative, idempotent monoids and the category $\underline{\mathcal{Z}}$ of compact Hausdorff zero-dimensional $\underline{\text{Sl}}_0$ -objects. The latter duality was treated in great detail by Hoffman et al. [11]. It is summarized and adapted to (2.2) in Section 2.

The fundamental concept of the paper is the notion of a semilattice representation, as discussed in Section 3. A semilattice H may be ordered as a meet semilattice, and then viewed as a small category. A representation $R : H \rightarrow \mathfrak{A}$ is a contravariant functor from H to \mathfrak{A} . Sheaf theorists may view it as an \mathfrak{A} -sheaf on the space H equipped with the Alexandrov topology.

Semilattice representations in \mathfrak{A} form a “semicolon category” $\tilde{\mathfrak{A}} = (\underline{\text{Sl}}; \mathfrak{A}^{\text{op}})$. For a \mathfrak{B} -space G , certain representations in $\tilde{\mathfrak{A}}$ have a property of “ \mathfrak{B} -continuity”. The category $\tilde{\mathfrak{X}}$ is the category of \mathfrak{B} -continuous representations of \mathfrak{B} -spaces. The main result of the paper, Theorem 4.3, then provides a duality $\tilde{D} : \tilde{\mathfrak{A}} \rightleftarrows \tilde{\mathfrak{X}} : \tilde{E}$ (4.1) between the categories $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{X}}$. The proof of the theorem is the content of Section 5. If the initial duality (2.1) arose from a schizophrenic object T , then results of Section 6 show how the new duality (4.1) arises from a related schizophrenic object T^∞ .

Section 7 discusses application of the technique. Suppose that \mathfrak{A} is a strongly irregular variety of algebras without constants. Then by Płonka’s Theorem 3.1, the category $\tilde{\mathfrak{A}}$ is essentially the regularization of \mathfrak{A} – the variety of algebras satisfying all the regular identities satisfied by algebras from \mathfrak{A} . (Pedantically speaking, this description of $\tilde{\mathfrak{A}}$ relies on identification of sheaves with bundles.) In this context, the technique of the paper may be described as the *regularization* of the initial duality (2.1) to obtain the new duality (4.1). Example 7.1, duality for left normal bands, is the regularization of Lindenbaum–Tarski duality between sets and complete atomic Boolean algebras. Lindenbaum–Tarski duality shows that Boolean logic is the logic for sets in the classical sense. Regularized Lindenbaum–Tarski duality then shows that weak Boolean logic is the logic for semilattice-ordered systems of sets. Guzmán [9] has recently discussed the role of weak Boolean logic in the semantics of programming languages such as PASCAL that decline to evaluate expressions as soon as any possibility of doubt arises. Regularized Lindenbaum–Tarski duality recovers Guzmán’s characterization [9, Theorem 3’] of functions representable in weak Boolean logic. Example 7.2 recalls the duality for doubly distributive dissemilattices obtained [8] by using “the technique of Płonka sums in duality theory” [8, p. 248]. Although this duality is regularized Priestley duality, it was noted [8, p. 248] that the “dualization of Płonka sums (could not be made) completely explicit” at the time of writing [8]. The present paper arose from the attempt to make the dualization of Płonka sums more explicit.

The final Section 8 of the paper regularizes Pontryagin duality to obtain a duality for commutative inverse monoids that are semilattices of groups (Clifford semigroups). This regularization has to be based on the commutative idempotent monoid duality $\underline{\text{Sl}}_0 \rightleftarrows \underline{\mathcal{Z}}$ rather than the commutative idempotent semigroup duality $\underline{\text{Sl}} \rightleftarrows \mathfrak{B}$. In a paper laying the foundation for harmonic analysis on commutative semigroups, Hewitt

and Zuckerman wrote [10, p. 70]: “[W]e state no theorems concerning ... analogues of the Pontryagin duality theorem We hope to deal with th[is] topic in a subsequent communication.” Austin’s 1962 Ph.D. thesis provided a basis for that subsequent communication [1], but did not present a full duality. Regularized Pontryagin duality (8.2) now gives full duality for strong semilattices of abelian groups.

The category theory used in this paper generally follows [16], while [25] provides a reference for universal algebraic notions such as regularity and replication. With the exception of inverse images, limits, and infima, maps and functors are generally written to the right of their arguments (as in the factorial) or to the upper right (as in the square). The advantages of this break with unfortunate convention are the elimination of brackets and the ease of reading concatenated mappings, e.g. (4.9), in the natural order from left to right.

2. Duality for algebras and semilattices

The archetypal form of duality considered is represented as

$$D : \mathfrak{A} \rightleftarrows \mathfrak{X} : E. \tag{2.1}$$

Here \mathfrak{A} is a complete and cocomplete concrete category of *algebras*, e.g. a variety of algebras considered as a category with homomorphisms as arrows, while \mathfrak{X} is a concrete category of *representation spaces* for \mathfrak{A} -algebras. There are (covariant) functors $D : \mathfrak{A} \rightarrow \mathfrak{X}^{\text{op}}$ and $E : \mathfrak{X}^{\text{op}} \rightarrow \mathfrak{A}$ furnishing an (*adjoint*) *equivalence* between \mathfrak{A} and \mathfrak{X}^{op} (as in [16, Theorem IV.4.1]). Normally, one considers $D : \mathfrak{A} \rightarrow \mathfrak{X}$ and $E : \mathfrak{X} \rightarrow \mathfrak{A}$ as contravariant functors.

One example of (2.1) is fundamental: that of semilattices. It takes the form

$$C : \underline{\text{Sl}} \rightleftarrows \mathfrak{B} : F. \tag{2.2}$$

Here $\underline{\text{Sl}}$ is the variety of semilattices. A \mathfrak{B} -*space*, i.e. an object G of \mathfrak{B} , has traditionally been defined topologically, as a compact Hausdorff zero-dimensional topological bounded semilattice. A *bounded semilattice* in this sense is a meet-semilattice having a least element 0 and greatest element 1 selected by nullary operations. The \mathfrak{B} -morphisms are continuous homomorphisms of bounded semilattices. The two-element meet-semilattice $\underline{2} = \{0 \leq 1\}$ is an object of $\underline{\text{Sl}}$. Equipped with the discrete topology and nullary operations $0, 1$, it becomes an object $\underline{2}$ of \mathfrak{B} . For a semilattice H , the \mathfrak{B} -space HC is defined to be the closed subspace $\underline{\text{Sl}}(H, \underline{2})$ of the product space $\underline{2}^H$. Elements of HC are called *characters* of H . For a semilattice homomorphism $f : H_1 \rightarrow H_2$, the \mathfrak{B}^{op} -morphism fC is defined as $H_2C \rightarrow H_1C; \theta \mapsto f\theta$. For a \mathfrak{B} -space G , the semilattice GF is defined to be the subsemilattice $\mathfrak{B}(G, \underline{2})$ of the semilattice reduct of the product $\underline{2}^G$. For a \mathfrak{B} -morphism $f : G_1 \rightarrow G_2$, the semilattice homomorphism fF (strictly: $f^{\text{op}}F$ [16, Section II.2]) is defined as $G_2F \rightarrow G_1F : \theta \mapsto f\theta$. The duality

between \underline{S} and \mathfrak{B} is covered explicitly in [7, p. 158; 6, p. 28]. It is closely related to the duality between the variety \underline{S}_0 (cf. [22, p. 142]) of monoids whose semigroup reducts are semilattices and the category \underline{Z} of compact Hausdorff zero-dimensional \underline{S}_0 -algebras. The latter duality is treated (concisely in [14, Section VI.3.6] and) very thoroughly in [11], albeit in terminology less well suited to the present context. Semilattices are called “protosemilattices” there, while the term “semilattice” refers to \underline{S}_0 -algebras. The category \underline{S}_0 is denoted there by \underline{S} . Many results carry over from [11], mutatis mutandis, as summarized below.

The characteristic function of a subset Θ of a semilattice (H, \cdot) is a character of (H, \cdot) iff the subset Θ is a wall of (H, \cdot) , i.e. iff

$$\forall h, k \in H, (h \cdot k \in \Theta) \Leftrightarrow (h \in \Theta, k \in \Theta) \tag{2.3}$$

[27, Prop. 2.2]. In meet semilattices, walls are often described as “filters” (cf. [11, Definition II.2.1]), while in join semilattices they are often described as “ideals” (cf. [26, Definition 4.1]). Let HW denote the set of walls of H .

Proposition 2.1. *Under intersection, HW forms a subsemilattice of the power set of H . Moreover, there is a natural \mathfrak{B} -isomorphism*

$$HC \rightarrow HW; \chi \mapsto \chi^{-1}\{1\}. \tag{2.4}$$

Proof. Cf. [11, Proposition II.2.4(ii)]. The zero element of HW is the empty wall. □

As a partially ordered set, each \mathfrak{B} -space is a complete (indeed algebraic) lattice [11, p. 39]. A subset X of a \mathfrak{B} -space G is a cover of an element g of G iff $g \leq \sup X$. An element c of G is compact if it is non-zero, and if each cover of c contains a finite subcover. Let GK denote the set of compact elements of G .

Proposition 2.2. *In a \mathfrak{B} -space G , the set GK of compact elements forms a join semilattice (i.e. a finitely cocomplete subcategory of the meet semilattice G , but not necessarily having an initial object).*

Proof. See [3, Theorem VIII.8; 11, Proposition II.1.10, Theorem II.3.3] or [25, p. 64]. □

For an element h of a semilattice H , the wall $[h]$ is the intersection of all the walls of H containing h . Such a wall is called *principal*.

Proposition 2.3. *For a semilattice H , an element Θ of HW is compact iff it is principal. Then there is a natural isomorphism $H \rightarrow HWK; h \mapsto [h]$.*

Proof. See [11, Proposition II.3.8; 25, 344(i)], or [26, Theorem 5.1]. The latter two references apply on considering the variety of stammered semilattices [25, 327]. □

In the other direction, one has the following.

Proposition 2.4. *For a \mathfrak{B} -space G , there is a natural isomorphism*

$$\gamma_G : G \rightarrow GKW; g \mapsto GK \cap \downarrow g. \tag{2.5}$$

Proof. Cf. [11, Proposition II.3.9]. The zero element of G maps to the empty wall: each compact element of G is strictly bigger than zero. \square

Given a \mathfrak{B} -morphism $f : G_1 \rightarrow G_2$, a semilattice homomorphism is defined by

$$f^K : G_2K \rightarrow G_1K; c \mapsto \inf f^{-1} \uparrow c. \tag{2.6}$$

Proposition 2.5. *There is a contravariant functor $K : \mathfrak{B} \rightarrow \underline{\mathbf{S}}$ given by Proposition 2.2 and (2.6). For a \mathfrak{B} -space G , define*

$$\kappa_G : GF \rightarrow GK; \theta \mapsto \inf \theta^{-1} \{1\}. \tag{2.8}$$

Then $\kappa : F \dashrightarrow K$ is a natural isomorphism.

Proof. See [11, Theorem II.3.7 and Proposition II.3.20]. \square

A final result on \mathfrak{B} -morphisms is needed.

Proposition 2.6. *Let $f : G_1 \rightarrow G_2$ be a \mathfrak{B} -morphism, and let c be a compact element of G_2 . Then $cf^Kf = \inf \{(f^{-1} \uparrow c)f\} \geq c$.*

Proof. See [11, Theorem II.3.22]. \square

3. Płonka sums, sheaves, and bundles

Let H be a meet semilattice. Now H may be considered as an algebra (H, \cdot) , as a partially ordered set (H, \leq) , or as the set of objects of a small category with $H(h, k) = \{h \rightarrow k\}$ for $h \leq k$ and $|H(h, k)| = 0$ otherwise. Additionally, H is a topological space under the *Alexandrov topology* $\Omega(H)$ on the dual poset (H, \geq) [14, II.1.8] consisting of subordinate subsets of (H, \leq) . The poset (H, \leq) may be identified with the subposet $\{\downarrow h \mid h \in H\}$ of $(\Omega(H), \subseteq)$ consisting of principal subordinate subsets. In this guise, the semilattice H reappears as a basis for the Alexandrov topology $\Omega(H)$. Since the elements of H are join-irreducible in $\Omega(H)$, the condition [17, II.1(9)] for a presheaf on H to be a sheaf is trivially satisfied. By the Comparison Lemma for Grothendieck topoi [17, Theorem II.1.3 and App., Corollary 3(a)], the functor category $\widehat{H} = \underline{\mathbf{Set}}^{H^{op}}$ (of presheaves on H) is equivalent to the category $\underline{\mathbf{Sh}}(H)$ of sheaves (of sets) over the space H under the Alexandrov topology $\Omega(H)$. By [17, Corollary II.6.3], the category $\underline{\mathbf{Sh}}(H)$ is in turn equivalent to the category $\underline{\mathbf{E}tale}H$ of étale bundles $\pi : E \rightarrow H$ over

the space H . Given a representation (presheaf) $R : H \rightarrow \underline{\text{Set}}$, the corresponding bundle $\pi : E \rightarrow H$ (or more loosely just the total space E) is the bundle RA of germs of the sheaf $R : \Omega(H) \rightarrow \underline{\text{Set}}$.

An alternative, purely algebraic description of the equivalence between semilattice representations and étale bundles may be given. The variety $\underline{\text{Lz}}$ of *left trivial* or *left zero bands* is the variety of semigroups satisfying

$$x * y = x \tag{3.1}$$

[12, p. 119; 25, 225]. The category $\underline{\text{Lz}}$ is isomorphic to the category $\underline{\text{Set}}$ of sets. In one direction, forget the multiplication (3.1). In the other direction, the multiplication on any set is just the projection $(x, y) \mapsto x$ from the direct square. The variety $\underline{\text{Ln}}$ of *left normal bands* is the variety of idempotent semigroups (bands) satisfying

$$x * y * z = x * z * y \tag{3.2}$$

[12, p. 119; 25, 223]. The *bundle* $\pi : E \rightarrow H$ of a left normal band $(E, *)$ is its projection onto its semilattice replica $(H, *)$ [25, p. 17]. One obtains a corresponding representation

$$E\Gamma : H \rightarrow \underline{\text{Set}}; h \mapsto \pi^{-1}\{h\}, \tag{3.3}$$

defined on the morphism level by $(h \rightarrow k)E\Gamma : \pi^{-1}\{k\} \rightarrow \pi^{-1}\{h\}; x \mapsto x * y$ for any y in $\pi^{-1}\{h\}$. In the other direction, a presheaf or representation

$$R : H \rightarrow \underline{\text{Set}} \tag{3.4}$$

gives a contravariant functor

$$R : H \rightarrow (*) \tag{3.5}$$

from the poset category (H, \leq_*) to the category $(*)$ of groupoids or magmas (sets with a binary multiplication) and homomorphisms. Here (3.5) is obtained from (3.4) on interpreting each set hR as the left zero band $(hR, *)$. The functor (3.5) summarizes the data for a construction known in semigroup theory as a “strong semilattice” [5; 12, p. 90] and in general algebra as a *Płonka sum* [18; 25, 236; 22]. Defining $\pi = \dot{\bigcup}_{h \in H} (hR \rightarrow \{h\})$ and

$$x * y = x(x^\pi * y^\pi \rightarrow x^\pi)^R \tag{3.6}$$

for x, y in $E = \dot{\bigcup}_{h \in H} hR$ gives the bundle

$$RA : E \rightarrow H \tag{3.7}$$

of a left normal band $(E, *)$. The constructions Γ of (3.3) and A of (3.7) extend to functors providing an equivalence

$$A : \widehat{H} \rightleftarrows (\underline{\text{Ln}}, H) : \Gamma \tag{3.8}$$

between the presheaf category \widehat{H} for a fixed semilattice H and the comma category $(\underline{\text{Ln}}, H)$ of left normal bands over H (cf. [16, Section II.6]). The equivalence (3.8) is an algebraic analogue of the equivalence

$$\Lambda : \underline{\text{Sh}}H \rightleftarrows \underline{\text{Etale}}H : \Gamma \tag{3.9}$$

of sheaf theory [14, Corollary V.1.5(i); 17, Corollary II.6.3].

The algebraic equivalence (3.8) may be extended. Let \underline{C} be a category whose objects are small categories and whose morphisms are functors. Let \underline{D} be a category. Then one may define a new category $(\underline{C}; \underline{D})$, called a “lax comma category” or *semicolon category*, as follows. Its objects are covariant functors $R : C \rightarrow \underline{D}$ from an object C of \underline{C} to \underline{D} . Given two such objects $R : C \rightarrow \underline{D}$ and $R' : C' \rightarrow \underline{D}$, a morphism $(\sigma, f) : R \rightarrow R'$ is a pair consisting of a \underline{C} -morphism $f : C \rightarrow C'$ and a natural transformation $\sigma : R \rightarrow fR'$. The composition of morphisms in $(\underline{C}; \underline{D})$ is defined by

$$(\sigma, f)(\tau, g) = (\sigma(f\tau), fg) \tag{3.10}$$

(cf. [11, Section 0.1; 16, Example V.2.5(b)]. Note that Mac Lane used the name “supercomma” and the symbol \Downarrow in place of the semicolon). For a concrete category \underline{D} , let $(\underline{\text{Sl}}; \underline{D}^{\text{op}})'$ denote the full subcategory of $(\underline{\text{Sl}}; \underline{D}^{\text{op}})$ comprising the representation $p \rightarrow \underline{D}^{\text{op}}$ and representations of non-empty semilattices in the full subcategory of $\underline{D}^{\text{op}}$ consisting of non-empty \underline{D} -objects. Then the equivalence (3.8) may be extended to the equivalence

$$\Lambda : (\underline{\text{Sl}}; \underline{\text{Set}}^{\text{op}})' \rightleftarrows \underline{\text{Ln}} : \Gamma. \tag{3.11}$$

Consider a left normal band morphism $F : E \rightarrow E'$, with semilattice replica $f : H \rightarrow H'$. Define $R = E\Gamma : H \rightarrow \underline{\text{Set}}$ and $R' = E'\Gamma : H' \rightarrow \underline{\text{Set}}$. A natural transformation $\varphi : R \rightarrow fR'$ is defined by its components

$$\varphi_h : hR \rightarrow hfR'; x \mapsto xF \tag{3.12}$$

at objects h of H . The $(\underline{\text{Sl}}; \underline{\text{Set}}^{\text{op}})$ -morphism $F\Gamma : E\Gamma \rightarrow E'\Gamma$ is then defined as the pair (φ, f) . Conversely, given such a pair as an $(\underline{\text{Sl}}; \underline{\text{Set}}^{\text{op}})$ -morphism, a left normal band morphism $F = (\varphi, f)\Lambda : R\Lambda \rightarrow R'\Lambda$ is defined as the disjoint union of the components (3.12).

The general equivalence (3.11) may be lifted to other contexts. Of particular interest in the current context is the case of a strongly irregular variety \mathfrak{B} of finitary algebras, considered as a category with homomorphisms as morphisms. A variety \mathfrak{B} is said to be *strongly irregular* if the \mathfrak{B} -algebra structure reduces to a left trivial semigroup [22, Section 4.8]. For example, the variety of groups (as usually presented with multiplication and inversion) is strongly irregular by virtue of $x * y = (xy)y^{-1}$. The *regularization* $\widetilde{\mathfrak{B}}$ of any variety \mathfrak{B} of finitary algebras is defined to be the variety of algebras satisfying each regular identity of \mathfrak{B} . (Recall that an identity is *regular* if it involves exactly the same set of arguments on each side [25, p. 13].) Then Płonka’s Theorem describing regularizations of strongly irregular varieties [18; 19; 22, 4.8 and 7.1; 25, 239] may be formulated as follows.

Theorem 3.1. *Let \mathfrak{B} be a strongly irregular variety of algebras whose type contains no constants. Then the equivalence (3.11) lifts to an equivalence*

$$\Lambda : (\underline{\text{Sl}}; \mathfrak{B}^{\text{op}})' \rightleftarrows \tilde{\mathfrak{B}} : \Gamma. \tag{3.13}$$

Corollary 3.2 (Płonka [21] and Płonka and Romanowska [22]). *Let \mathfrak{B}_0 be a strongly irregular variety of algebras whose type contains exactly one constant. Then the equivalence (3.11) lifts to an equivalence*

$$\Lambda : (\underline{\text{Sl}}_0; \mathfrak{B}_0^{\text{op}}) \rightleftarrows \tilde{\mathfrak{B}}_0 : \Gamma. \tag{3.14}$$

Now consider the duality (2.1). Motivated by (3.13), one writes

$$\tilde{\mathfrak{U}} = (\underline{\text{Sl}}; \mathfrak{U}^{\text{op}}) \tag{3.15}$$

for the category of (contravariant) representations of semilattices in \mathfrak{U} . Such representations R are often implicitly identified with the corresponding bundles RA . For a \mathfrak{B} -space G , a representation $R : G \rightarrow \mathfrak{X}$ is said to be \mathfrak{B} -continuous if

$$gR = \varinjlim (R : GK \cap \downarrow g \rightarrow \mathfrak{X}) \tag{3.16}$$

for each element g of G . (The limit on the right-hand side of (3.16) is the limit of the restriction of R to the upwardly directed ordered subset of G consisting of compact elements below g .) The category $\tilde{\mathfrak{X}}$ is then defined to be the full subcategory of $(\mathfrak{B}; \mathfrak{X}^{\text{op}})$ consisting of \mathfrak{B} -continuous (contravariant) representations of \mathfrak{B} -spaces in \mathfrak{X} . As for $\tilde{\mathfrak{U}}$, such representations R are often identified with the corresponding bundles RA .

4. Duality for semilattice representations

Given a duality (2.1) between complete and cocomplete concrete categories \mathfrak{U} and \mathfrak{X} , this section sets up a duality

$$\tilde{D} : \tilde{\mathfrak{U}} \rightleftarrows \tilde{\mathfrak{X}} : \tilde{E} \tag{4.1}$$

between the category $\tilde{\mathfrak{U}}$ of representations in \mathfrak{U} and the category $\tilde{\mathfrak{X}}$ of \mathfrak{B} -continuous representations of \mathfrak{B} -spaces in \mathfrak{X} . The main task is to define the contravariant functors \tilde{D} and \tilde{E} .

To begin, consider an $\tilde{\mathfrak{U}}$ -algebra given by a contravariant Płonka functor $R_2 : H_2 \rightarrow \mathfrak{U}$. For each character θ of the semilattice H_2 , this functor restricts to a contravariant functor R_2 from $\theta^{-1}\{1\}$ to the cocomplete category \mathfrak{U} . The dual object $(R_2 : H_2 \rightarrow \mathfrak{U})\tilde{D}$ is then defined as the representation $H_2C \rightarrow \mathfrak{X}$ sending each character θ to the dual space $[\varinjlim (R_2 : \theta^{-1}\{1\} \rightarrow \mathfrak{U})]D$ of the colimit of the corresponding restriction. Thus,

$$(R_2 : H_2 \rightarrow \mathfrak{U})\tilde{D} = (H_2C \rightarrow \mathfrak{X}; \theta \mapsto [\varinjlim (R_2 : \theta^{-1}\{1\} \rightarrow \mathfrak{U})]D). \tag{4.2}$$

Now consider an $\tilde{\mathfrak{A}}$ -homomorphism f , i.e. a pair (φ, f^π) consisting of the semilattice replica $f^\pi : H_1 \rightarrow H_2$ of f and a natural transformation $\varphi : R_1 \rightarrow f^\pi R_2$ between the domain representation $R_1 : H_1 \rightarrow \mathfrak{A}$ and the composite of f^π with the codomain representation $R_2 : H_2 \rightarrow \mathfrak{A}$. The dual morphism $(\varphi, f^\pi)^{\tilde{D}}$ is a pair $(\tau, f^{\pi C})$ consisting of the dual $f^{\pi C} : H_2 C \rightarrow H_1 C$ of f^π and a natural transformation $\tau : R_2 \tilde{D} \rightarrow f^{\pi C} R_1 \tilde{D}$. For each character θ of H_2 , consider the functors

$$\begin{array}{ccc} (f^\pi \theta)^{-1}\{1\} & \xrightarrow{R_1} & \mathfrak{A} \\ f^\pi \downarrow & & \\ \theta^{-1}\{1\} & \xrightarrow{R_2} & \mathfrak{A} \end{array} \tag{4.3}$$

together with the natural transformation $\varphi : R_1 \rightarrow f^\pi R_2$. They determine an \mathfrak{A} -homomorphism

$$\underline{\text{Lim}}(\varphi, f^\pi)_\theta : \underline{\text{lim}}(R_1 : (f^\pi \theta)^{-1}\{1\} \rightarrow \mathfrak{A}) \rightarrow \underline{\text{lim}}(R_2 : \theta^{-1}\{1\} \rightarrow \mathfrak{A}) \tag{4.4}$$

(cf. [11, Lemma 0.1.3; 16, Example V.2.5]). Applying D to (4.4) yields an \mathfrak{X} -morphism $\underline{\text{Lim}}(\varphi, f^\pi)_\theta D$ which gives the definition of τ_θ . Thus,

$$(\varphi, f^\pi)^{\tilde{D}} = (\underline{\text{Lim}}(\varphi, f^\pi)D, f^{\pi C}). \tag{4.5}$$

One obtains

Proposition 4.1. *There is a contravariant functor $\tilde{D} : \tilde{\mathfrak{A}} \rightarrow \tilde{\mathfrak{X}}$ defined by an object part (4.2) and a morphism part (4.5).*

Proof. It must first be shown that the representation defined on the right hand side of (4.2) is an object of $\tilde{\mathfrak{X}}$, i.e. that $\theta R_2^{\tilde{D}} = \underline{\text{lim}}(R_2^{\tilde{D}} : HCK \cap \downarrow \theta \rightarrow \mathfrak{X})$ for θ in $H_2 C$. To this end, it is convenient to identify characters θ of H with the walls $\theta^{-1}\{1\}$ that they determine, according to (2.4). Under the identification, HCK is the set of principal walls (Proposition 2.3). For an element h of H , with corresponding principal wall $[h]$, one has $[h]R_2^{\tilde{D}} = [\underline{\text{lim}}(R_2 : [h] \rightarrow \mathfrak{A})]D = hR_2 D$. Then for an arbitrary wall Θ of H , it follows that $\underline{\text{lim}}(R_2^{\tilde{D}} : HCK \cap \downarrow \Theta \rightarrow \mathfrak{X}) = \underline{\text{lim}}(R_2^{\tilde{D}} : \{[h] | h \in \Theta\} \rightarrow \mathfrak{X}) = \underline{\text{lim}}(R_2 D : \Theta \rightarrow \mathfrak{X}) = [\underline{\text{lim}}(R_2 : \Theta \rightarrow \mathfrak{A})]D = \Theta R_2^{\tilde{D}}$, as required. The penultimate equality holds since the functor $D : \mathfrak{A} \rightarrow \mathfrak{X}^{\text{op}}$, having $E : \mathfrak{X}^{\text{op}} \rightarrow \mathfrak{A}$ as a right adjoint, preserves colimits [16, Section V.5].

Explicitly, (4.2) only defines the object part of a representation $R_2 \tilde{D} : H_2 C \rightarrow \mathfrak{X}$. Thus, consider an $H_2 C$ -morphism $\chi \rightarrow \theta$, corresponding to an embedding $j : \chi^{-1}\{1\} \rightarrow \theta^{-1}\{1\}$. Together with the identical natural transformation $\iota : R_2 \rightarrow jR_2$ between $R_2 : \chi^{-1}\{1\} \rightarrow \mathfrak{A}$ and $\chi^{-1}\{1\} \xrightarrow{j} \theta^{-1}\{1\} \xrightarrow{R_2} \mathfrak{A}$, the embedding j determines an \mathfrak{A} -morphism

$$\underline{\text{Lim}}(\iota, j) : \underline{\text{lim}}(R_2 : \chi^{-1}\{1\} \rightarrow \mathfrak{A}) \rightarrow \underline{\text{lim}}(R_2 : \theta^{-1}\{1\} \rightarrow \mathfrak{A}). \tag{4.6}$$

Applying D to (4.6) yields the \mathfrak{X} -morphism that is the image of $\chi \rightarrow \theta$ under $R_2\tilde{D}$. The naturality of the transformation $\tau = \underline{\text{Lim}}(\varphi, f^\pi)D$ of (4.5), in the form of the commuting diagram

$$\begin{array}{ccc}
 \chi f^{\pi C} R_1 \tilde{D} & \xleftarrow{\tau_{\mathfrak{X}}} & \chi R_2 \tilde{D} \\
 j_{R_1 \tilde{D}} \downarrow & & \downarrow j_{R_2 \tilde{D}} \\
 \theta f^{\pi C} R_2 \tilde{D} & \xleftarrow{\tau_{\theta}} & \theta R_2 \tilde{D}
 \end{array} \tag{4.7}$$

in \mathfrak{X} , is obtained from the commuting diagram

$$\begin{array}{ccc}
 (R_1 : (f^\pi \chi)^{-1}\{1\} \rightarrow \mathfrak{A}) & \xrightarrow{(\varphi, f^\pi)} & (R_2 : \chi^{-1}\{1\} \rightarrow \mathfrak{A}) \\
 \downarrow (i, j) & & \downarrow (i, j) \\
 (R_1 : (f^\pi \theta)^{-1}\{1\} \rightarrow \mathfrak{A}) & \xrightarrow{(\varphi, f^\pi)} & (R_2 : \theta^{-1}\{1\} \rightarrow \mathfrak{A})
 \end{array} \tag{4.8}$$

by successive applications of the functors $\underline{\text{Lim}}$ and D . The functoriality of \tilde{D} is routine. □

The definition of \tilde{E} on an object $R : G \rightarrow \mathfrak{X}$ of $\tilde{\mathfrak{X}}$ is quite direct. Recall the natural isomorphism $\kappa_G : GF \rightarrow GK$ between the meet semilattice GF dual to the \mathfrak{B} -space G and the join semilattice GK of compact elements of G (Proposition 2.5). The composite of κ_G with the order-preserving embedding $j : GK \rightarrow G$ of the poset GK in G gives a contravariant functor $\kappa_G j : GF \rightarrow G$. The definition of \tilde{E} on objects is then given by

$$(R : G \rightarrow \mathfrak{X})\tilde{E} = (\kappa_G j R E : GF \rightarrow \mathfrak{A}). \tag{4.9}$$

The right-hand side of (4.9), as the composite of three contravariant functors, is contravariant. It thus forms a representation of the semilattice GF in \mathfrak{A} , determining an $\tilde{\mathfrak{A}}$ -algebra. Now consider an $\tilde{\mathfrak{X}}$ -morphism

$$f = (\varphi, f^\pi) : (R_1 : G_1 \rightarrow \mathfrak{X}) \rightarrow (R_2 : G_2 \rightarrow \mathfrak{X})$$

comprising a \mathfrak{B} -morphism $f^\pi : G_1 \rightarrow G_2$ and a natural transformation $\varphi : R_1 \rightarrow f^\pi R_2$. The morphism part of \tilde{E} will be specified by

$$(\varphi, f^\pi)^{\tilde{E}} = (\tau, f^{\pi F}), \tag{4.10}$$

where $f^{\pi F}$ is the dual of f^π and $\tau : \kappa_{G_2} j_{R_2} E \rightarrow f^{\pi F} \kappa_{G_1} j_{R_1} E$ is a natural transformation. Towards the definition of τ , recall the commutative diagram (cf. Proposition 2.5)

$$\begin{array}{ccc}
 G_2 F & \xrightarrow{\kappa_{G_2}} & G_2 K \\
 f^{\pi F} \downarrow & & \downarrow f^{\pi K} \\
 G_1 F & \xrightarrow{\kappa_{G_1}} & G_1 K
 \end{array} . \tag{4.11}$$

For a compact element c of G_2 , one has $c f^{\pi K} = \inf f^{\pi^{-1}} \uparrow c$ (2.6). Moreover, $c f^{\pi K} f^\pi \geq c$ by Proposition 2.6. Define σ_c to be the composite of the component of φ at $c f^{\pi K}$ with the image of the G_2 -morphism $c \rightarrow c f^{\pi K} f^\pi$ under R_2 . Then σ_c is the component at c of a natural transformation $\sigma : f^{\pi K} R_1 \rightarrow R_2$. Indeed, for a $G_2 K$ -morphism $d \rightarrow c$, there is a commutative diagram

$$\begin{array}{ccccc}
 c f^{\pi K} R_1 & \xrightarrow{\varphi_{c f^{\pi K}}} & c f^{\pi K} f^\pi R_2 & \xrightarrow{(c \rightarrow c f^{\pi K} f^\pi) R_2} & c R_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 d f^{\pi K} R_1 & \xrightarrow{\varphi_{d f^{\pi K}}} & d f^{\pi K} f^\pi R_2 & \xrightarrow{(d \rightarrow d f^{\pi K} f^\pi) R_2} & d R_2.
 \end{array} \tag{4.12}$$

The left-hand square commutes by the naturality of $\varphi : (R_1 : G_1 \rightarrow \mathfrak{X}) \rightarrow (f^\pi R_2 : G_1 \rightarrow \mathfrak{X})$. The right-hand square is the image under R_2 of the commutative square

$$\begin{array}{ccc}
 c f^{\pi K} f^\pi & \longleftarrow & c \\
 \uparrow & & \uparrow \\
 d f^{\pi K} f^\pi & \longleftarrow & d
 \end{array} \tag{4.13}$$

in G_2 . The natural transformation τ of (4.10) is now given as

$$\tau = \kappa_{G_2} \sigma E : \kappa_{G_2} j R_2 E \rightarrow \kappa_{G_2} f^{\pi K} j R_1 E = f^{\pi F} \kappa_{G_1} j R_1 E, \tag{4.14}$$

the identity of the codomain functors following by the commuting of (4.11). In summary,

Proposition 4.2. *There is a contravariant functor $\tilde{E} : \tilde{\mathfrak{X}} \rightarrow \tilde{\mathfrak{A}}$. Its object part is given by (4.9). Its morphism part is given by (4.10) and (4.14). \square*

The main theorem of the paper may now be formulated. Its proof is presented separately in the following section.

Theorem 4.3. *Suppose given a duality (2.1) between a complete, cocomplete concrete category \mathfrak{A} of algebras and a concrete category \mathfrak{X} of representation spaces. Then the functors \tilde{D} of Proposition 4.1 and \tilde{E} of Proposition 4.2 yield a duality (4.1) between the category $\tilde{\mathfrak{A}} = (\underline{\mathbf{S}}\mathfrak{I}; \mathfrak{A}^{\text{op}})$ and the full subcategory $\tilde{\mathfrak{X}}$ of $(\mathfrak{B}; \mathfrak{X}^{\text{op}})$ consisting of \mathfrak{B} -continuous representations.*

5. Proof of the main theorem

This section is devoted to the proof of the main Theorem 4.3. In the given duality (2.1), there is a natural isomorphism $\varepsilon : 1_{\mathfrak{A}} \rightarrow DE$. In the duality (2.2) for semilattices, there is a natural isomorphism $\iota : 1_{\underline{\mathbf{S}}\mathfrak{I}} \rightarrow CF$. The first half of the proof shows that there is a natural isomorphism $\tilde{\varepsilon} : 1_{\tilde{\mathfrak{A}}} \rightarrow \tilde{D}\tilde{E}$ given by its component

$$\tilde{\varepsilon}_R = (R\varepsilon, \iota_H) \tag{5.1}$$

at an object $R : H \rightarrow \mathfrak{A}$ of $\widetilde{\mathfrak{A}}$. Consider the diagram

$$\begin{array}{ccc} H & \xrightarrow{R} & \mathfrak{A} \\ \downarrow \iota_H & & \\ HCF & \xrightarrow{R^{\widetilde{D}\widetilde{E}}} & \mathfrak{A} \end{array}, \tag{5.2}$$

in which the bottom row factorizes as

$$HCF \xrightarrow{\kappa_{HC}} HCK \xrightarrow{j} HC \xrightarrow{R^{\widetilde{D}}} \mathfrak{X} \xrightarrow{E} \mathfrak{A} \tag{5.3}$$

according to (4.9). Start with an element h of H , and chase it along $\iota_H R^{\widetilde{D}\widetilde{E}}$. In passing from H to HC , it is again helpful (as in the proof of Proposition 4.1) to realize HC as the meet semilattice of walls of H according to (2.4). Thus, h appears in HCK as the principal wall $[h]$ (Proposition 2.3). Under j , this wall maps to the element

$$([h] \rightarrow \{1\}) \dot{\cup} ((H - [h]) \rightarrow \{0\}) \tag{5.4}$$

of HC . Under $R^{\widetilde{D}}$, (5.4) is represented as the \mathfrak{A} -algebra $[\varinjlim (R : [h] \rightarrow \mathfrak{A})]D = hRD$. Thus

$$h\iota_H R^{\widetilde{D}\widetilde{E}} = hRDE, \tag{5.5}$$

whence $(R\varepsilon)_h = (\varepsilon_{hR} : hR \rightarrow h\iota_H R^{\widetilde{D}\widetilde{E}})$ is an isomorphism.

To verify the naturality of $\widetilde{\varepsilon}$, consider an \mathfrak{A} -morphism

$$(\varphi, f) : (R : H \rightarrow \mathfrak{A}) \rightarrow (R' : H' \rightarrow \mathfrak{A}). \tag{5.6}$$

The commuting of the diagram

$$\begin{array}{ccc} R & \xrightarrow{\widetilde{\varepsilon}_R} & R\widetilde{D}\widetilde{E} \\ (\varphi, f) \downarrow & & \downarrow (\tau, f^{CF}) \\ R' & \xrightarrow{\widetilde{\varepsilon}_{R'}} & R'\widetilde{D}\widetilde{E} \end{array} \tag{5.7}$$

is required, where $(\tau, f^{CF}) = (\varinjlim (\varphi, f)D, f^C)\widetilde{E}$. For each element h of H , this amounts to requiring the equality $\varepsilon_{hR}\tau_{h\iota_H} = \varphi_h\varepsilon_{hfR'}$. Since $\varphi_h\varepsilon_{hfR'} = \varepsilon_{hR}\varphi_h^{DE}$, it suffices to prove that

$$\tau_{h\iota_H} = \varphi_h DE, \tag{5.8}$$

the domains of the respective sides of (5.8) being given by the corresponding sides of (5.5). Consider the commuting diagram

$$\begin{array}{ccc} hR & \xrightarrow{\varphi^h} & hfR' \\ \downarrow & & \parallel \\ \varinjlim (R : f^{-1}[hf] \rightarrow \mathfrak{A}) & \xrightarrow{\varinjlim (\varphi, f)_{h/f}} & \varinjlim (R' : [hf] \rightarrow \mathfrak{A}) \end{array} \tag{5.9}$$

in \mathfrak{A} , where the first column is the natural map to the colimit. Applying the functor D to (5.9), one obtains the equality

$$(\kappa_{HC}\sigma)_{h_H} = (\varphi D)_h \tag{5.10}$$

involving the natural transformation σ used in the definition (4.14) of the first component τ of $(\tau, f^{CF}) = (\varinjlim(\varphi, f)D, f^C)\tilde{E}$. The required equality (5.8) is then obtained by applying the functor E to (5.10).

On the other side, the duality (2.1) includes a natural isomorphism $\eta : 1_{\mathfrak{X}} \rightarrow ED$. The duality (2.2) for semilattices includes a natural isomorphism $\gamma : 1_{\mathfrak{G}} \rightarrow FC$ (cf. Proposition 2.4). The second half of the proof shows that there is a natural isomorphism $\tilde{\eta} : 1_{\tilde{\mathfrak{X}}} \rightarrow \tilde{E}\tilde{D}$ given by

$$\tilde{\eta}_R = (R\eta, \gamma_G) \tag{5.11}$$

at an object $R : G \rightarrow \mathfrak{X}$ of $\tilde{\mathfrak{X}}$. Consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{R} & \mathfrak{X} \\ \gamma_G \downarrow & & \\ GFC & \xrightarrow{R\tilde{E}\tilde{D}} & \mathfrak{X} \end{array} \tag{5.12}$$

in which the bottom row appears as

$$\begin{aligned} (GF \xrightarrow{\kappa_G} GK \xrightarrow{j} G \xrightarrow{R} \mathfrak{X} \xrightarrow{E} \mathfrak{A})\tilde{D} \\ = (GFC \rightarrow \mathfrak{X}; \theta \mapsto [\varinjlim(\kappa_G j RE : \theta^{-1}\{1\} \rightarrow \mathfrak{A})]D). \end{aligned} \tag{5.13}$$

Start with an element g of G , and chase it along $\gamma_G R \tilde{E} \tilde{D}$. Set $\theta_g = g\gamma_G$, i.e. $\chi\theta_g = 1 \Leftrightarrow g\chi = 1$ for χ in GF . Note $\{\chi \in GF \mid g\chi = 1\}\kappa_G = GK \cap \downarrow g$. Then

$$\begin{aligned} g\gamma_G R \tilde{E} \tilde{D} &= \theta_g R \tilde{E} \tilde{D} = [\varinjlim(\kappa_G j RE : \theta_g^{-1}\{1\} \rightarrow \mathfrak{A})]D \\ &= [\varinjlim(\kappa_G j RE : \{\chi \mid g\chi = 1\} \rightarrow \mathfrak{A})]D \\ &= [\varinjlim(RE : GK \cap \downarrow g \rightarrow \mathfrak{A})]D \\ &= [\varinjlim(R : GK \cap \downarrow g \rightarrow \mathfrak{X})]ED = gRED. \end{aligned}$$

The penultimate equality holds since $E : \mathfrak{X} \rightarrow \mathfrak{A}^{\text{op}}$, having $D : \mathfrak{A}^{\text{op}} \rightarrow \mathfrak{X}$ as a left adjoint, preserves limits [16, Theorem V.5.1]. Thus $(R\eta)_g = \eta_{gR} : gR \rightarrow g\gamma_G R \tilde{E} \tilde{D}$ is an isomorphism.

To verify the naturality of $\tilde{\eta}$, consider an \mathfrak{X} -morphism

$$(\varphi, f) : (R : G \rightarrow \mathfrak{X}) \rightarrow (R' : G' \rightarrow \mathfrak{X}). \tag{5.14}$$

The commuting of the diagram

$$\begin{array}{ccc}
 R & \xrightarrow{\tilde{\eta}_R} & R\tilde{E}\tilde{D} \\
 (\varphi, f) \downarrow & & \downarrow (\underline{\text{Lim}}(\tau, f^F)D, f^{FC}) \\
 R' & \xrightarrow{\tilde{\eta}_{R'}} & R'\tilde{E}\tilde{D}
 \end{array} \tag{5.15}$$

is required, where $(\tau, f^F) = (\varphi, f)\tilde{E}$. For each element g of G , this amounts to requiring the equality $\eta_{gR}[\underline{\text{Lim}}(\tau, f^F)D]_{g\gamma_G} = \varphi_g \eta_{gfR'}$. Since $\varphi_g \eta_{gfR'} = \eta_{gR} \varphi_g^{ED}$, it suffices to prove that

$$[\underline{\text{Lim}}(\tau, f^F)D]_{g\gamma_G} = \varphi_g ED, \tag{5.16}$$

the domains of the respective sides of (5.16) being equal as noted above. Let k be a compact element of G' below gf . Since $g \in f^{-1} \uparrow k$, one has $kf^K = \inf f^{-1} \uparrow k \leq g$ in G . Consider the diagram

$$\begin{array}{ccccc}
 \underline{\text{lim}}(R : GK \cap \downarrow g \rightarrow \mathfrak{X}) = gR & \xrightarrow{\varphi_g} & gfR' & \xlongequal{\quad} & \underline{\text{lim}}(R' : G'K \cap \downarrow (gf) \rightarrow \mathfrak{X}) \\
 \downarrow & & \downarrow & & \downarrow \\
 kf^K R = gR & \xrightarrow{\varphi_{kf^K}} & kf^K fR' & \xrightarrow{\quad} & kR'
 \end{array} \tag{5.17}$$

in \mathfrak{X} , where all four unlabelled morphisms are representations of unique morphisms in the semilattices G and G' . The left-hand square commutes by naturality of $\varphi : R \rightarrow fR'$, while the right hand square commutes since R' represents G' in \mathfrak{X} . The equalities hold since R and R' are objects of \mathfrak{X} . Now the bottom row of (5.17) is the component at k of the natural transformation $\sigma : f^K R \rightarrow R'$ used in the definition (4.14) of $(\tau, f^F) = (\varphi, f)\tilde{E}$. Considering the diagram (5.17) as k ranges over $G'K \cap \downarrow (gf)$, one obtains $\varphi_g = [\underline{\text{Lim}}(\sigma, f^K)]_{(GK \cap \downarrow g)}$. Applying the limit-preserving functor $E : \mathfrak{X} \rightarrow \mathfrak{A}^{\text{op}}$ gives $\varphi_g E = [\underline{\text{Lim}}(\tau, f^F)]_{g\gamma_G}$. Applying D then gives the required equality (5.16), completing the proof of the theorem.

6. Schizophrenic objects

In many cases, the functors of the duality (2.1) are represented by a *schizophrenic object* [6; 14, Section VI.4.1]. (The terminology is attributed to Simmons [14, p. 268].) The schizophrenic object T appears simultaneously as an object \underline{T} of \mathfrak{A} and as an object \tilde{T} of \mathfrak{X} , in such a way that there are natural isomorphisms

$$\lambda : D \rightarrow \mathfrak{A}(\quad, \underline{T}) \tag{6.1}$$

and

$$\rho : E \rightarrow \mathfrak{X}(\quad, \tilde{T}). \tag{6.2}$$

Moreover, the underlying sets of \underline{T} and \tilde{T} coincide (with T). Also $\infty := T^0$ is “schizophrenic”, as the terminal object of \mathfrak{A} or \mathfrak{X} . Examples are furnished by the Two-element semilattice $\underline{2}$ in the duality (2.2) and the one-dimensional Torus \mathbb{R}/\mathbb{Z} in Pontryagin duality. Given a schizophrenic object T for the duality (2.1), and the terminal object ∞ of \mathfrak{A} or \mathfrak{X} , there is a representation

$$\begin{array}{ccc}
 1 & \longrightarrow & T \\
 \uparrow J & & \downarrow \\
 0 & \longrightarrow & \infty
 \end{array} \tag{6.3}$$

that may be interpreted either as an object \underline{J} or $\underline{T}^\infty (= \underline{J}\mathcal{A})$ representing $\underline{2}$ in \mathfrak{A} , or as an object \tilde{J} or $\tilde{T}^\infty (= \tilde{J}\mathcal{A})$ representing $\underline{2}$ in \mathfrak{X} . The common underlying set $\mathcal{J}\mathcal{A}$ is denoted T^∞ .

Theorem 6.1. *If T is a schizophrenic object for the duality (2.1), then T^∞ is a schizophrenic object for the duality (4.1).*

Proof. The first half of the proof sets up a natural isomorphism

$$\tilde{\lambda} : \tilde{D} \longrightarrow \tilde{\mathfrak{A}}(\underline{\quad}, \underline{J}). \tag{6.3}$$

For a representation $R : H \rightarrow \mathfrak{A}$ in $\tilde{\mathfrak{A}}$, the $\tilde{\mathfrak{X}}$ -object $\tilde{\mathfrak{A}}(R, \underline{J})$ is the representation

$$\tilde{\mathfrak{A}}(R, \underline{J}) : HC \rightarrow \mathfrak{X}; \theta \mapsto \mathfrak{A}(\varinjlim_{\theta^{-1}\{1\}} R, \underline{T}). \tag{6.4}$$

Indeed, in the corresponding fibration of $\tilde{\mathfrak{A}}(R, \underline{J})$ over $\underline{\text{S}}(H, \underline{2}) = HC$, the fibre over a character θ of H is $\text{nat}(R, \theta \underline{J}) \times \{\theta\} \cong \text{nat}(R, \theta \underline{J})$

$$\begin{aligned}
 &\cong \text{nat}(R|_{\theta^{-1}\{1\}}, (\theta \underline{J})|_{\theta^{-1}\{1\}}) \\
 &\cong \text{nat}(R|_{\theta^{-1}\{1\}}, \Delta_{\underline{T}} : \theta^{-1}\{1\} \rightarrow \mathfrak{A}) \\
 &\cong \mathfrak{A}(\varinjlim_{\theta^{-1}\{1\}} R, \underline{T}).
 \end{aligned}$$

The second natural isomorphism follows since ∞ is terminal in \mathfrak{A} , while the last natural isomorphism follows from the definition of the colimit [16, III.4(3)]. Then the component of (6.4) at R is an $\tilde{\mathfrak{X}}$ -morphism

$$\tilde{\lambda}_R = (\tau, 1_{HC}). \tag{6.5}$$

Here the natural transformation $\tau : R\tilde{D} \longrightarrow 1_{HC} \tilde{\mathfrak{A}}(R, \underline{J})$ has the natural isomorphism

$$\lambda_{\theta(R\tilde{D})} : (\varinjlim_{\theta^{-1}\{1\}} R)D \longrightarrow \mathfrak{A}(\varinjlim_{\theta^{-1}\{1\}} R, \underline{T}) \tag{6.6}$$

as its component at the character θ of H .

The other half of the proof sets up a natural isomorphism

$$\tilde{\rho} : \tilde{E} \longrightarrow \mathfrak{X}(\underline{\quad}, \underline{T}). \tag{6.7}$$

For a representation $R : G \rightarrow \mathfrak{X}$ in $\tilde{\mathfrak{X}}$, the $\tilde{\mathfrak{U}}$ -object $\tilde{\mathfrak{X}}(R, \underline{J})$ is the representation

$$\tilde{\mathfrak{X}}(R, \underline{J}) : GF \rightarrow \mathfrak{U}; \theta \mapsto \mathfrak{X}(\theta \kappa_G j R, \underline{T}). \tag{6.8}$$

Indeed, in the corresponding fibration of $\tilde{\mathfrak{X}}(R, \underline{J})$ over $\mathfrak{B}(G, \underline{2}) = GF$, the fibre over a character θ of G is $\text{nat}(R, \theta \underline{J}) \times \{\theta\} \cong \mathfrak{X}(\varinjlim R|_{\theta^{-1}\{1\}}, \underline{T}) \cong \mathfrak{X}((\inf \theta^{-1}\{1\})R, \underline{T}) \cong \mathfrak{X}(\theta \kappa_G j R, \underline{T})$, where the first natural isomorphism follows as above. Then the component of (6.8) at R is an $\tilde{\mathfrak{U}}$ -morphism

$$\tilde{\rho}_R = (\varphi, 1_{GF}). \tag{6.9}$$

Here the natural transformation $\varphi : R\tilde{E} \longrightarrow 1_{GF}\tilde{\mathfrak{X}}(R, \underline{J})$ has the natural isomorphism

$$\rho_{\theta(R\tilde{E})} : \theta \kappa_G j R E \longrightarrow \mathfrak{X}(\theta \kappa_G j R, \underline{T}) \tag{6.10}$$

as its component at the character θ of G . \square

7. Applications

This section discusses two typical applications of the general Theorems 4.3 and 6.1. In these applications, the category \mathfrak{U} of (2.1) is a strongly irregular variety \mathfrak{B} of finitary algebras without nullary operations, so that (by Theorem 3.1) $\tilde{\mathfrak{U}}$ comprises the regularization $\tilde{\mathfrak{B}}$ of the variety defined by the regular identities of \mathfrak{B} . Moreover, the duality (2.1) arises from a schizophrenic object T , so that \mathfrak{B} is the closure $\text{ISP}\{\underline{T}\}$ of the singleton class $\{\underline{T}\}$ under the closure operations P of power, S of subalgebra, and I of isomorphic copy. The representation space \underline{T} is a compact Hausdorff algebra with closed relations. These relations on \underline{T} are subalgebras of powers of T with respect to the algebra structure on T given by \underline{T} . Moreover, operations $\omega : T^n \rightarrow T$ from \underline{T} give elements of $\mathfrak{X}(T^n, \underline{T})$. Then the category \mathfrak{X} is the closure $\text{ISP}\{\underline{T}\}$ under the closure operations P of power, S of (closed) substructure, and I of isomorphic (homeomorphic) copy. (Thus the duality (2.1) is of the type described in [6, Section 1], although Davey concentrates on the case of finite T .) By Theorem 4.3, there is a duality (4.1) for the category $\tilde{\mathfrak{U}}$. By Theorem 6.1, this duality arises from the schizophrenic object T^∞ . As T^∞ in $\tilde{\mathfrak{X}}$, the schizophrenic object is a compact Hausdorff (topological) algebra with closed relations, and $\tilde{\mathfrak{X}} = \text{ISP}\{T^\infty\}$. This characterization of $\tilde{\mathfrak{X}}$ describes the class in terms of properties of T^∞ . As illustrated in the following examples, one may

often isolate a finite set of such properties to obtain appropriate axiomatizations of the representation spaces.

Example 7.1 (*Duality for left normal bands*). Here $\widetilde{\mathfrak{B}}$ is the variety $\underline{\text{Ln}}$ of left normal bands (3.2) [12, p. 119; 25, 223], while \mathfrak{B} is the variety $\underline{\text{Lz}}$ of left zero bands (3.1) [12, p. 119; 25, 225], isomorphic to the category $\underline{\text{Set}}$ of sets. Thus the duality (2.1) is the duality between sets and compact Hausdorff zero-dimensional Boolean algebras [14, VI.4.6(b)]. The schizophrenic object T is the two-element left zero band \underline{T} or the two-element discrete topological Boolean algebra \widetilde{T} . Now Boolean algebras may be described as algebras with binary meet and join operations and a unary complementation, satisfying the identities for distributive lattices along with

$$\begin{cases} (x \vee y)' = x' \wedge y'; \\ x'' = x; \\ x \vee (x \wedge x') = x \end{cases} \tag{7.1}$$

and

$$x \wedge x' = y \wedge y'. \tag{7.2}$$

Then the regularization $\widetilde{\text{Bool}}$ of the variety Bool of Boolean algebras consists of doubly distributive dissemilattices [25, p. 109] (i.e. non-absorptive distributive lattices) equipped with an additional unary operation $'$ satisfying (7.1) and

$$(x \wedge x') \vee (y \wedge y') = x \wedge x' \wedge y \wedge y' \tag{7.3}$$

[20; 22, 7.6]. The schizophrenic object T^∞ of Theorem 6.1 has three elements. As \underline{T}^∞ , it is a left normal band. As \widetilde{T}^∞ , it is a finite discrete topological regularized Boolean algebra, with its three elements α (“tilt”) from ∞ and 0 (“false”), 1 (“true”) from T selected as constants by nullary operations c_α, c_0 and c_1 respectively. Thus $\widetilde{\mathfrak{X}}$ is axiomatized as the class $\underline{\text{Stone}} \underline{\text{Bool}}_3$ of compact Hausdorff zero-dimensional $\underline{\text{Bool}}$ -algebras with 3 nullary operations c_α, c_0, c_1 satisfying

$$\begin{cases} x \vee (x \wedge c_0) = x; \\ c_0 \vee x = x = x \wedge c_1; \\ c_\alpha \vee x = c_\alpha = x \wedge c_\alpha; \\ c_0 \vee (c_0 \wedge x) = c_1 \vee x \Rightarrow x = c_\alpha. \end{cases} \tag{7.4}$$

One obtains the duality

$$\underline{\text{Ln}}(\ , \underline{T}^\infty) : \underline{\text{Ln}} \rightleftarrows \underline{\text{Stone}} \underline{\text{Bool}}_3 : \underline{\text{Stone}} \underline{\text{Bool}}_3(\ , \widetilde{T}^\infty) \tag{7.5}$$

for left normal bands that may be regarded as the regularization of Lindenbaum–Tarski duality (cf. [14, VI.4.6(a),(b)]). The algebra \widetilde{T}^∞ is well known under various names such as the “weak extension of Boolean logic” [9, 15] and the “Bochvar system

of logic” [2, #082#; 4]. Guzmán [9, Theorem 3’] characterizes the clone of (the algebra reduct of) \underline{T}^∞ as the clone of operations preserving a certain ternary relation ρ_W . As

$$\rho_W = \{(a_1, a_2, a_3) \mid a_1 = a_2 * a_3 \text{ in } (\underline{T}^\infty, *)\}, \quad (7.6)$$

this characterization arises as a consequence of the role of the schizophrenic object T^∞ in the duality (7.5).

Example 7.2 (Regularized Priestley duality). In [8, Theorem 7.5] a duality was obtained for the variety $\underline{\text{Dd}}$ of doubly distributive dissemilattices (cf. Example 7.1 above, [25, p. 109]) that is the regularization of the variety of distributive lattices. This duality is a regularization (in the current sense) of Priestley duality between distributive lattices and compact Hausdorff zero-dimensional partially ordered spaces. The example was critical in motivating the current work.

8. Regularized Pontryagin duality

The examples of Section 7 illustrated how Theorems 4.3 and 6.1 could be used to obtain regularizations of Lindenbaum–Tarski and Priestley duality. The main theorems are based on the duality (2.2) between $\underline{\text{Sl}}$ and \mathfrak{B} . The theorems applied to the regularizations $\tilde{\mathfrak{B}}$ of Section 7 because the algebras in the original varieties \mathfrak{B} there had no nullary operations in their type. In attempting an analogous regularization of Pontryagin duality, it would be necessary to construe abelian groups without constants (cf. [19]). The variety \mathfrak{B} would then include the empty model as initial object. In essence, \mathfrak{B} would be the variety of abelian quasigroups in the sense of [13, Section 2]. Taking the circle group $T = \mathbb{R}/\mathbb{Z}$ as schizophrenic object, however, Pontryagin duality does not apply to this variety \mathfrak{B} . For example, both the empty and the trivial quasigroup have a unique \mathfrak{B} -morphism to T . Thus both are represented by the trivial quasigroup. Indeed, one is left with the

Problem 8.1. Find a duality for abelian quasigroups.

The attempt to regularize Pontryagin duality may be renewed by consideration of some history. In 1956, Hewitt and Zuckerman raised the question of extending Pontryagin duality to commutative semigroups [10, p. 70]. A partial answer was given by Austin [1] in his 1962 Ph.D. thesis. However, as Austin wrote [1, p. 253]: “Lacking [separation] results, we are forced to introduce the hypothesis that [the dual of a compact object] separates points, and we are unable to prove an analogue of [the isomorphism of a discrete object with its double dual].” Despite this lack, Hofmann et al. [11, p. 27] acknowledged Austin’s work as “the forerunner for a character and duality theory for semilattices” in their own very satisfactory duality theory between $\underline{\text{Sl}}_0$ and $\underline{\text{Z}}$ (cf. Section 2). Risking revisionism, one may view the duality between $\underline{\text{Sl}}_0$ and $\underline{\text{Z}}$ as part of the continuing search for extensions of Pontryagin duality. In this vein, one

could adapt Theorems 4.3 and 6.1 to be based on $\underline{\text{Sl}}_0 \rightleftharpoons \underline{\mathbb{Z}}$ duality instead of $\underline{\text{Sl}} \rightleftharpoons \underline{\mathfrak{B}}$ duality. The changes necessary would be comparable to those made in the reverse direction in Section 2.

With Theorems 4.3 and 6.1 suitably modified to be based on the $\underline{\text{Sl}}_0 \rightleftharpoons \underline{\mathbb{Z}}$ duality, one may consider the traditional variety $\underline{\text{Ab}}$ of abelian groups with the zero element selected as a constant. One then obtains the variety $\widetilde{\underline{\text{Ab}}}_0$ of commutative inverse semigroups with identity 0 satisfying

$$\begin{cases} -(-x) = x \\ -(x + y) = (-x) + (-y) \\ x - x + x = x \\ x + 0 = x \end{cases} \tag{8.1}$$

[21; 22, Section 11]. By Corollary 3.2, $\widetilde{\underline{\text{Ab}}}_0$, as the class of Płonka sums with constants of $\underline{\text{Ab}}$ -algebras over $\underline{\text{Sl}}_0$ -semilattices [21; 22, Theorem 11.1(b)], is equivalent to the semicolon category $(\underline{\text{Sl}}_0; \underline{\text{Ab}}^{\text{op}})$ of contravariant representations of $\underline{\text{Sl}}_0$ -semilattices in $\underline{\text{Ab}}$. The schizophrenic object $T = \mathbb{R}/\mathbb{Z}$ for Pontryagin duality $\underline{\text{Ab}} \rightleftharpoons \underline{\text{CH}} \underline{\text{Ab}}$ between $\underline{\text{Ab}}$ and the category $\underline{\text{CH}} \underline{\text{Ab}}$ of compact Hausdorff abelian groups [14, VI.4.9] then yields a schizophrenic object T^∞ (6.3). As an $\widetilde{\underline{\text{Ab}}}_0$ -object, T^∞ has 0 in $(\mathbb{R}/\mathbb{Z}, +, 0) = T$ selected by the nullary operation. Then T^∞ is a compact Hausdorff $\widetilde{\underline{\text{Ab}}}_0$ -algebra. The class $\underline{\text{CH}} \widetilde{\underline{\text{Ab}}}$ is thus axiomatized as the category of compact Hausdorff $\widetilde{\underline{\text{Ab}}}_0$ -algebras. One obtains

$$\widetilde{\underline{\text{Ab}}}_0(\quad, T^\infty) : \widetilde{\underline{\text{Ab}}}_0 \rightleftharpoons \underline{\text{CH}} \widetilde{\underline{\text{Ab}}} : \underline{\text{CH}} \widetilde{\underline{\text{Ab}}}(\quad, T^\infty) \tag{8.2}$$

as the regularization of Pontryagin duality. For an $\widetilde{\underline{\text{Ab}}}_0$ -monoid G , the dual $\widetilde{\underline{\text{Ab}}}_0(G, T^\infty)$ is the set \widehat{G}^* of semicharacters of G in the notation of [10, Definition 5.3].

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